Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 28-47, January 2008

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ON THE EXPONENT OF R-REGULAR PRIMITIVE MATRICES*

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Abstract. Let P_{nr} be the set of n-by-n r-regular primitive $(0,1)$-matrices. In this paper, an explicit formula is found in terms of n and r for the minimum exponent achieved by matrices in P_{nr}. Moreover, matrices achieving that exponent are given in this paper. Gregory and Shen conjectured that $b_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$ is an upper bound for the exponent of matrices in P_{nr}. Matrices achieving the exponent b_{nr} are presented for the case when n is not a multiple of r. In particular, it is shown that $b_{2r+1,r}$ is the maximum exponent attained by matrices in $P_{2r+1,r}$. When n is a multiple of r, it is conjectured that the maximum exponent achieved by matrices in P_{nr} is strictly smaller than b_{nr}. Matrices attaining the conjectured maximum exponent in that set are presented. It is shown that the conjecture is true when $n = 2r$.

Key words. r-Regular matrices, Primitive matrices, Exponent of primitive matrices.

AMS subject classifications. 05C20, 05C50, 15A36.

1. Introduction.

A nonnegative square matrix A is called *primitive* if there exists a positive integer k such that A^k is positive. The smallest such k is called the *exponent of* A. We denote the exponent of a primitive matrix A by $exp(A)$.

A (0,1)-matrix A is said to be r-regular if every column sum and every row sum is constantly r.

Consider the set P_{nr} of all primitive r-regular $(0,1)$-matrices of order n, where $2 \leq r \leq n$. Notice that, for $n > 1$, n-by-n 1-regular matrices are permutation matrices, which are not primitive. An interesting problem is to find the following two positive integers:

$$l_{nr} = \min\{exp(A) : A \in P_{nr}\}, \quad \text{and} \quad u_{nr} = \max\{exp(A) : A \in P_{nr}\},$$

as well as finding matrices attaining those exponents. In this paper, we call the integers l_{nr} and u_{nr} the optimal lower bound and the optimal upper bound for the exponent of matrices in P_{nr}, respectively.

*Received by the editors May 01, 2007. Accepted for publication January 15, 2008. Handling Editor: Michael J. Tsatsomeros.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 28-47, January 2008

In the literature, numerous papers can be found about good upper bounds for the exponent of general primitive matrices A of order n. In [8] Wielandt stated, without proof, that

$$exp(A) \leq (n-1)^2 + 1.$$

Recently, the proof was found in Wielandt's unpublished diaries and published in [5]. There are many improvements of Wielandt's bound for special classes of primitive matrices. The problem of finding an upper bound for the exponent of matrices in P_{nr} has been considered by several authors in Discrete Mathematics, in particular, by some researchers in Graph Theory [2, 4, 6, 7]. In the literature, several such bounds can be found. In [4], it is shown that $exp(A) \leq \frac{2n(3n-2)}{(r+1)^2} - \frac{n+2}{r+1}$. In [7], it is shown that, if $A \in P_{nr}$, then $exp(A) \leq 3n^2/r^2$. Also, it is conjectured there that, if $A \in P_{nr}$, then $exp(A) \leq \lfloor \frac{n}{r} \rfloor^2 + 1$, where $\lfloor\ \rfloor$ denotes the *floor* function, that rounds a number to the next smaller integer. J. Shen proved that this conjecture is true when $r = 2$ [6], however it remains open for $r > 2$.

In this paper, we give an explicit expression for l_{nr} in terms of n and r, and construct matrices attaining that exponent. We also construct matrices whose exponent is $\lfloor \frac{n}{r} \rfloor^2 + 1$ when $n = gr + c$, with $0 < c < r$, which proves that $u_{nr} \geq \lfloor \frac{n}{r} \rfloor^2 + 1$ in those cases. Moreover, we prove that $u_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$ when $g = 2$ and $c = 1$. When $n = gr$, with $g = 2$, we determine u_{nr}; when $g \geq 3$, we give a conjecture for the value of u_{nr} and present matrices achieving the conjectured optimal upper bound exponent. According to this conjecture, u_{nr} would be smaller than $\lfloor \frac{n}{r} \rfloor^2 + 1$.

2. Notation and Auxiliary Results. In the sequel we will use the following notation: If A is an n-by-m matrix, we denote by $A(i,j)$ the entry of A in the position (i,j). By $A(i_1 : i_2, j_1 : j_2)$, with $i_2 \geq i_1$ and $j_2 \geq j_1$, we denote the submatrix of A lying in rows $i_1, i_1+1, \ldots, i_2$ and columns $j_1, j_1+1 \ldots, j_2$. We abbreviate $A(i_1 : i_1, j_1 : j_2)$ to $A(i_1, j_1 : j_2)$ and $A(1 : n, j_1 : j_2)$ to $A(:, j_1 : j_2)$. Similar abbreviations are used for the columns of A. The m-by-n matrix whose entries are all equal to one is denoted by J_{mn}. Unspecified entries in matrices are represented by a $*$.

Some of the proofs in this paper involve the concept of digraph associated with a $(0,1)$- matrix.

Definition 2.1. *Let A be a $(0,1)$-matrix of size n-by-n. The digraph $G(A)$ associated with A is the directed graph with vertex set $V = \{1, 2, ..., n\}$ and arc set E where $(i,j) \in E$ if and only if $A(i,j) = 1$.*

Notice from the previous definition that A is the adjacency matrix of $G(A)$.

A digraph G is said to be r-regular if and only if its adjacency matrix is an

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 28-47, January 2008

r-regular matrix. Note that the outdegree and the indegree of each vertex of an r-regular digraph are exactly r. A digraph is said to be primitive if and only if its adjacency matrix is primitive. Clearly, for $A \in P_{nr}$, $exp(A) = k$ if and only if any two vertices in $G(A)$ are connected by a walk of length k and, if $k > 1$, there are at least two vertices that are not connected by a walk of length $k - 1$.

It is important to notice that if A is an r-regular primitive matrix and $B = P^T AP$ for some permutation matrix P, then, for any positive integer k, $B^k = P^T A^k P$. Thus, $exp(A) = exp(B)$. Also $G(A)$ and $G(B)$ are isomorphic digraphs. Therefore, throughout the paper, we will work on the set of equivalence classes under permutation similarity. Notice also that $A \in P_{nr}$ if and only $A^t \in P_{nr}$.

Next we include some simple observations about r-regular primitive matrices that will be useful to prove some of the main results in the paper.

Lemma 2.2. *Let $A \in P_{n,r}$ and let k be any positive integer. Then, every row of A^k contains at most r^k nonzero entries.*

Proof. We prove the result by induction on k. Let $A \in P_{n,r}$. Then, every row of A contains r nonzero entries since A is r-regular. Therefore, the result is true for $k = 1$.

