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ON THE SPECTRA OF JOHNSON GRAPHS*

MIKE KREBS† AND ANTHONY SHAHEEN†

Abstract. The spectrum of a Johnson graph is known to be given by the Eberlein polynomial. In this paper, a straightforward representation-theoretic derivation of this fact is presented. Also discussed are some consequences of this formula, such as the fact that infinitely many of them are Ramanujan.

Key words. Young subgroups, Spherical functions, Finite symmetric spaces, Ramanujan graphs, Symmetric groups, Representations, Characters, Spectral graph theory, Gelfand pair.

AMS subject classifications. 20B35, 05C99, 53C35, 43A90.

1. Introduction. Johnson graphs are heavily studied objects. The spectrum of any graph—that is, the multiset of eigenvalues of its adjacency matrix—is an important invariant from which much information about the graph can be ascertained. It is known that the spectrum of a Johnson graph is given by the Eberlein polynomial. This has been derived in the context of association schemes [1, 4, 9], q-analogs [6, 8], and wreath products [7]. See also [5, 18, 19, 11, 15, 16]. While some of the above papers prove this result in a representation theoretic way, in this paper, we present an alternate proof, one which combines combinatorial and representation-theoretic techniques.

Given a group G and a subgroup K, we say that the pair (G, K) is "double-coset-inversion-stable" (DCIS), if $KzK = Kz^{-1}K$ for all z in G. This condition allows us to form the Cayley graph $\mathrm{Cay}(G, KzK)$ of G generated by KzK. One can then define the quotient graph $\mathrm{Cay}(G, KzK)/K$ of $\mathrm{Cay}(G, KzK)$ as follows: the vertices of $\mathrm{Cay}(G, KzK)/K$ are the elements of G/K, and two vertices xK and yK are connected by an edge if and only if $x^{-1}y \in KzK$. (One may interpret the set $\{KzK\}$ of double cosets as a set of distances—i.e., $x^{-1}y \in KzK$ means that xK has distance KzK from yK.) We remark that any DCIS pair (G, K) is necessarily a Gelfand pair or "finite symmetric space," meaning that the set $L^2(K\backslash G/K)$ of K-bi-invariant complex-valued functions on G is commutative under convolution, or equivalently that the Hecke algebra $\mathcal{H}(K, G)$ is commutative [13, 20].

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†Department of Mathematics, California State University - Los Angeles, 5151 State University Drive, Los Angeles, California 90032, USA (mkrebs@calstatela.edu, ashahee@calstatela.edu).

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In this paper, we consider the case where S_ℓ is the symmetric group on ℓ letters, and $Y_\lambda = S_{\lambda_1} \times \ldots \times S_{\lambda_k}$ is a Young subgroup of S_ℓ for some partition $\lambda_1 + \ldots + \lambda_k = \ell$. We shall restrict our attention to the case $k = 2$, as the pair (S_ℓ, Y_λ) is DCIS if and only if $k \leq 2$. Therefore fix $m \leq n$, and let $Y = S_m \times S_n$. A system of double coset representatives for Y in S_{m+n} is given by τ_i for $i = 0, \ldots, m$, where τ_i is the product of the i disjoint transpositions $(1, m+1), \ldots, (i, m+i)$. It is easily verified that the graphs $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ are precisely the Johnson graphs (see Remark 3.4).

Define $c(i, m', n') = \binom{m'}{i}\binom{n'}{i}$ if $0 \leq i \leq m'$ and $c(i, m', n') = 0$ otherwise. Define $E(i, j, m', n')$ recursively by $E(i, j, m', n') = c(i, m', n')$ if $j = m'$ and $E(i, j, m', n') = E(i, j, m'-1, n'-1) - E(i-1, j, m'-1, n'-1)$ if $j < m'$. (So E is the Eberlein polynomial.)

The main fact about the spectra of Johnson graphs is that every eigenvalue of the adjacency matrix of the graph $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ equals $E(i, j, m, n)$ for some $j \in \{0, 1, \ldots, m\}$. The theory of Gelfand pairs provides a means of computing these eigenvalues in terms of representations of the symmetric group; this is the approach we shall take.

We then discuss some consequences of our knowledge of the spectra. In particular, we note the well-known facts that these graphs are (in most cases) connected and nonbipartite. Moreover, we find as an immediate corollary that for all n, the graph $\mathrm{Cay}(S_{2+n}, Y\tau_2 Y)/Y$ is Ramanujan (see Definition 5.7 for a definition of Ramanujan graphs).

This construction does not yield as many Ramanujan graphs as might be desired. If $m = 1$, then the graphs $\mathrm{Cay}(S_{m+n}, Y\tau_1 Y)/Y$ are complete; only finitely many of these graphs, other than $\mathrm{Cay}(S_{1+n}, Y\tau_1 Y)/Y$ and $\mathrm{Cay}(S_{2+n}, Y\tau_2 Y)/Y$, seem to be Ramanujan. Moreover, as $n \to \infty$, the degree of $\mathrm{Cay}(S_{2+n}, Y\tau_2 Y)/Y$ also goes to infinity. (This is similar to the case of the finite upper half plane graphs [20].) One might prefer to construct a family of Ramanujan graphs with fixed degree.

2. Adjacency operators of DCIS pairs. In this section, we define the Cayley graph quotient $\mathrm{Cay}(G, KaK)/K$ of a DCIS pair (G, K), and we show that $\mathrm{Cay}(G, KaK)/K$ is a highly regular graph.

Definition 2.1. Let G be a finite group, and let K be a subgroup of G. Then (G, K) is *double-coset-inversion-stable* (DCIS) if $a^{-1} \in KaK$ for all $a \in G$.

Definition 2.2. Let (G, K) be a DCIS pair, and let $a \in G$. Let $\mathrm{Cay}(G, KaK)/K$ be the graph whose vertex set is the set G/K of left cosets of K in G, where two vertices xK and yK are connected by an edge if and only if $x^{-1}y \in KaK$.

