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ON THE BRUALDI-LIU CONJECTURE FOR THE EVEN PERMANENT[*]

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Abstract. Counterexamples are given to Brualdi and Liu's conjectured even permanent analogue of the van der Waerden-Egorychev-Falikman Theorem.

Key words. Even permanent, Doubly stochastic, Permutation matrix.

AMS subject classifications. 15A15.

For an $n \times n$ matrix $M = [m_{ij}]$ consider the sum

$$\sum_{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)}.$$

If the sum is taken over all permutations σ of $[n] = \{1, 2, \dots, n\}$ then we get $\mathrm{per}(M)$, the *permanent* of M. If, however, we only take the sum over all even permutations σ of $[n]$ then we get $\mathrm{per}^{\mathrm{ev}}(M)$, the *even permanent* of M.

Let Ω_n denote the set of doubly stochastic matrices (non-negative matrices with row and column sums 1). It is well known that Ω_n consists of all matrices which can be written as a convex combination of permutation matrices of order n. By analogy we define Ω_n^{ev} to be the set of all matrices which can be written as a convex combination of *even* permutation matrices of order n.

The famous van der Waerden-Egorychev-Falikman Theorem states that $\mathrm{per}(M) \geq n!/n^n$ for all $M \in \Omega_n$ with equality iff every entry of M equals $1/n$. Similarly, Brualdi and Liu [2] conjectured $\mathrm{per}^{\mathrm{ev}}(M) \geq \frac{1}{2}n!/n^n$ for all $M \in \Omega_n^{\mathrm{ev}}$ with equality iff every entry of M equals $1/n$. They claimed their conjecture was true for $n \leq 3$. We show below that their conjecture is false for $n \in \{4, 5\}$, although we leave open the possibility that it is true for larger n. For background on all of the above, see Brualdi's new book [1].

Let

$$C_4 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

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Then $C_4 \in \Omega_4^{\text{ev}}$ since

$$C_4 = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

To show that C_4 is a counterexample we consider the more general problem of finding $\text{per}^{\text{ev}}(C_n)$ where C_n is the $n \times n$ matrix with zeroes on the main diagonal and every other entry equal to $1/(n-1)$. Clearly $\text{per}(C_n) = D_n/(n-1)^n$ where

$$D_n = n!\Big(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \cdots (-1)^n \frac{1}{n!}\Big)$$

is the number of derangements (fixed point free permutations) of $[n]$. Using the cards-decks-hands method of Wilf [4] it can be shown that $(1-x)^{-y}e^{-yx}$ is a generating function in which the coefficient of $\frac{1}{n!}x^n y^k$ is the number of derangements of $[n]$ with exactly k cycles. It can then be deduced that the number of even derangements is $\frac{1}{2}\big(D_n + (-1)^n(1-n)\big)$ (this result is probably well-known, certainly it is obtained in [3]). Hence

$$\frac{\text{per}^{\text{ev}}(C_n)}{\frac{1}{2}n!/n^n} = \frac{\big(D_n + (-1)^n(1-n)\big)}{n!}\Big(\frac{n}{n-1}\Big)^n > \Big(\frac{1}{e} - \frac{1}{n(n-2)!}\Big)\exp\Big(1 + \frac{1}{2n}\Big) > 1$$

for $n \geq 5$. It follows that C_n is not a counterexample to the Brualdi-Liu conjecture for any $n \geq 5$. However, $\text{per}^{\text{ev}}(C_4) = 1/27 < 3/64$ so C_4 is a counterexample.

Two further counterexamples arise from the following family of matrices. Let T_n denote the mean of the $(n-1)(n-2)$ permutation matrices corresponding to 3-cycles which move the point 1. For example,

$$T_5 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{12} & \frac{1}{2} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{2} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} & \frac{1}{2} \end{bmatrix}.$$

Then $T_n \in \Omega_n^{\text{ev}}$ by construction. Now given that $\text{per}^{\text{ev}}(T_4) = 5/108 < 3/64$ and $\text{per}^{\text{ev}}(T_5) = 11/576 < 12/625$, both T_4 and T_5 are counterexamples to the Brualdi-Liu conjecture. That the family $\{T_n\}$ contains no further counterexamples is easy to show. The permutation matrices corresponding to 3-cycles alone contribute at least

$$(n-1)(n-2)\frac{1}{(n-1)^2}\Big(\frac{n-3}{n-1}\Big)^{n-3}\frac{1}{(n-1)(n-2)} = \frac{(n-3)^{n-3}}{(n-1)^{n-1}} \sim \frac{1}{(en)^2}$$

to $\text{per}^{\text{ev}}(T_n)$.

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