Assume that every row of A^{k-1} contains at most r^{k-1} nonzero entries. Then, any $r \times n$ submatrix of A^{k-1}has at most r^k nonzero columns. Because $A^k = AA^{k-1}$, the result follows. □

Lemma 2.3. *Let $A \in P_{nr}$ and let $k > 1$ be a positive integer. If $A^k(i, j) = 0$, then there are at least r zero entries in the i-th row of A^{k-1}; also there are at least r zero entries in the j-th column of A^{k-1}.*

Proof. Notice that $A^k(i, j) = A^{k-1}(i, :)A(:, j) = 0$. Since A is r-regular, r entries of $A(:, j)$ are ones. Taking into account that $A^{k-1}(i, :) \geq 0$, the first result follows. The second claim can be proven in a similar way taking into account that $A^k(i, j) = A(i, :)A^{k-1}(:, j) = 0$. □

Lemma 2.4. *Let $A \in P_{nr}$ and $i \in \{1, \ldots, n\}$. Then, the number of nonzero entries in the i-th row (column) of A^k, $k \geq 1$, is a nondecreasing sequence in k.*

Proof. Suppose that in the i-th row of A^k there are exactly s nonzero entries. We want to show that in the i-th row of A^{k+1} there are at least s nonzero entries. Denote by S the set $\{j \in \{1, ..., n\} : A^k(i, j) \neq 0\}$. Since the outdegree of each node of $G(A)$ is exactly r, there are rs arcs with origin in the vertices in S. Since the indegree of each node of G is exactly r, then the rs arcs with origin in S have their terminus in at least $rs/r = s$ vertices. Thus, with origin in the i-th node of $G(A)$, there are walks of length $k + 1$ to at least s distinct vertices. The result for columns follows taking

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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into account that $A^t \in P_{nr}$. □

Note that the last lemma implies that each row (column) of A^k has at least r nonzero entries.

If $i \in \{1, ..., n\}$ is such that $A(i,i) = 1$, then Lemma 2.4 may be refined. We consider this situation in the next lemma, as it will allow us to get an interesting corollary. We assume that $n \geq 2r$ since, by Lemma 2.3, if $n < 2r$, $A^2(i,:)$ is positive.

LEMMA 2.5. *Let $A \in P_{nr}$, with $n \geq 2r$, and $i \in \{1, \ldots, n\}$. Suppose that $A(i,i) = 1$. Let s_k be the number of nonzero entries in $A^k(i,:)$, $k \geq 1$. If $s_k < n$, then the number of nonzero entries in the i-th row of A^{k+1} is at least $s_k + 1$. In particular, the i-th row of A^{n-2r+3} is positive.*

Proof. By a possible permutation similarity of A, we assume that $i = 1$ and $A(1,:) = \begin{bmatrix} J_{1r} & 0 \end{bmatrix}$. Let $k \in \{2, \ldots, n\}$. Clearly, the first r entries of $A^k(1,:)$ are nonzero. If $k = 2$, since A is not reducible, $A^2(1,:)$ has more than r nonzero entries. Now suppose that $k > 2$ and $s_k < n$. With a possible additional permutation similarity, we assume, without loss of generality, that $A^k(1,:) = \begin{bmatrix} a_1 & \cdots & a_{s_k} & 0 \end{bmatrix}$, where $a_i > 0$, $i = 1, \ldots, k$. We show that $s_{k+1} \geq s_k + 1$. Suppose that $A^{k-1}(1,:) = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$, where $b_1, b_2, \ldots, b_r, b_{i_1}, \ldots, b_{i_{s_{k-1}-r}}$ are positive integers, with $r < i_1 < \cdots < i_{s_{k-1}-r} \leq n$. Because $A^k = AA^{k-1}$, then $i_{s_{k-1}-r} \leq s_k$; also, as $A^k = A^{k-1}A$ then

$$A = \begin{bmatrix} J_{1r} & 0 & 0 \\ * & R_{11} & 0 \\ * & R_{21} & R_{22} \\ * & R_{31} & R_{32} \end{bmatrix},$$

for some blocks R_{ij}, where R_{11} and R_{22} are $(r-1)$-by-$(s_k - r)$ and $(s_k - r)$-by-$(n - s_k)$ matrices, respectively. Since all the entries of

$$\begin{bmatrix} b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} R_{11}^t & R_{21}^t & R_{31}^t \end{bmatrix}^t$$

are nonzero, then also all the entries of

$$\begin{bmatrix} a_2 & \cdots & a_{s_k} & 0 \end{bmatrix} \begin{bmatrix} R_{11}^t & R_{21}^t & R_{31}^t \end{bmatrix}^t$$

are nonzero, which implies that $A^{k+1}(1,i) \neq 0$ for $i = 1, ..., s_k$. Since A is not reducible, it also follows that R_{22} is nonzero. Therefore, $A^{k+1}(1,:)$ has at least $s_k + 1$ nonzero entries. Clearly, $A^{n-2r+2}(1,:)$ has at most $r - 1$ zero entries, which implies, by Lemma 2.3, that $A^{n-2r+3}(1,:)$ is positive. □

The next result is a simple consequence of Lemma 2.5. It gives an upper bound for the exponent of matrices in P_{nr} with nonzero trace. Another such upper bound

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can be found in [4]: if $A \in P_{nr}$ has p nonzero diagonal entries, then $exp(A) \leq \max\{2(n-r+1)-p, n-r+1\}$. It is easy to check that there are values of n and r for which the upper bound given in Corollary 2.6 for the exponent of matrices with nonzero trace is smaller than those in [4] and [7]. Check with n=30 and r=15, for instance.

Corollary 2.6. *Let $A \in P_{nr}$, with $n \geq 2r$, and suppose that $trace(A) \neq 0$. Then, $exp(A) \leq 2n - 4r + 6$.*

Proof. Let $i \in \{1, \ldots, n\}$ be such that $A(i,i) \neq 0$. According to Lemma 2.5, the i-th row and the i-th column of A^{n-2r+3} have no zero entries. Therefore, from any vertex in $G(A)$ there is a walk of length $n-2r+3$ to vertex i; also, there is a walk of length $n-2r+3$ from vertex i to any vertex. Thus, any two vertices are connected by a walk of length $2n-4r+6$. □

Finally, we include the following technical lemma.

Lemma 2.7. *Let D_{rk}, $k < r$, denote an r-by-k matrix with exactly $r-1$ nonzero entries in each column. Then, at least one row of D_{rk} has no zero entries. Moreover, if $k < r-1$, then at least two rows of D_{rk} have no zero entries.*

Proof. Notice that the number t of nonzero entries in D_{rk} is $k(r-1)$ since every column contains $r-1$ nonzero entries. Assume that all rows of D_{rk} have at least one zero entry. Then, the number m of zero entries in D_{rk} would be at least r. This implies that

$$t = rk - m \leq rk - r < k(r-1),$$

which is a contradiction. The second claim can be proven in a similar way. □

3. Optimal lower bound. In this section, we determine the optimal lower bound l_{nr} for the exponent of matrices in P_{nr} in terms of n and r. We also present matrices achieving this exponent.