Given a DCIS pair (G, K), let $\tilde{A}_{KaK}$ denote the adjacency operator of

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ELA
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$\text{Cay}(G, KaK)/K$. Given a function

$$f \in L^2(G/K) = \{f : G \to \mathbb{C} \mid f(gk) = f(g), \forall g \in G, k \in K\},$$

we have that

$$(\tilde{A}_{KaK} f)(x) = \sum_{\substack{y:G/K \\ y^{-1}x \in KaK}} f(y),$$

(The notation $y : G/K$ indicates that y runs through a set of representatives for the left cosets of K in G.) Note that if $x_1K = x_2K$ and $f \in L^2(G/K)$, then $(\tilde{A}_{KaK} f)(x_1) = (\tilde{A}_{KaK} f)(x_2)$. Therefore, $\tilde{A}_{KaK} f \in L^2(G/K)$.

Note that $y^{-1}x \in KaK$ if and only if $x = ys$ for some $s \in KaK$. Therefore, if $f \in L^2(G/K)$, then

$$(\tilde{A}_{KaK} f)(x) = \sum_{s:KaK/K} f(xs).$$

One can re-phrase the above equation using a "distance" function for the graph $\text{Cay}(G, KaK)/K$. Define a distance function $d : G/K \times G/K \to S$, where S is the set of K-double cosets in G, as follows: if $xK, yK \in G/K$, let $d(xK, yK) = Ky^{-1}xK$. It is easy to see that $d(gzK, gwK) = d(zK, wK)$ for all $g \in G$ and $zK, wK \in G/K$. Also, $d(zK, wK) = d(wK, zK)$ since (G, K) is a DCIS pair. We have that

$$(\tilde{A}_{KaK} f)(z) = \sum_{\substack{w:G/K \\ d(zK,wK)=KaK}} f(w).$$

Consider the subspace

$$L^2(K\backslash G/K) = \{f : G \to \mathbb{C} \mid f(k_1gk_2) = f(g), \forall g \in G, k_1, k_2 \in K\}$$

of $L^2(G/K)$. If $f \in L^2(K\backslash G/K)$, $k \in K$, and $z \in G$, then

$$(\tilde{A}_{KaK} f)(kz) = \sum_{\substack{w:G/K \\ d(kzK,wK)=KaK}} f(w) = \sum_{\substack{w:G/K \\ d(zK,k^{-1}wK)=KaK}} f(w) = \sum_{\substack{kw:G/K \\ d(zK,wK)=KaK}} f(kw) = \sum_{\substack{w:G/K \\ d(zK,wK)=KaK}} f(w).$$

Therefore, $\tilde{A}_{KaK} : L^2(K\backslash G/K) \to L^2(K\backslash G/K)$.
Let $A_{KaK} = \tilde{A}_{KaK} \mid_{L^2(K\backslash G/K)}$. We call A_{KaK} the *collapsed adjacency operator*.

Remark 2.3. There is a connection between the collapsed adjacency operator and the Hecke algebra $\mathcal{H}(K, G)$, given as follows. For any $f_1, f_2 \in L^2(K\backslash G/K)$, define their convolution $f_1 * f_2 \in L^2(K\backslash G/K)$ by $(f_1 * f_2)(g) = \sum_{ab=g} f_1(a)f_2(b)$.

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For any $S \subset G$, define $\delta_S : G \to \mathbb{C}$ by $\delta_S(g) = 1$ if $g \in S$ and $\delta_S(g) = 0$ if $g \notin S$. There is an isomorphism from $\mathcal{H}(K,G)$ to $L^2(K\backslash G/K)$ that maps the double coset KaK to $\frac{1}{\#(K)}\delta_{KaK}$ [13, p. 26]. If $f \in L^2(K\backslash G/K)$, then

$$(\delta_{KaK} * f)(g) = (f * \delta_{KaK})(g) = \sum_{h\in G} f(h)\delta_{KaK}(h^{-1}g) = \sum_{\substack{h\in G\\ h^{-1}g\in KaK}} f(h)$$

$$= \sum_{\substack{h\in G\\ g^{-1}h\in KaK}} f(h) = \#(K)\sum_{\substack{h:G/K\\ g^{-1}h\in KaK}} f(h) = \#(K)\sum_{s\in KaK/K} f(gs) = \#(K)(\tilde{A}_{KaK}f)(g).$$

Therefore, the operator A_{KaK} can be thought of as the K-double coset KaK acting on $\mathcal{H}(K,G)$ by left multiplication.

Definition 2.4 ([2]). A graph with vertex set V is called *highly regular with collapsed adjacency matrix* $C = (c_{ij})$ if for every vertex $x \in V$ there is a partition of V into non-empty sets $V_1 = \{x\}, V_2, \ldots, V_p$ such that each vertex $y \in V_i$ is adjacent to exactly c_{ij} vertices in V_j.

Note that our definition of the collapsed adjacency matrix is the transpose of that in [2], and that we do not require the graph to be connected.

Lemma 2.5. *If (G,K) is a DCIS pair, then* $\mathrm{Cay}(G,KaK)/K$ *is highly regular. Moreover, $\tilde{A}_{KaK}$ and A_{KaK} have the same minimal polynomial (and in particular, the same eigenvalues).*

Proof. Let $x = gK$. Let $a_1, \ldots, a_p$ be a set of representatives for the double cosets of K in G, with $a_1 = e$. For $1 \le j \le p$, let $V_j = \{ghK \mid h \in Ka_jK\}$. Then the V_j's partition the vertex set G/K of $\mathrm{Cay}(G,KaK)/K$. Let $B = (b_{ij})$ be the matrix for A_{KaK} with respect to the standard basis $\{\delta_{Ka_1K}, \ldots, \delta_{Ka_nK}\}$ of $L^2(K\backslash G/K)$. It follows from the definitions that $\mathrm{Cay}(G,KaK)/K$ is highly regular with collapsed adjacency matrix B. It is shown in [2, pp. 272–273] that the adjacency matrix of any highly regular graph has the same minimal polynomial as its collapsed adjacency matrix. $\square$

3. The graphs $\mathrm{Cay}(S_{m+n}, Y\tau_kY)/Y$.

Let S_ℓ denote the symmetric group on ℓ letters. Let $Y_{m,n} = S_m \times S_n$. (We may sometimes write Y instead of $Y_{m,n}$.)

Lemma 3.1. *Suppose $1 \le m \le n$, and let $\tau_k = (1, m+1)\ldots(k, m+k)$ for $0 \le k \le m$. (τ_0 is the identity element.) Then a complete set of double-coset representatives for Y in S_{m+n} is given by $\tau_0, \ldots, \tau_m$. Moreover, (S_{m+n}, Y) is a DCIS pair.*

Proof. Krieg [13, p. 58] shows that there are exactly $m+1$ double cosets YaY for a in S_{m+n}, namely $Y\tau_0Y, \ldots, Y\tau_mY$. This shows that (S_{m+n}, Y) is DCIS. $\square$

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Volume 17, pp. 154-167, March 2008

Remark 3.2. If (G,K) is any DCIS pair, then $\mathrm{Cay}(G,KaK)/K$ has $[G:K]$ vertices and degree $\mathrm{ind}(KaK)$, where $\mathrm{ind}(KaK)=\frac{\#(KaK)}{\#K}$.

Remark 3.3. If (G,K) is any DCIS pair, then $\mathrm{Cay}(G,KeK)/K$ consists of nothing but loops at each vertex.

Remark 3.4. Let $J(a,b,c)$ be the Johnson graph where the vertices are the subsets of $\{1,2,\ldots,a\}$ of size b, and where the subsets X and Y are adjacent if and only if $\#(X\cap Y)=c$. We now show that $\mathrm{Cay}(S_{m+n},Y\tau_k Y)/Y$ and $J(m+n,m,m-k)$ are isomorphic as graphs. In this paper, we multiply as follows: if $\sigma,\tau\in S_{m+n}$, then $\sigma\tau=\tau\circ\sigma$. Krieg [13, p. 60] shows that the sets of the form:

$$R_T=\{\pi\in S_{m+n}\mid \pi^{-1}\{1,\ldots,m\}=T\},$$

where $T\subseteq\{1,\ldots,m+n\}$ give the set of left cosets for S_{m+n}/Y. Krieg describes these as right cosets, but we multiply in the opposite order as Krieg, so they are left cosets. This gives a correspondence between the vertices of $\mathrm{Cay}(S_{m+n},Y\tau_k Y)/Y$ and $J(m+n,m,m-k)$. It is easy to see that xY is adjacent to yY in $\mathrm{Cay}(S_{m+n},Y\tau_k Y)/Y$ if and only if $\#((y^{-1}x)(\{1,\ldots,m\})\cap\{m+1,\ldots,m+n\})=k$. Therefore, xY is adjacent to yY if and only if $\#(T_1\cap T_2)=m-k$ where $T_1=y^{-1}(\{1,\ldots,m\})$ and $T_2=x^{-1}(\{1,\ldots,m\})$.

Lemma 3.5.

1. *$\mathrm{Cay}(S_{n+m},Y\tau_k Y)/Y$ has degree $\binom{m}{k}\binom{n}{k}$.*
2. *If $n=m$, then $\mathrm{Cay}(S_{n+m},Y\tau_m Y)/Y$ has $\frac{(2m)!}{2(m!)^2}$ components, each of which consists of two vertices and an edge.*

Proof. Krieg [13, p. 60] shows that $\binom{m}{k}\binom{n}{k}$ equals $\mathrm{ind}(Y\tau_k Y)$, which, by Remark 3.2, is the degree of $\mathrm{Cay}(S_{n+m},Y\tau_k Y)/Y$.

If $n=k$ (which is only possible if $n=m$), then $Y\tau_k Y$ is the set of all permutations $\zeta\in S_{m+n}$ such that $\zeta(M)=N$ and $\zeta(N)=M$. It follows that a vertex xK in $\mathrm{Cay}(S_{m+n},Y\tau_k Y)/Y$ is connected to $\tau_k xK$ and to no other vertex. Hence the number of components in this graph is $\frac{1}{2}\cdot\frac{|S_{m+n}|}{|Y|}=\frac{(2m)!}{2(m!)^2}$. ∎

Remark 3.6. Remark 3.3 and Lemma 3.5(b) show that $\mathrm{Cay}(S_{n+m},Y\tau_0 Y)/Y$ and $\mathrm{Cay}(S_{2m},Y\tau_m Y)/Y$ are both disconnected, and that the latter is bipartite. In the sequel, we will see that these are exceptional cases. The case $m=n=k$ is reminiscent of the case $a=4\delta$ in the finite upper half plane graphs.

The next lemma gives us a recursive formula which completely determines all eigenvalues and all eigenfunctions of the collapsed adjacency operator for $\mathrm{Cay}(S_{m+n},Y\tau_1 Y)/Y$. We shall see in the sequel that this will enable us to determine the spectrum of $\mathrm{Cay}(S_{m+n},Y\tau_i Y)/Y$ for all i.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 17, pp. 154-167, March 2008

Lemma 3.7. *Let* $Y_1 = S_m \times S_n$ *and* $Y_2 = S_{m+1} \times S_{n+1}$*. Consider the basis* $\beta_1 = \{\delta_{Y_1\tau_0 Y_1}, \ldots, \delta_{Y_1\tau_m Y_1}\}$ *for* $L^2\left(Y_1 \backslash S_{m+n}/Y_1\right)$*, and the basis* $\beta_2 = \{\delta_{Y_2\tau_0 Y_2}, \ldots, \delta_{Y_2\tau_{m+1} Y_2}\}$ *for* $L^2\left(Y_2 \backslash S_{(m+1)+(n+1)}/Y_2\right)$ *.*