Lemma 3.1. *Let $A \in P_{nr}$. Then,*

$$exp(A) \geq \lceil log_r(n) \rceil.$$

Proof. Taking into account Lemma 2.2, each row of A has at most r^k nonzero entries. Since $r^k \geq n$ if and only if $k \geq log_r(n)$, the result follows. □

Next we prove that there exist matrices in P_{nr} whose exponent is $\lceil log_r(n) \rceil$.

Definition 3.2. *Let $B = [b_{ij}]$ be an m-by-n real (complex) matrix. We call the*

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A publication of the International Linear Algebra Society
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indicator matrix of B, *which we denote by* $M(B)$, *the* m*-by-*n $(0,1)$*-matrix* $[\mu_{ij}]$, *with* $\mu_{ij} = 1$ *if* $b_{ij} \neq 0$ *and* $\mu_{ij} = 0$ *if* $b_{ij} = 0$.

Definition 3.3. *Let* $v = (v_1, v_2, ..., v_n)$ *be a row vector in* $\mathbb{R}^n$. *Let* s *be an integer such that* $0 < s \leq n$. *Define the* s*-shift operator* $f_s : \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$f_s(v_1, v_2, ..., v_n) = (v_{n-s+1}, v_{n-s+2}, ..., v_n, v_1, v_2, ..., v_{n-s}).$$

The s*-generalized circulant matrix associated with* v *is the* n*-by-*n *matrix whose* k*-th row is given by* $f_s^{k-1}(v)$, *for* $k = 1, ..., n$, *where* f_s^{k-1} *denotes the composition of* f_s *with itself* $k-1$ *times.*

Note that $f_s^n(v_1, ..., v_n) = (v_1, ..., v_n)$, as the position of v_1 after n s-shifts is $ns+1$ modulo n, that is, 1.

Let $0 < s \leq r$ be an integer. We denote by T_s^{nr} the s-generalized circulant matrix associated with $u_r = \sum_{i=1}^r e_i^t$, where e_i denotes the i-th column of the n-by-n identity matrix. For instance,

$$T_1^{52} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Lemma 3.4. *For* $r \geq 2$, *the matrix* T_r^{nr} *is* r*-regular and primitive. Moreover,* $exp(T_r^{nr}) = \lceil log_r(n) \rceil$.

Proof. First we prove that T_r^{nr} is an r-regular matrix. By construction, it is easy to see that the row sum is constantly r. In order to determine the column sum note that there are exactly nr entries equal to one in T_r^{nr}. We denote by s_i, $i \geq 1$, the remainder of the division of i by n, if i is not a multiple of n, and $s_i = n$ otherwise. By construction again, the ones in the i-th row occur in positions $s_{(i-1)r+1}, ..., s_{ir}$. The sequence of columns in which the ones occur, starting in the first row, then the second row and so on, is just the sequence $s_1, s_2, s_3, ..., s_{nr}$, that is, $1, ..., n, 1, ..., n, ..., 1, ..., n$. Clearly, each $j \in \{1, 2, ..., n\}$ appears exactly r times in that sequence.

Now we prove that T_r^{nr} is primitive by computing its exponent. We first show, by induction on k, that the first $\min\{n, r^k\}$ entries of the first row of $(T_r^{nr})^k$ are nonzero and, if $r^k < n$, the last $n - r^k$ entries of the first row of $(T_r^{nr})^k$ are zero. If $k = 1$, this claim is trivially true. Now suppose that the claim is valid for $k = p$. Note that, for each integer $1 \leq k \leq n$, all the columns of the submatrix of T_r^{nr} indexed by the first r^k rows and the first $\min\{n, r^{k+1}\}$ columns are nonzero. Also, if $r^{k+1} < n$, the

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submatrix of T_r^{nr} indexed by the first r^k rows and the last $n - r^{k+1}$ columns is 0. Taking into account this observation, it follows that the first $\min\{n, r^{p+1}\}$ entries of $(T_r^{nr})^{p+1}(1:) = (T_n^{nr})^p(1,:)T_r^{nr}$ are nonzero while the last $n - \min\{n, r^{p+1}\}$ are zero.

Using similar arguments, we can show that, in general, the i-th row of $M((T_r^{nr})^k)$ is $f_r^{(i-1)r^{k-1}}(u_k)$, where $u_k = \sum_{j=1}^{\min\{r^k,n\}} e_j^t$.

Therefore, any row of $(T_r^{nr})^k$ has exactly $\min\{r^k, n\}$ nonzero entries. Thus, $(T_r^{nr})^k$ is positive if and only if $r^k \geq n$, which implies the result. ☐

THEOREM 3.5. *Suppose that* $2 \leq r \leq n$. *Then,* $l_{rn} = \lceil log_r(n) \rceil$.

Proof. Follows from Lemma 3.1 and Lemma 3.4. ☐

4. Optimal upper bound.

Although stated in terms of graphs, the following conjecture is given in [7]: If $A \in P_{nr}$, then $exp(A) \leq \lfloor \frac{n}{r} \rfloor^2 + 1$. In [6] this conjecture was proven for $r = 2$. Notice that this conjecture is trivially true for $r \geq \frac{n+1}{2}$. Hence, in the sequel we assume that $n \geq 2r$.

Given any $g \geq 2$, an r-regular primitive digraph with $n = gr + 1$ vertices whose exponent is $\lfloor \frac{n}{r} \rfloor^2 + 1$ can be found in [7]. A matrix with such a graph is the following:

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{1r} & 0 & 0 \\ 0 & 0 & \cdots & T_1^{r,r-1} & J_{r1} & 0 \end{bmatrix}. \tag{4.1}$$

In the next two subsections we generalize the structure of the matrix A by defining the matrices E_{nr} for all possible combinations of n and r.

4.1. The case in which n is not a multiple of r.

Generalizing the structure of the matrix in (4.1), in this section we define the n-by-n matrices E_{nr}, when $n = gr+c$ for some positive integers $g \geq 2$ and $0 < c < r$, as follows:

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$$E_{nr} = \begin{bmatrix} 0 & 0 & J_{rr} \\ J_{cr} & 0 & 0 \\ T_1^{r,r-c} & J_{rc} & 0 \end{bmatrix}, \quad \text{if } n = 2r + c, \tag{4.2}$$

$$E_{nr} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{rr} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_{cr} & 0 & 0 \\ 0 & 0 & \cdots & 0 & T_1^{r,r-c} & J_{rc} & 0 \end{bmatrix}, \tag{4.3}$$

$$\text{if } n = gr + c, \text{ with } g \geq 3. \tag{4.4}$$

Note that we can replace $T_1^{r,r-c}$ by any matrix in $P_{r,r-c}$ without changing the exponent of E_{nr}.