Let $A_{m,n}$ *be the collapsed adjacency operator for* $\mathrm{Cay}(S_{m+n}, Y_1\tau_1 Y_1)/Y_1$*, and let* $A_{m+1,n+1}$ *be the collapsed adjacency operator for* $\mathrm{Cay}(S_{(m+1)+(n+1)}, Y_2\tau_1 Y_2)/Y_2$*. If*

$$[f_{j,m,n}]_{\beta_1} = \left(\ 1, \frac{a_1}{\binom{m}{1}\binom{n}{1}}, \ldots, \frac{a_m}{\binom{m}{m}\binom{n}{m}}\ \right)^T$$

is an eigenfunction of $A_{m,n}$ *with eigenvalue* a_1*, then*

$$[f_{j,m+1,n+1}]_{\beta_2} = \left(\ 1, \frac{a_1 - 1}{\binom{m+1}{1}\binom{n+1}{1}}, \ldots, \frac{a_m - a_{m-1}}{\binom{m+1}{m}\binom{n+1}{m}}, \frac{-a_m}{\binom{m+1}{m+1}\binom{n+1}{m+1}}\ \right)^T$$

is an eigenfunction of $A_{m+1,n+1}$ *with eigenvalue* $a_1 - 1$*.*

Proof. $a_0 = 1$. From [13, p. 60] we have that $A_{m,n}\delta_{Y_1\tau_0 Y_1} = \delta_{Y_1\tau_1 Y_1}$,

$$A_{m,n}\delta_{Y_1\tau_k Y_1} = (m+1-k)(n+1-k)\delta_{Y_1\tau_{k-1}Y_1} + k(m+n-2k)\delta_{Y_1\tau_k Y_1} + (k+1)^2\delta_{Y_1\tau_{k+1}Y_1}$$

for $0 < k < m$, and $A_{m,n}\delta_{Y_1\tau_m Y_1} = (n+m-1)\delta_{Y_1\tau_{m-1}Y_1} + m(n-m)\delta_{Y_1\tau_m Y_1}$. The result then follows from induction. ◻

4. Gelfand Pairs and Spherical Functions.

Definition 4.1. If G is a finite group and K is a subgroup of G, then we say that (G, K) is a *Gelfand pair* if $L^2\left(K\backslash G/K\right)$ is a commutative algebra under convolution.

We now briefly note some of the main facts about Gelfand pairs; more information can be found in [20, Ch. 19]. Given a Gelfand pair (G, K), let $\hat{G}$ be the set of all irreducible representations of G, modulo equivalence. Let $Ind_K^G(1)$ denote the representation on G induced by the trivial representation on K. Define $\hat{G}^K = \{\pi \in \hat{G} \mid \pi \text{ occurs in } Ind_K^G(1)\}$. Given $\pi \in \hat{G}^K$, let χ_π is the character associated with π, and define the *spherical function* $h_\pi \in L^2\left(K\backslash G/K\right)$ by $h_\pi(x) = \frac{1}{\#(K)}\sum_{k\in K}\chi_\pi(kx)$. The set $\{h_\pi \mid \pi \in \hat{G}^K\}$ is an orthogonal basis for $L^2\left(K\backslash G/K\right)$.

Lemma 4.2. *If* (G, K) *is a Gelfand pair, then* $Ind_K^G(1)$ *is multiplicity free; that is, no representation occurs more than once in the decomposition* $Ind_K^G(1) = \pi_1 \oplus \ldots \oplus \pi_r$*.*

Proof. See [20, pg. 344]. ◻

Remark 4.3. Every DCIS pair is a Gelfand pair.

Remark 4.4. Suppose that h_π is a spherical function for a DCIS pair (G,K). Then by [20, p. 343] we have $\frac{1}{\#(K)}\sum_{k\in K} h_\pi(xky) = h_\pi(x)h_\pi(y)$, $\forall x, y \in G$. Suppose that $x \in G$ and KyK equals the disjoint union $y_1 K \coprod \ldots \coprod y_n K$. Let $k_i \in K$ such

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ELA
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that $y_iK = k_iyK$. Let $ind(KyK) = n = \frac{\#(KyK)}{|K|}$. Then

$$(\delta_{KyK} * h_\pi)(x) = \sum_{ab=x} \delta_{KyK}(a)h_\pi(b) = \sum_{a\in KyK} h_\pi(a^{-1}x) = \sum_{a\in KyK} h_\pi(ax)$$

$$= \sum_{i=1}^{n} \sum_{a\in y_iK} h_\pi(ax) = \sum_{i=1}^{n} \sum_{k\in K} h_\pi(k_iykx) = ind(KyK) \sum_{k\in K} h_\pi(ykx)$$

$$= ind(KyK)\#(K)h_\pi(y)h_\pi(x).$$

By Remark 2.3, $(A_{KyK}h_\pi)(x) = (ind(KyK)h_\pi(y))\, h_\pi(x)$. Hence every eigenvalue of $\tilde{A}_{KyK}$ is of the form $ind(KyK)h_\pi(y)$ for some $\pi \in \hat{G}^K$.

We now apply these general facts about Gelfand pairs to the specific case of $(S_{m,n}, Y)$. The representation theory of the symmetric group is classical. (We take [10, §4] and [12] as general references for it.) We shall make use of the correspondence between Young diagrams and irreducible representations of the symmetric group, as in [10, §4]. Let $G = S_{m+n}$, and let $K = Y = S_m \times S_n$.

Lemma 4.5. *For any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of $m + n$, let π_λ be the representation of G whose associated Young diagram corresponds to λ. Then $\hat{G}^K = \{\pi_{(m+n)}\} \cup \{\pi_{(n+j,m-j)} \,|\, 0 \le j < m\}$.*

Proof. This follows from Young's Rule [10, p. 57], [12, p. 51]. □

In other words, Lemma 4.5 says that $\hat{G}^K$ equals the set of all representations whose Young diagram has either one or two rows, where in the latter case the second row has no more than m boxes. This gives us a one-to-one correspondence between spherical functions of the Gelfand pair (G, K) and partitions $(n + j) + (m - j)$ for $0 \le j \le m$. Let $s_{j,m,n}$ be the spherical function corresponding to the partition $(n + j) + (m - j)$.