Next we show that $\exp(E_{nr}) = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$, which implies that $u_{nr} \geq \left\lfloor \frac{n}{r} \right\rfloor^2 + 1$. We then prove the equality when $g = 2$ and $c = 1$.

Lemma 4.1. *If $n = 2r + c$, where $0 < c < r$, then $exp(E_{nr}) = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1 = 5$.*

Proof. It is easy to check that

$$M(E_{nr}^2) = \begin{bmatrix} J_{rr} & J_{rc} & 0 \\ 0 & 0 & J_{cr} \\ J_{rr} & 0 & J_{rr} \end{bmatrix}, \quad M(E_{nr}^3) = \begin{bmatrix} J_{rr} & 0 & J_{rr} \\ J_{cr} & J_{cc} & 0 \\ J_{rr} & J_{rc} & J_{rr} \end{bmatrix},$$

$$M(E_{nr}^4) = \begin{bmatrix} J_{rr} & J_{rc} & J_{rr} \\ J_{cr} & 0 & J_{cr} \\ J_{rr} & J_{rc} & J_{rr} \end{bmatrix}.$$

Finally, we get that $M(E_{nr}^5) = J_{nn}$, which implies the result. ∎

Lemma 4.2. *If $n = gr+c$, with $g \geq 3$ and $0 < c < r$, then $exp(E_{nr}) = \left\lfloor \frac{n}{r} \right\rfloor^2 + 1 = g^2 + 1$.*

Proof. Consider the digraph G associated with E_{nr}. Let us group the vertices of G in the following way: We call B_1 the set of vertices from $(g-1)r + c + 1$ to $gr + c$; we call B_2 the set of vertices from $(g-1)r + 1$ to $(g-1)r + c$; we call B_i, $i = 3, \dots, g+1$, the set of vertices from $(g-i+1)r + 1$ to $(g-i+2)r$.

Suppose that u and v are two vertices in the same block B_i. Then there is a path from u to v of length g and another one of length $g+1$, except if $u, v \in B_2$,

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in which case there is just a path of length $g+1$. Therefore, a walk from u to v has length t if and only $t = \alpha g + \beta(g+1)$, for some nonnegative integers α, β, with $\beta > 0$ if $u, v \in B_2$. In particular, no vertex in B_2 lies on a closed walk of length g^2 since $\alpha g + \beta(g+1) = g^2$ implies $\beta = 0$. Thus, $exp(E_{nr}) > g^2$.

Because

$$g^2 + 1 = (g-1)g + (g+1),$$

it follows that there is a walk of length g^2+1 from any vertex to any other in the same block B_i, $i = 1, , \ldots, g+1$.

Now consider a vertex u in B_i and a vertex v in B_j, where $i, j \in \{1, ..., g+1\}$ and $i \neq j$. Let s be the distance from u to v. Note that $s \leq g$. We will show that there is a walk of length g^2+1 from u to v. Suppose that $s > 1$. In this case we have

$$g^2 - s + 1 = (s-2)g + (g-s+1)(g+1).$$

Thus, u lies on a closed walk of length $g^2 - s + 1$, which implies that there is a walk of length g^2+1 from u to v.

Now suppose that $s = 1$. If $u \notin B_2$, u lies on a closed walk of length g^2, which implies that there is a walk of length g^2+1 from u to v. If $u \in B_2$, then $v \in B_3$ and v lies on a close walk of length g^2, which implies that there is a walk of length g^2+1 from u to v.

We have shown that the vertices in B_2 do not lie on any closed walk of length g^2. On the other hand, between any two vertices there is a walk of length g^2+1. Thus $E_{nr}^{g^2}$ is not positive, while $E_{nr}^{g^2+1}$ is positive. Therefore, $\exp(E_{nr}^{g^2+1}) = g^2+1$. □

The following theorem follows in a straightforward way from Lemmas 4.1 and 4.2.

THEOREM 4.3. *If $n = gr + c$, with $0 < c < r$, then $u_{nr} \geq \lfloor \frac{n}{r} \rfloor^2 + 1$.*

We now show that, when $n = 2r+1$, $u_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1$.

THEOREM 4.4. *Let $n = 2r+1$. Then, $u_{nr} = \lfloor \frac{n}{r} \rfloor^2 + 1 = 5$.*

Proof. Clearly, by Theorem 4.3, $u_{nr} \geq 5$. We now show that if $A \in P_{nr}$ and $\exp(A) > 4$, then $\exp(A) = 5$, which means that there are no matrices in P_{nr} with exponent greater than 5, and, therefore, $u_{nr} = 5$. The strategy we follow allows us to characterize, up to a permutation similarity, all the matrices in P_{nr} that achieve exponent 5.
Suppose that $\exp(A) \geq 5$. Then, there is a zero entry in A^4. Without loss of generality, we can assume that $A^4(1, i) = 0$ for some $i \in \{1, ..., n\}$. Applying Lemma 2.3 repeatedly, we deduce that there are at least r zero entries in the first row of A^3 and A^2.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 28-47, January 2008

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By a convenient permutation similarity on A, we can reduce the proof to the next two cases (and subcases). Throughout the proof, we denote by D_{rk} an r-by-k matrix with exactly $r-1$ nonzero entries in each column and by C_{rr} a matrix in $P_{r,r-1}$.

Case 1. Let us assume that $A(1,:) = [J_{1r} \quad 0]$. Then, $A^2(1,i) \neq 0$ for $i = 1,...,r$ and we can assume that $A^2(1, r+2:n) = 0$. Therefore,

$$A = \begin{bmatrix} J_{1r} & 0 & 0_{1r} \\ * & R_1 & 0_{r-1,r} \\ * & * & D_{r+1,r} \end{bmatrix},$$

for some $(r-1)$-by-1 block R_1. If R_1 is zero, clearly A is reducible, which is a contradiction. If R_1 is nonzero, then $M(A^2)(1,:) = [J_{1,r+1} \quad 0_{1,r}]$ and $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$ for $i = 1,...,r+1$. Since $A^3(1,:)$ contains at least r zero entries then $M(A^3)(1,:) = [J_{1,r+1} \quad 0_{1,r}]$, which implies that $D_{r+1,r}(1,:) = 0$. Thus,

$$A = \begin{bmatrix} J_{1r} & 0 & 0_{1r} \\ C_{rr} & J_{r1} & 0_{rr} \\ 0_{rr} & 0_{r1} & J_{rr} \end{bmatrix}$$

is reducible, which is again a contradiction.

Case 2. Let us assume now that $A(1,:) = [0 \quad J_{1r} \quad 0_{1r}]$. Notice that there is $i \in \{r+2,...,n\}$ such that $A^2(1,i) \neq 0$, otherwise $A(1:r+1, r+2:n) = 0$, and A would be reducible. This observation leads to the following subcases:

Subcase 2.1. Assume that $A^2(1,i) = 0$ for $i = 1, r+2,...,n-1$. Then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ 0 & C_{rr} & 0 & J_{r1} \\ J_{r1} & 0 & J_{r,r-1} & 0 \end{bmatrix}.$$

A calculation shows that $\exp(A) = 3$, which is a contradiction.

Subcase 2.2. Let us assume that $A^2(1,i) = 0$ for $i = 1,...,k+1, r+2,...,2r-k$, with $0 < k < r-1$. Then,

$$A = \begin{bmatrix} 0 & J_{1k} & J_{1,r-k} & 0_{1,r-k-1} & 0_{1,k+1} \\ 0_{r1} & 0_{rk} & R_1 & 0_{r,r-k-1} & R_2 \\ J_{r1} & D_{rk} & * & J_{r,r-k-1} & R_3 \end{bmatrix},$$

for some blocks R_i, $i = 1,2,3$. Taking into account Lemma 2.7, each column of R_1 and R_2 is nonzero, which implies that $A^2(1,i) \neq 0$ for $i = k+2,...,r+1, 2r-$

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A publication of the International Linear Algebra Society
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$k+1, ..., n$. Since $A^2(1,:)$ has at least r entries equal to zero, then $M(A^2)(1,:) = [0_{1,k+1} \quad J_{1,r-k} \quad 0_{r-k-1} \quad J_{1,k+1}]$. Note that the submatrix of $\begin{bmatrix} R_2^t & R_3^t \end{bmatrix}^t$ indexed by rows $k+1, \ldots, r, 2r-k, \ldots, 2r$ has all columns nonzero, otherwise A would not be r-regular. Thus, $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$ for $i = 1, .., k+1, r+2, ..., n$, and $A^3(1,:)$ would not have r zero entries, a contradiction.

Subcase 2.3. Let us assume that $A^2(1,i) = 0$ for $i = 1, ..., r$. Then,

$$A = \begin{bmatrix} 0 & J_{1,r-1} & 1 & 0_{1r} \\ 0_{r1} & 0_{r,r-1} & R_1 & R_2 \\ J_{r1} & D_{r,r-1} & * & * \end{bmatrix},$$

for some blocks R_i, $i = 1, 2$. Taking into account Lemma 2.7, all columns of R_2 are nonzero, which implies that $A^2(1,i) \neq 0$ for $i = r+2, ..., n$. If $R_1 = 0$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix},$$

and $exp(A) = 5$. If R_1 is nonzero, then, $M(A^2)(1,:) = [0_{1r} \quad J_{1,r+1}]$ and $A^3(1,:) = A^2(1,:)A$ has at most one nonzero entry, which is a contradiction. (Note that the last row of $[R_1 R_2]$ has exactly one zero entry.)

Subcase 2.4. Assume that $A^2(1,i) = 0$ for $i = 2, ..., r+1$. Then,

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ * & 0 & * \\ * & D_{rr} & * \end{bmatrix}.$$

Note that, by Lemma 2.4, $A^2(1,:)$ has at least r nonzero entries.

- Let us assume that $A^2(1,:)$ has exactly r nonzero entries. If $M(A^2)(1,:) = [0_{1,r+1} \quad J_{1r}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix}; \tag{4.5}$$

if $M(A^2)(1,:) = [1 \quad 0_{1,r+1} \quad J_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ J_{r1} & 0 & 0 & J_{r,r-1} \\ 0 & C_{rr} & J_{r1} & 0 \end{bmatrix}. \tag{4.6}$$

A straightforward computation shows that in both cases $\exp(A) = 5$.

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- Let us assume that $A^2(1,:)$ has exactly $r+1$ nonzero entries. Then, $M(A^2)(1,:) = [1 \quad 0_{1r} \quad J_{1r}]$ and A has the form

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ R_1 & 0 & R_2 \\ R_3 & D_{rr} & R_4 \end{bmatrix}, \tag{4.7}$$

where R_1 and R_2 are r-by-1 and r-by-r matrices, respectively, with all columns nonzero. Notice also that, since not all rows of D_{rr} sum r, either R_3 or some column in R_4 is nonzero. A calculation shows that $A^3(1,i) \neq 0$ for $i = 2, \ldots, r+1$. Moreover, there is another nonzero entry in $A^3(1,:)$. If $A^3(1,:) = [J_{1,r+1} \quad 0_{1r}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 \\ 0 & 0 & J_{rr} \\ J_{r1} & C_{rr} & 0 \end{bmatrix};$$

if $A^3(1,:) = [0 \quad J_{1,r+1} \quad 0_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0 \\ J_{r1} & 0 & 0 & J_{r,r-1} \\ 0 & C_{rr} & J_{r1} & 0 \end{bmatrix}.$$

In both cases, $\exp(A) = 5$.

Subcase 2.5. Let us assume that $A^2(1,i) = 0$ for $i = 2, ..., k+1, r+2, ..., 2r-k+1$, with $0 < k < r$. Then,

$$A = \begin{bmatrix} 0 & J_{1k} & J_{1,r-k} & 0_{1,r-k} & 0_{1k} \\ R_1 & 0_{rk} & * & 0_{r,r-k} & R_2 \\ * & D_{rk} & * & J_{r,r-k} & * \end{bmatrix},$$

for some blocks R_i, $i = 1,2$. Taking into account Lemma 2.7, each column of R_1 and R_2 is nonzero. Then, $A^2(1,i) \neq 0$ for $i = 1, 2r-k+2, ..., n$, which implies that $A^3(1,i) = A^2(1,:)A(:,i) \neq 0$, for $i = 2, ..., 2r-k+1$. Since $A^3(1,:)$ has at least r zero entries, then $r-1 \leq k < r$, that is, $k = r-1$. Therefore,

$$M(A^2)(1,:) = [1 \quad 0_{1,r-1} \quad * \quad 0 \quad J_{1,r-1}].$$

- If $M(A^2)(1,:) = [1 \quad 0_{1,r-1} \quad 0 \quad 0 \quad J_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1r} & 0 & 0_{1,r-1} \\ J_{r1} & 0_{rr} & 0_{r1} & J_{r,r-1} \\ 0_{r1} & C_{rr} & J_{r1} & 0_{r,r-1} \end{bmatrix}.$$

A calculation shows that $\exp(A) = 5$.