Lemma 4.6. *For all $y_1, y_2 \in Y$, we have $s_{j,m,n}(y_1\tau_1y_2) = \frac{(m-k)(n-k)-k}{mn}$, where $k = m - j$.*

Proof. Let $\chi_{(m+n-k,k)}$ be the character of the representation associated to the Young diagram with two rows, $m + n - k$ boxes in the top row and k boxes in the bottom row. To save space, we will sometimes denote $\chi_{(m+n-k,k)}$ by χ. For any set T, let S_T be the group of all permutations of T. Define

$$H_i^M = S_{\{1\}} \times S_{\{2\}} \times \ldots \times S_{\{i\}} \times S_{\{i+1,\ldots,m\}}$$
$$H_t^N = S_{\{m+1\}} \times S_{\{m+2\}} \times \ldots S_{\{m+t\}} \times S_{\{m+t+1,\ldots,m+n\}}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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where we let $H_m^M = H_{m-1}^M$ and $H_n^N = H_{n-1}^N$. Since S_m equals the disjoint union $H_1^M \cup H_1^M(1,2) \cup \ldots \cup H_1^M(1,m)$, we have that

$$\begin{aligned}
\#(Y)s_{j,m,n}(\tau_1) &= \sum_{\sigma\in Y\tau_1} \chi_{(m+n-j,j)}(\sigma) \\
&= \sum_{\sigma\in(H_1^M\times S_n)\tau_1} \chi(\sigma) \;+ \sum_{\sigma\in(H_1^M\times S_n)(1,2)\tau_1} \chi(\sigma) \;+\; \ldots \;+ \sum_{\sigma\in(H_1^M\times S_n)(1,m)\tau_1} \chi(\sigma) \\
&= \sum_{\sigma\in(H_1^M\times S_n)\tau_1} \chi(\sigma) \;+\; (m-1) \sum_{\sigma\in(H_1^M\times S_n)(1,2)\tau_1} \chi(\sigma).
\end{aligned}$$

Continuing in this fashion, first by using $H_1^M = H_2^M \cup H_2^M(2,3) \cup \ldots \cup H_2^M(2,m)$ and so on, we find that $\#(Y)s_{j,m,n}(\tau_1)$ equals

$$\begin{aligned}
&\sum_{\sigma\in(H_1^M\times S_n)\tau_1} \chi(\sigma) + (m-1) \sum_{\sigma\in(H_2^M\times S_n)(1,2)\tau_1} \chi(\sigma) + \\
&\qquad (m-1)(m-2) \sum_{\sigma\in(H_3^M\times S_n)(2,3)(1,2)\tau_1} \chi(\sigma) + \ldots + \\
&\qquad (m-1)(m-2)\cdots(2) \sum_{\sigma\in(H_{m-1}^M\times S_n)(m-2,m-1)\cdots(1,2)\tau_1} \chi(\sigma) + \\
&\qquad (m-1)! \sum_{\sigma\in(H_m^M\times S_n)(m-1,m)(m-2,m-1)\cdots(1,2)\tau_1} \chi(\sigma).
\end{aligned}$$

Let $H_{i,t} = H_i^M \times H_t^N$,

$$\sigma_{i,t} = \begin{cases} \tau_1, & i=t=1 \\ (i-1,i)\cdots(1,2)\tau_1, & t=1, i>1 \\ (m+t-1,m+t)\cdots(m+1,m+2)\tau_1, & t>1, i=1 \\ (m+t-1,m+t)\cdots(m+1,m+2)(i-1,i)\cdots(1,2)\tau_1, & t,i>1 \end{cases}$$

$$c_{i,t} = \begin{cases} 1, & i=t=1 \\ (m-1)(m-2)\cdots(m-i+1), & t=1, i>1 \\ (n-1)(n-2)\cdots(n-t+1), & t>1, i=1 \\ (m-1)(m-2)\cdots(m-i+1)(n-1)(n-2)\cdots(n-t+1), & t,i>1 \end{cases}$$

and

$$S_{i,t} = c_{i,t} \sum_{\sigma\in H_{i,t}\sigma_{i,t}} \chi(\sigma).$$

If now we expand $s_{j,m,n}$ in terms of S_n then we find that $\#(Y)s_{j,m,n}(\tau_1)$ equals the

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 154-167, March 2008

following sum:

$$
\begin{array}{ccccccccccccc}
S_{1,1} & + & S_{2,1} & + & S_{3,1} & + & \dots & S_{m-1,1} & + & S_{m,1} & + \\
S_{1,2} & + & S_{2,2} & + & S_{3,2} & + & \dots & S_{m-1,2} & + & S_{m,2} & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
S_{1,n-1} & + & S_{2,n-1} & + & S_{3,n-1} & + & \dots & S_{m-1,n-1} & + & S_{m,n-1} & + \\
S_{1,n} & + & S_{2,n} & + & S_{3,n} & + & \dots & S_{m-1,n} & + & S_{m,n} &
\end{array}
\tag{4.1}
$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product for group characters. We now evaluate $\sum_{i,t} S_{i,t}$. By the Murnaghan-Nakayama Rule and Frobenius Reciprocity, we have that

$$S_{i,t} = (n-1)!(m-1)!\,[\,OH(i,t) - TR(i,t) + BR(i,t)\,]$$

where

$$OH(i,t) = \begin{cases} \left\langle \chi_{(n+m-k-(i+t),k)}, 1_{S_{m-i}\times S_{n-t}}^{S_{(m-i)+(n-t)}} \right\rangle & , \quad i+t \le n+m-2k \\ 0 & , \quad otherwise \end{cases}$$

$$TR(i,t) = \begin{cases} \left\langle \chi_{(k-1,k-(i+t-(n+m-2k+1)))}, 1_{S_{m-i}\times S_{n-t}}^{S_{(m-i)+(n-t)}} \right\rangle & , \quad n+m-2k+1 < i+t \le n+m-(k-1) \\ 0 & , \quad otherwise \end{cases}$$

and

$$BR(i,t) = \begin{cases} \left\langle \chi_{(n+m-k,k-(i+t))}, 1_{S_{m-i}\times S_{n-t}}^{S_{(m-i)+(n-t)}} \right\rangle & , \quad i+t \le k \\ 0 & , \quad otherwise \end{cases} .$$