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- If $M(A^2)(1,:) = [1 \quad 0_{1,r-1} \quad 1 \quad 0 \quad J_{1,r-1}]$, then

$$A = \begin{bmatrix} 0 & J_{1,r-1} & 1 & 0 & 0_{1,r-1} \\ * & 0_{r,r-1} & * & 0_{r1} & * \\ * & D_{r,r-1} & * & J_{r1} & * \end{bmatrix}$$

  and $M(A^3)(1,2:r+2) = J_{1,r+1}$. Because $A^3(1,:)$ has at least r zero entries, it follows that $M(A^3)(1,:) = [0 \quad J_{1,r+1} \quad 0_{1,r-1}]$. Since $A^2(1,r+1) \neq 0$, then $A^3(1,i) = A^2(1,:)A(:,i) = 0$ implies $A(r+1,i) = 0$. Thus, $A(r+1,i) = 0$, for $i = 1,..,r,r+2,...,n$, and the $(r+1)$-th row of A would have at least $2r$ entries equal to 0, which contradicts the fact that A is r-regular. □

Notice that, according to the proof of Theorem 4.4, the only "types" of matrices in $P_{2r+1,r}$ (up to a permutation similarity) that achieve maximum exponent are

$$A_1 := \begin{bmatrix} 0 & 0 & J_{rr} \\ J_{1r} & 0 & 0 \\ C_{rr} & J_{r1} & 0 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} 0 & 0 & J_{rr} \\ C_{rr} & J_{r1} & 0 \\ J_{1r} & 0 & 0 \end{bmatrix}.$$

Clearly, if C_{rr} is chosen equal to $T_1^{r,r-1}$, then $A_1 = E_{2r+1,r}$.

Note that the matrix A_2 has nonzero trace and has maximum exponent among the matrices in $P_{2r+1,r}$. However, Corollary 2.6 shows that, for most combinations of n and r, u_{nr} is not attained by matrices with nonzero trace. In particular, this is true if $n = gr + c$, with $0 < c < r$ and $g > r + \sqrt{r^2 - 4r + 5 + 2c}$, as $2n - 4r + 6 < g^2 + 1$ and, by Theorem 4.3, $u_{nr} \geq g^2 + 1$.

4.2. The case in which n is a multiple of r. Suppose that $n = gr$, for some positive integer $g \geq 2$. Denote by E_{nr} the $n \times n$ matrix given by

$$E_{nr} = H_{2r,r}, \quad \text{if } n = 2r,$$

$$E_{nr} = \begin{bmatrix} 0 & J_{rr} \\ H_{2r,r} & 0 \end{bmatrix}, \quad \text{if } n = 3r,$$

$$E_{nr} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{rr} & 0 & 0 \\ 0 & 0 & \cdots & 0 & H_{2r,r} & 0 \end{bmatrix}, \quad \text{if } n = gr, \text{ with } g \geq 4,$$

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where

$$H_{2r,r} = \begin{bmatrix} J_{r-1,r-1} & J_{r-1,1} & 0_{r-1,1} & 0_{r-1,r-1} \\ J_{1,r-1} & 0 & 1 & 0_{1,r-1} \\ 0_{1,r-1} & 1 & 0 & J_{1,r-1} \\ 0_{r-1,r-1} & 0_{r-1,1} & J_{r-1,1} & J_{r-1,r-1} \end{bmatrix}.$$

We will show that $u_{2r,r} = exp(E_{2r,2})$. Taking into account the result of some numerical experiments, we also conjecture that, when $n = gr$ for some $g \geq 3$, the matrices E_{nr} achieve the maximum exponent in the set P_{nr}. This conjecture is also reinforced by the following observation. Let us say that the exponent of an n-by-n r-regular matrix A is infinite if A is not primitive. Given $n = gr$, with $g \geq 3$, consider the following cyclic matrix:

$$P_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & J_{rr} \\ J_{rr} & 0 & \cdots & 0 & 0 \\ 0 & J_{rr} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_{rr} & 0 \end{bmatrix}$$

which is irreducible but not primitive and, therefore, has infinite exponent. In [3] it was proven that given two n-by-n r-regular matrices A and B, then B can be gotten from A by a sequence of interchanges on 2-by-2 submatrices of A:

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix E_{nr} we have constructed has been obtained by applying just one of these interchanges to P_1. Notice, however, that not any arbitrary interchange in P_1 produces a matrix with maximum exponent.

In particular, our conjecture implies that $u_{nr} < \lfloor \frac{n}{r} \rfloor^2 + 1$. It is worth to point out that Shen [6] proved that $u_{n2} < \lfloor \frac{n}{2} \rfloor^2 + 1$.

Next we show that, if $n = 2r$, then $u_{nr} = \frac{n(n-r)}{2r^2} + 2 = 3$.

Theorem 4.5. *Let $r \geq 2$. Then, $u_{2r,r} = 3$.*

Proof. Let $A \in P_{2r,r}$ and suppose that $exp(A) > 3$. Then, there must exist a zero entry in A^3. Without loss of generality, we can assume that $A^3(1, i) = 0$ for some $i \in \{1, ..., n\}$. Applying Lemma 2.3, we deduce that there must be at least r zero entries in the first row of A^2. Without loss of generality, we can assume that one of the next cases holds.

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Case 1. Suppose that $A(1,:) = [J_{1r} \quad 0_{1r}]$. Then, for A to have exponent larger than 3, $M(A^2)(1,:) = [J_{1r} \quad 0_{1r}]$. Taking into account the position of the zeros in the first row of A^2, we deduce that

$$A = \begin{bmatrix} J_{rr} & 0_{rr} \\ 0_{rr} & J_{rr} \end{bmatrix},$$

which is a reducible matrix.

Case 2. Suppose that $A(1,:) = [0 \quad J_{1r} \quad 0_{1,r-1}]$. If $A^2(1,1) = 0$ or $A^2(1,i) = 0$ for some $i \geq r+1$, then A would not be r-regular. Therefore, for A to have exponent larger than 3, $M(A^2)(1,:) = [1 \quad 0_{1r} \quad J_{1,r-1}]$. Then,

$$A = \begin{bmatrix} 0 & J_{1r} & 0_{1,r-1} \\ J_{r1} & 0_{rr} & J_{r,r-1} \\ 0_{r-1,1} & J_{r-1,r} & 0_{r-1,r-1} \end{bmatrix},$$

which is reducible.

In both cases, we get a contradiction. Thus, for any $A \in P_{2r,r}$, $exp(A) \leq 3$. Since $E_{2r,r}^2$ is not positive, then $exp(E_{2r,r}) = 3 = u_{2r,r}$. □

Next we give the exponent of the matrices E_{nr} when $n = gr$ for some positive integer $g \geq 3$. Before we prove the result, we include a preliminary result.

Let $a_1, a_2, ..., a_p$ be positive integers such that $gcd(a_1, .., a_p) = 1$. The Frobenius-Schur index, $\phi(a_1, ..., a_p)$, is the smallest integer such that the equation $x_1a_1 + ... + x_pa_p = l$ has a solution in nonnegative integers $x_1, x_2, ..., x_p$ for all $l \geq \phi(a_1, ..., a_p)$. The following result is due to Brauer in 1942.