By Lemma 4.2 and Lemma 4.5 we have that

$$OH(i,t) = \begin{cases} 1 & , \quad i+t \le n+m-2k,\ i \le m-k,\ t \le n-k \\ 0 & , \quad otherwise \end{cases}$$

$$TR(i,t) = \begin{cases} 1 & , \quad n+m-2k+1 < i+t \le n+m-(k-1),\ i \ge m-k+1,\ t \ge n-k+1 \\ 0 & , \quad otherwise \end{cases}$$

and

$$BR(i,t) = \begin{cases} 1 & , \quad i+t \le k \\ 0 & , \quad otherwise \end{cases} .$$

By arranging the terms in a grid, as in equation (4.1), one can see that

$$\sum_{i=1}^{m} \sum_{t=1}^{n} OH(i,t) = \binom{n+m-2k}{2} - \binom{n-k}{2} - \binom{m-k}{2},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 17, pp. 154-167, March 2008
ELA
http://math.technion.ac.il/iic/ela

$$\sum_{i=1}^{m}\sum_{t=1}^{n} TR(i,t) = \binom{2k}{2} - 3\binom{k}{2},$$

and

$$\sum_{i=1}^{m}\sum_{t=1}^{n} BR(i,t) = \binom{k}{2}.$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{m}\sum_{t=1}^{n} S_{i,t} &= (m-1)!(n-1)!\left[\tbinom{n+m-2k}{2} - \tbinom{n-k}{2} - \tbinom{m-k}{2} - \tbinom{2k}{2} + 4\tbinom{k}{2}\right]\\ &= (m-1)!(n-1)!\left((m-k)(n-k)-k\right). \qquad \square\end{aligned}$$

To illustrate the proof of Lemma 4.6, we consider the case $n = 8$, $m = 5$, and $j = 2$. Then $nm \cdot s_{2,5,8}(\tau_1)$ equals the sum in the table below, obtained by filling in equation (4.1):

$t\backslash i$	1		2		3		4		5	
1	2	+	2	+	0	+	0	+	0	+
2	2	+	1	+	0	+	0	+	0	+
3	1	+	1	+	0	+	0	+	0	+
4	1	+	1	+	0	+	0	+	0	+
5	1	+	1	+	0	+	0	+	0	+
6	0	+	0	+	-1	+	-1	+	-1	+
7	0	+	0	+	-1	+	-1	+	0	+
8	0	+	0	+	-1	+	0	+	0	

Add the “triangle of 2’s” to the “triangle of -1’s.” One is then left with an $(m-k)\times(n-k)$ rectangle of 1’s plus a diagonal of length k, consisting of -1’s.

One might attempt to evaluate $s_{j,m,n}(\tau_i)$ for all i by using a technique similar to that in the proof of Lemma 4.6. However, the sum analogous to (4.1) becomes rather complicated. In the next section, we will instead use an indirect method to evaluate $s_{j,m,n}(\tau_i)$ for all i.

5. Spectra of Johnson graphs, and some consequences. In this section, we conclude our proof of the fact that the spectra of the Johnson graphs are given by the Eberlein polynomial. Our first result provides a recursive formula for the spherical functions $s_{j,m,n}$ of the Gelfand pair (S_{m+n}, Y) and thereby determines the spectra of the Johnson graphs $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$. We then discuss some consequences; for example, we conclude that, modulo a few exceptional cases, these graphs are

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 154-167, March 2008

ELA
http://math.technion.ac.il/iic/ela

connected and nonbipartite. Moreover, we find that the graphs $\mathrm{Cay}(S_{2+n}, Y\tau_2 Y)/Y$ are Ramanujan for all n.

THEOREM 5.1. *Define the following functions $f_{j,m,n} : S_{m+n} \to \mathbb{C}$ recursively on m. For $m = 1$, define*

$$f_{0,1,n}(g) = \begin{cases} 1 & , \quad g \in Y_{1,n}\tau_0 Y_{1,n} \\ \frac{-1}{n} & , \quad g \in Y_{1,n}\tau_1 Y_{1,n} \end{cases} \qquad \textit{and} \qquad f_{1,1,n}(g) \equiv 1$$

For $m > 1$, define

$$f_{j,m+1,n+1}(g) = \begin{cases} 1, & g\in Y_{m+1,n+1}\,\tau_0\,Y_{m+1,n+1} \\ \frac{\binom{m}{i}\binom{n}{i} f_{j,m,n}(\tau_i) - \binom{m}{i-1}\binom{n}{i-1} f_{j,m,n}(\tau_{i-1})}{\binom{m+1}{i}\binom{n+1}{i}}, & g\in Y_{m+1,n+1}\tau_i Y_{m+1,n+1},\ 1\le i\le m \\ \frac{-\binom{m}{m}\binom{n}{m} f_{j,m,n}(\tau_m)}{\binom{m+1}{m+1}\binom{n+1}{m+1}}, & g\in Y_{m+1,n+1}\,\tau_{m+1}\,Y_{m+1,n+1} \end{cases}$$

for $0 \le j \le m$, and define $f_{m+1,m+1,n+1}(g) \equiv 1$. Let

$$\begin{aligned} E(i,j,m,n) &= \sum_{k=max\{0,i-j\}}^{min\{m-j,i\}} (-1)^k \binom{m-j}{k}\binom{j}{i-k}\binom{n-(m-j)}{i-k} \\ &= \binom{m}{i}\binom{n}{i} f_{j,m,n}(\tau_i). \end{aligned}$$

Then:

(a) For all j, m, n, we have $s_{j,m,n} = f_{j,m,n}$.

(b) Every eigenvalue of the adjacency operator $\tilde{A}_{Y_{m,n}\tau_i Y_{m,n}}$ of $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ equals $E(i,j,m,n)$ for some $j \in \{0,1,\ldots,m\}$.