Proposition 4.6. [1] *Let y be a positive integer. Then*

$$\phi(y, y+1, ..., y+j-1) = y \left\lfloor \frac{y+j-3}{j-1} \right\rfloor.$$

Lemma 4.7. *Let $y > 1$ be a positive integer. Then,*

$$\phi(y, y+1, y+2) = \begin{cases} \frac{1}{2}y^2, & \text{if } y \text{ is even} \\ \frac{1}{2}(y-1)y, & \text{if } y \text{ is odd.} \end{cases}$$

Moreover, there are nonnegative integers a, b, c satisfying $\phi(y, y+1, y+2) - 2 = ay + b(y+1) + c(y+2)$ if and only if y is even. If y is odd, there are nonnegative integers a, b, c satisfying $\phi(y, y+1, y+2) - 3 = ay + b(y+1) + c(y+2)$.

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Proof. The first claim follows from Proposition 4.6. Now we show the second claim. Clearly, if y is even, $\phi(y, y+1, y+2) - 2 = (\frac{y}{2} - 1)(y+2)$ can be written as $ay + b(y+1) + c(y+2)$ for some nonnegative numbers a, b, c. If y is odd

$$\phi(y, y+1, y+2) - 3 = \frac{1}{2}(y-1)y - 3 = \left(\frac{y-1}{2} - 1\right)(y+2).$$

which implies that $\phi(y, y+1, y+2) - 3$ can be written as $ay + b(y+1) + c(y+2)$ for some nonnegative integers a, b, c. To see that there are no nonnegative integers a, b, c such that

$$\phi(y, y+1, y+2) - 2 = ay + b(y+1) + c(y+2),$$

notice that the largest number of the form $ay+b(y+1)+c(y+2)$, for some nonnegative integers a, b, c, smaller than $\phi(y, y+1, y+2)$ is $\left(\frac{y-1}{2} - 1\right)(y+2)$ and

$$\left(\frac{y-1}{2} - 1\right)(y+2) < \left(\frac{y-1}{2} - 1\right)(y+2) + 3 - 2 = \phi(y, y+1, y+2) - 2. \qquad \square$$

THEOREM 4.8. *Let $n = gr$, with $g \geq 3$ and $r \geq 2$. Then,*

$$\exp(E_{nr}) = \begin{cases} \frac{n(n-r)}{2r^2} + 2, & \text{if } \frac{n}{r} \text{ is even} \\ \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right), & \text{if } \frac{n}{r} \text{ is odd.} \end{cases}$$

Proof. Consider the digraph G associated with E_{nr}. We group the vertices of G in the following way: for $i = 1, \ldots, g$, we call block B_i the set of vertices from $(g-i)r+1$ to $(g-i+1)r$. For convenience, we denote the vertices $n-3r+1, \ldots, n-2r$ in B_3 by $w_1, \ldots, w_r$, resp; the vertices $n-2r+1, \ldots, n-r$ in B_2 by $v_1, \ldots, v_r$, resp., and the vertices $n-r+1, \ldots, n$ in B_1 by $u_1, \ldots, u_r$, resp. Let $B_1' = \{u_2, \ldots, u_r\}$, $B_2' = \{v_2, \ldots, v_{r-1}\}$ and $B_3' = \{w_1, \ldots, w_{r-1}\}$. Note that B_2' is empty if $r = 2$. The digraph G is given in Figure 4.1.

A directed edge in this graph from a set S_1 to a set S_2 means that there is an arc from each vertex in S_1 to each vertex in S_2.

Let G' be the subgragh of G induced by the vertices in $B_1 \cup B_2 \cup B_3$. The following table gives the possible lengths of a walk in G' from a vertex in B_1 to a vertex in B_3.

¿From	To	Possible lengths
u_1	any vertex in B_3'	$2, 3$
u_1	w_r	$1, 2$ (if $r > 2$), 3
any vertex in B_1'	any vertex in B_3'	$2, 3$
any vertex in B_1'	w_r	$2, 3$

Table 1.

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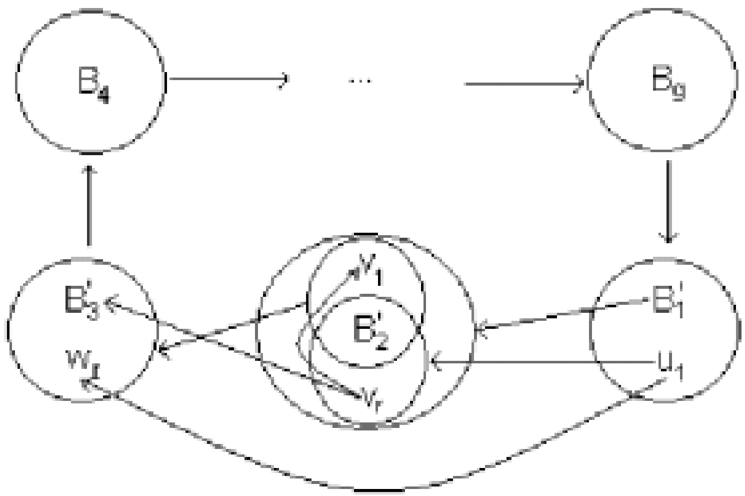

Fig. 4.1.

Thus, for any $i \in \{1, \ldots, g\}\backslash\{2\}$, any walk in G from a vertex $u \in B_i$ to a vertex $v \in B_i$ has length t if and only if

$$t = a\left[(g-2)+1\right] + b\left[(g-2)+2\right] + c\left[(g-2)+3\right], \qquad (4.8)$$

for some nonnegative integers a, b, c, with $b + c > 0$ if either $u \in B_1'$ or $v \in B_3'$.

Taking into account Lemma 4.7, the smallest nonnegative integer t_0 such that, for any $t \geq t_0$, (4.8) holds for some nonnegative integers a, b, c is

$$t_0 = \begin{cases} \frac{1}{2}(g-1)^2, & \text{if } g \text{ is odd} \\ \frac{1}{2}(g-2)(g-1), & \text{if } g \text{ is even.} \end{cases}$$

We will show that, if g is odd, any two vertices u, v in G are connected by a walk of length $t_0 + g$ but not of length $t_0 + g - 1$; if g is even, any two vertices u, v in G are connected by a walk of length $t_0 + g + 1$ but not of length $t_0 + g$. Denote by $d(u, v)$ the distance from the vertex u to the vertex v. Clearly, $d(u, v) \leq g$.

If $u, v \in B_i$ for some $i \in \{1, \ldots, g\}\backslash\{2\}$, with $u = u_1$ if $i = 1$, and $v = w_r$ if $i = 3$, then, for any $t \geq t_0$, there is a walk of length t from u to v.