Proof. By Remark 4.4 and Lemma 4.6, the eigenvalues of $A_{Y_{m,n}\tau_1 Y_{m,n}}$ are all distinct. Hence every eigenspace of $A_{Y_{m,n}\tau_1 Y_{m,n}}$ is one-dimensional. By Lemma 3.1, $L^2(Y\backslash S_{m+n}/Y)$ has dimension $m+1$. By Remark 4.4, Lemma 4.5, and Lemma 4.6, $\{s_{j,m,n} \,|\, 0 \le j \le m\}$ is a set of $m+1$ eigenfunctions of $A_{Y_{m,n}\tau_1 Y_{m,n}}$ with distinct eigenvalues. Therefore, up to a constant, every eigenfunction of $A_{Y_{m,n}\tau_1 Y_{m,n}}$ equals $s_{j,m,n}$ for some j. Lemma 3.7 shows that for $0 \le j \le m$, we have that $f_{j,m,n}$ is an eigenfunction of $A_{Y_{m,n}\tau_1 Y_{m,n}}$. Frobenius Reciprocity implies that $s_{j,m,n}(\tau_0) = 1 = f_{j,m,n}(\tau_0)$ for all j, m, n. By Lemma 4.6, we have that $f_{j,m,n}(\tau_1) = s_{j,m,n}(\tau_1)$ for all j, m, n. This proves (a). To prove (b), use Lemma 2.5, Remark 4.4, and part (a). ∎

REMARK 5.2. We claim that $E(i,0,m,n) = (-1)^i\binom{m}{i}$, and we quickly sketch a proof of this fact. In [20], Terras shows that $s_{0,m,n}(g) = \langle \pi_{(m,n)}(g) \cdot v, v \rangle$, where V

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 154-167, March 2008

ELA
http://math.technion.ac.il/iic/ela

is a complex vector space on which $\pi_{(m,n)}$ acts unitarily, $v \in V$ is a vector of norm 1 fixed by the action of $\pi_{(m,n)}(y)$ for all $y \in Y$, and $\langle \cdot, \cdot \rangle$ is an inner product on V with respect to which $\pi_{(m,n)}$ acts unitarily. Consider the Young diagram with two rows, n boxes in the top row and m in the bottom. Let $a_{n,m}$ and $c_{n,m}$ be as in [10, p. 46]. Let $V = c_{n,m}\mathbb{C}S_{n+m}$. Then V has a natural inner product with respect to which $\pi_{(m,n)}$ acts unitarily. Let $v = \frac{a_{n,m}}{||a_{n,m}||}$; then v is a Y-fixed vector of norm 1.

THEOREM 5.3. *If $0 < i \leq m$ and $i \neq n$, then $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ is connected and nonbipartite.*

Proof.

We first claim that if $0 \leq j < m$, then $|E(i,j,m,n)| < \binom{m}{i}\binom{n}{i}$, where $E(i,j,m,n)$ is as in Theorem 5.1. Note that $E(i,j,m,n) = E(i,j,m-1,n-1) - E(i-1,j,m-1,n-1)$. Since by Lemma 3.5 the graphs $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ have degree $\binom{m}{i}\binom{n}{i}$, we see that

$$\begin{aligned} |E(i,j,m,n)| &\leq |E(i,j,m-1,n-1)| + |E(i-1,j,m-1,n-1)| \\ &\leq \binom{m-1}{i}\binom{n-1}{i} + \binom{m-1}{i-1}\binom{n-1}{i-1}. \end{aligned}$$

We need to show that

$$\binom{m-1}{i}\binom{n-1}{i} + \binom{m-1}{i-1}\binom{n-1}{i-1} < \binom{m}{i}\binom{n}{i}.$$

This inequality reduces to $2i^2 < (m+n)i$, which further reduces (since $i > 0$) to $2i < m+n$, which is true unless $i = m = n$. This proves our claim.

It then follows from Theorem 5.1(b) that $-\binom{m}{i}\binom{n}{i}$ is not an eigenvalue of $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$. Therefore $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ is nonbipartite [2].

By Theorem 5.1, our claim also shows that if $j < m$, then $s_{j,m,n}(\tau_i) \neq s_{m,m,n}(\tau_i)$.

Consequently (see [20] or [3]), the multiplicity of $E(i,m,m,n) = \binom{m}{i}\binom{n}{i}$ as an eigenvalue of $\tilde{A}_{Y_{m,n}\tau_i Y_{m,n}}$ equals the degree of the representation associated to $s_{m,m,n}$. But $s_{m,m,n}$ comes from the representation $\pi_{(n+m)}$ induced from the Young diagram with a single row consisting of $n+m$ boxes. As this is the trivial representation, this multiplicity is 1. Therefore $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$ is connected [2]. ◻

REMARK 5.4. We saw in section 3 that the graphs $\mathrm{Cay}(S_{m+n}, Y\tau_0 Y)/Y$ and $\mathrm{Cay}(S_{2m}, Y\tau_m Y)/Y$ are disconnected, and that the latter is bipartite.

COROLLARY 5.5. *If $0 < i \leq m$ and $i \neq n$, then S_{m+n} is generated by $Y\tau_i Y$.*

Proof. This is equivalent to connectedness of $\mathrm{Cay}(S_{m+n}, Y\tau_i Y)/Y$. ◻

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 154-167, March 2008

ELA
http://math.technion.ac.il/iic/ela

Remark 5.6. It is not difficult to show directly that $Y\tau_iY$ generates S_{m+n} whenever $0 < i \le m$ and $i \neq n$. However, we find it interesting that one can prove this fact by estimating eigenvalues.

Definition 5.7 ([14]). A k-regular graph is *Ramanujan* if every eigenvalue μ of its adjacency matrix satisfies $|\mu| = k$ or $|\mu| \le 2\sqrt{k-1}$.

An inequality of Alon-Boppana and Serre [17] shows that this bound is asymptotically the best possible. Ramanujan graphs are precisely those whose Ihara zeta functions satisfy the Riemann hypothesis [20]. See, for example, [17] for a survey paper on Ramanujan graphs.