Suppose that $u, v \in B_2$. Clearly, there is a walk of length 1 from u to some vertex

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A publication of the International Linear Algebra Society
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in B_3. Also, there is a vertex v' in B_1 such that there is a walk of length 1 from v' to v. Taking into account these observations, and the fact that, for $t \geq t_0$, there is a walk of length t from any vertex in B_3 to w_r, it follows that there is a walk of length $t + (g-2) + 2 = t + g$ from u to v.

Suppose that $u \in B_1'$ and $v \in B_1$. Notice that there is a walk of length g from u to u_1. Since, for $t \geq t_0$, there is a walk of length t from u_1 to v, it follows that there is a walk of length $t + g$ from u to v.

Let $u \in B_3$ and $v \in B_3'$. Then, there is a walk of length g from w_r to v. Since, for $t \geq t_0$, there is a walk of length t from u to w_r, then there is a walk of length $t + g$ from u to v.

Now suppose that $u \in B_i$ and $v \in B_j$, with $i \neq j$.

Suppose that $u \notin B_1' \cup B_2$. Let $w = u$ if $i \neq 3$, and $w = w_r$ otherwise. Then, for $t \geq t_0$, since $g - d(w, v) > 0$, $t + g - d(w, v) \geq t_0$ and there is a walk of length $t + g - d(w, v)$ from u to w. This implies that there is a walk of length $t + g$ from u to v.

Suppose that $u \in B_1'$ and $v \notin B_2 \cup B_3$. Note that $d(w_r, v) \leq g - 2$. Also, there is a walk of length 2 from u to w_r. As, for $t \geq t_0$, w_r lies on a closed walk of length $t+g-d(w_r, v)-2$, then there is a walk of length $2+(t+g-d(w_r, v)-2)+d(w_r, v) = t+g$ from u to v.

Suppose that $u \in B_2$ and $v \notin B_3'$. Then $d(w_r, v) \leq g - 1$. As, for $t \geq t_0$, there is a walk of lenth $t + g - d(w_r, v) - 1$ from any vertex in B_3 to w_r, then there is a walk of length $1 + (t + g - d(w_r, v) - 1) + d(w_r, v) = t + g$ from u to v.

We have shown that, for any $t \geq t_0$, there is a walk of length $t + g$ from u to v, unless either $u \in B_2$ and $v \in B_3'$, or $u \in B_1'$ and $v \in B_2 \cup B_3$.

In order to determine the exponent of E_{nr}, we now consider two cases, depending on the parity of g.

Case 1. Suppose that g is odd. Notice that every walk in G from v_1 to v_r of length $t > g$ contains a subgraph which is a walk of length $t - g$ from a vertex in B_3 to a vertex in B_3. Because there is no walk of length $t_0 - 1$ from a vertex in B_3 to a vertex in B_3, then there is no walk of length $t_0 + g - 1$ from v_1 to v_r.

We have already proven that there is a walk of length $t_0 + g$ from any vertex u to any vertex v, unless either $u \in B_2$ and $v \in B_3'$, or $u \in B_1'$ and $v \in B_2 \cup B_3$, in which cases there is a walk of length s_1 from u to some vertex in B_3 and there is a walk of length s_2 from some vertex in B_1 to v, with $s_1 + s_2 = 4$. By Lemma 4.7, there are

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 28-47, January 2008

nonnegative integers a, b, c such that

$$t_0 - 2 = \frac{1}{2}(g-1)^2 - 2 = a(g-1) + bg + c(g+1).$$

Thus, from any vertex in B_3, there is a walk to w_r of length $t_0 - 2$, which implies that there is a walk of length $(t_0 - 2) + (g-2) + 4 = t_0 + g$ from u to v. Therefore,

$$\exp(E_{n,r}) = t_0 + g = \frac{1}{2}(g^2+1) = \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right).$$

Case 2. Suppose that g is even. First, consider the case $u \in B_1'$ and $v \in B_3$. Clearly, there is a walk of length 3 from u to w_r; also, there is a walk of length 3 from some vertex in B_1 to v. Taking into account Lemma 4.7, w_r lies on a closed walk of length t_0-3, which implies that there is a walk of length $(t_0-3)+(g-2)+6 = t_0+g+1$ from u to v.

Now suppose that either $u \in B_2$ and $v \in B_3'$, or $u \in B_1'$ and $v \in B_2$. Then, there is a walk of length s_1 from u to some vertex in B_3 and there is a walk of length s_2 from some vertex in B_1 to v, with $s_1 + s_2 = 3$. As, from any vertex in B_3, there is a walk of length t_0 to w_r, then there is a walk of length $t_0 + (g-2) + 3 = t_0 + g + 1$ from u to v.

Now we show that there are two vertices not connected by a walk of length $t_0 + g$. Note that $t_0 + g > g + 2$. Also, every walk of length $t > g + 2$ from $u \in B_1'$ to v_r contains a subgraph which is a walk of length $t - g - 1$ or $t - g - 2$ from a vertex in B_3 to a vertex in B_3. By Lemma 4.7, for $k \in \{1, 2\}$, there are no nonnegative integers such that $t_0 - k = a(g-1) + bg + c(g+1)$. So, there is no walk of length $t_0 + g$ from $u \in B_1'$ to v_r.

Thus,

$$\exp(E_{n,r}) = t_0 + g + 1 = \frac{1}{2}(g^2 - g) + 2 = \frac{n(n-r)}{2r^2} + 2. \square$$

If $n = r$, the only matrix in $P_{r,r}$ is J_n which has exponent 1. Note that $n/r = 1$ is odd and $u_{rr} = \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right) = 1$. If $n = 2r$, by Theorem 4.5, $u_{nr} = \frac{n(n-r)}{2r^2} + 2 = 3$. If $n = gr$, with $g \geq 3$ and $r \geq 2$, it follows from Theorem 4.8 that $u_{nr} \geq \exp(E_{nr})$. We conjecture that in this case the equality also holds. Note that $\exp(E_{nr}) < \lfloor \frac{n}{r} \rfloor^2 + 1$.

CONJECTURE 1. *Let $n = gr$ with $g \geq 1$ and $r \geq 2$. Then,*

$$u_{nr} = \begin{cases} \frac{n(n-r)}{2r^2} + 2, & \text{if } \frac{n}{r} \text{ is even} \\ \frac{1}{2}\left(\left(\frac{n}{r}\right)^2 + 1\right), & \text{if } \frac{n}{r} \text{ is odd.} \end{cases}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 28-47, January 2008

Acknowledgment We would like to thank Prof. Chi-Kwong Li, from The College of William and Mary, for mentioning this problem to us and for the fruitful discussions we had with him while working on this paper.

We also thank the referee for his constructive observations and suggestions.

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