There are two infinite families of Ramanujan graphs amongst the graphs $\mathrm{Cay}(S_{m+n}, Y\tau_iY)/Y$. One is trivial: $\mathrm{Cay}(S_{1+n}, Y\tau_1Y)/Y$ is the complete graph of degree n on $n+1$ vertices. As for the other, we have:

Theorem 5.8. *For all n, the graph* $\mathrm{Cay}(S_{2+n}, Y\tau_2Y)/Y$ *is Ramanujan.*

Proof. ¿From Lemma 3.5, $\mathrm{Cay}(S_{2+n}, Y\tau_2Y)/Y$ has degree $\binom{n}{2}$. From Theorem 5.1, its nontrivial eigenvalues are 1 and $1-n$. ∎

(In light of Theorem 5.8, it is tempting to call the graphs $\mathrm{Cay}(S_{2+n}, Y\tau_2Y)/Y$ "Ramanu-Johnson graphs.")

It is unfortunate that the degree of the graphs in Theorem 5.8 blows up as n goes to infinity. A similar phenomenon occurs with the finite upper half plane graphs.

Other than $\mathrm{Cay}(S_{1+n}, Y\tau_1Y)/Y$ and $\mathrm{Cay}(S_{2+n}, Y\tau_2Y)/Y$, the following are all the Ramanujan graphs of the form $\mathrm{Cay}(S_{m+n}, Y\tau_iY)/Y$ for $n \le 17$:

- If $2 \le n \le 11$, then $\mathrm{Cay}(S_{2+n}, Y\tau_1Y)/Y$ is Ramanujan.
- If $3 \le n \le 5$, then $\mathrm{Cay}(S_{3+n}, Y\tau_1Y)/Y$ is Ramanujan.
- If $3 \le n \le 11$, then $\mathrm{Cay}(S_{3+n}, Y\tau_2Y)/Y$ is Ramanujan.
- If $n = 4$, then $\mathrm{Cay}(S_{3+n}, Y\tau_3Y)/Y$ is Ramanujan.
- If $4 \le n \le 6$, then $\mathrm{Cay}(S_{4+n}, Y\tau_2Y)/Y$ is Ramanujan.
- If $8 \le n \le 10$, then $\mathrm{Cay}(S_{4+n}, Y\tau_3Y)/Y$ is Ramanujan.
- If $6 \le n \le 7$, then $\mathrm{Cay}(S_{5+n}, Y\tau_3Y)/Y$ is Ramanujan.

We now give a heuristic argument for why we expect that only finitely many of the graphs $\mathrm{Cay}(S_{m+n}, Y\tau_iY)/Y$ will be Ramanujan when $m \ge 3$. As a function of n, the degree of this graph is a polynomial of degree m. Because of the recurrence relation $E(i,j,m,n) = E(i,j,m-1,n-1) - E(i-1,j,m-1,n-1)$, the eigenvalue $E(i,m-1,m,n)$ will be a polynomial in n of degree $m-1$. For n large, this eigenvalue will violate the Ramanujan bound unless $m \ge 2(m-1)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
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REFERENCES

[1] E. Bannai and T. Ito. *Algebraic combinatorics. I. Association schemes*. Benjamin/Cummings Publishing Co., Menlo Park, CA, 1984.

[2] Béla Bollobás. *Modern Graph Theory.* Grad. Texts in Math. 184, Springer-Verlag, New York, 1998.

[3] G. Davidoff, P. Sarnak, and A. Valette. *Elementary Number Theory, Group Theory and Ramanujan Graphs.* London Mathematical Society Student Texts, 55. Cambridge University Press, Cambridge, 2003.

[4] Philippe Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl. 10*, 1973.

[5] Philippe Delsarte. Association schemes and t-designs in regular semilattices. *J. Combinatorial Theory Ser. A*, 20(2):230–243, 1976.

[6] Philippe Delsarte. Hahn polynomials, discrete harmonics, and t-designs. *SIAM J. Appl. Math.*, 34(1):157–166, 1978.

[7] Charles Dunkl. A Krawtchouk polynomial addition theorem and wreath products of symmetric groups. *Indiana Univ. Math. J.*, 25(4):335–358, 1976.

[8] Charles Dunkl. An addition theorem for some q-Hahn polynomials. *Monatsh. Math.*, 85(1):5–37, 1978.

[9] Charles Dunkl. Orthogonal functions on some permutation groups. *Relations between combinatorics and other parts of mathematics, (Proc. Sympos. Pure Math., Ohio State Univ., Columbus, Ohio, 1978)*, Proc. Sympos. Pure Math. XXXIV, Amer. Math. Soc., Providence, R.I., pp. 129–147, 1979.

[10] W. Fulton and J. Harris. *Representation theory.* Grad. Texts in Math. 129, Springer-Verlag, New York, 1991.

[11] H. Gollan and W. Lempken. An easy linear algebra approach to the eigenvalues of the Bernoulli-Laplace model of diffusion. *Arch. Math. (Basel)*, no. 2, 64(2):150–153, 1995.

[12] G. James. *The representation theory of the symmetric group.* Lecture Notes in Mathematics, 682, Springer, Berlin, 1978.

[13] Aloys Krieg. *Hecke Algebras.* Mem. Amer. Math. Soc. 87, no. 435, 1990.

[14] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[15] J. Marco and J. Parcet. Laplacian operators an Radon transforms on Grassmann graphs. *Monatsh. Math.*, 150(2):97–132, 2007.

[16] J. Marco and J. Parcet. On the natural representation of $S(\Gamma)$ into $L^2(P(\Gamma))$: discrete harmonics and Fourier transform. *J. Combin. Theory Ser. A*, 100(1):153–175, 2002.

[17] M. Ram Murty. Ramanujan graphs. *J. Ramanujan Math. Soc.*, 18(1):33–52, 2003.

[18] Dennis Stanton. Harmonics on posets. *J. Combin. Theory Ser. A*, 40(1):136–149, 1985.

[19] Dennis Stanton. Orthogonal polynomials and Chevalley groups. *Special Functions: group theoretical aspects and applications*, Math. Appl., Reidel, Dordrecht, pp. 87–128, 1984.

[20] Audrey Terras. *Fourier analysis on finite groups and applications.* London Math. Soc. Stud. Texts, 43, Cambridge University Press, Cambridge, 1999.