Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 17, pp. 316-332, July 2008

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EIGENVALUES OF PARTIALLY PRESCRIBED MATRICES*

MARIJA DODIG†

Abstract. In this paper, loop connections of two linear systems are studied. As the main result, the possible eigenvalues of a matrix of a system obtained as a result of these connections are determined.

Key words. Loop connections, Feedback equivalence, Eigenvalues.

AMS subject classifications. 93B05, 15A21.

1. Introduction. Consider two linear systems S_1 and S_2, given by the following equations:

$$S_i \quad \left\{ \begin{array}{lcll} \dot{x}_i & = & A_i x_i & + \quad B_i u_i \\ y_i & = & C_i x_i & \end{array} \right. \qquad i = 1, 2,$$

where $A_i \in \mathbb{K}^{n_i \times n_i}$ is usually called *the matrix of the system* S_i, $B_i \in \mathbb{K}^{n_i \times m_i}$, $C_i \in \mathbb{K}^{p_i \times n_i}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $i = 1, 2$. Also, x_i is the state, y_i is the output and u_i is the input of the system S_i, $i = 1, 2$; for details see [4].

By *loop (or closed) connections* of the linear systems S_1 and S_2 we mean connections where the input of S_2 is a linear function of the output of S_1, and the input of S_1 is a linear function of the output of S_2, i.e.,

$$u_2 = \bar{X}_1 y_1$$
$$u_1 = \bar{X}_2 y_2$$

where $\bar{X}_1 \in \mathbb{K}^{m_2 \times p_1}$ and $\bar{X}_2 \in \mathbb{K}^{m_1 \times p_2}$. As a result of this connection we obtain a system S with the state $\left[\begin{array}{cc} x_1^T & x_2^T \end{array} \right]^T$, and the matrix

$$\left[\begin{array}{cc} A_1 & B_1 \bar{X}_2 C_2 \\ B_2 \bar{X}_1 C_1 & A_2 \end{array} \right]. \tag{1.1}$$

Analogously to [1], we shall only consider the systems S_1 and S_2 with the properties $\operatorname{rank} B_1 = n_1$ and $\operatorname{rank} C_2 = n_2$. Hence, studying the properties of the system S gives the following matrix completion problem:

* Received by the editors June 5, 2007. Accepted for publication June 8, 2008. Handling Editor: Joao Filipe Queiro.

† Centro de Estruturas Lineares e Combinatórias, CELC, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal (dodig@cii.fc.ul.pt).

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Problem 1.1. *Let $\mathbb{F}$ be a field. Determine possible eigenvalues of the matrix*

$$\begin{bmatrix} A_1 & X_2 \\ B_2 X_1 C_1 & A_2 \end{bmatrix} \tag{1.2}$$

when matrices $X_1 \in \mathbb{F}^{m_2 \times p_1}$ and $X_2 \in \mathbb{F}^{n_1 \times n_2}$ vary.

Similar completion problems have been studied in papers by G. N. de Oliveira [6], [7], [8],[9], E. M. de Sá [10], R. C. Thompson [13] and F. C. Silva [11], [12]. In the last two papers, F. C. Silva solved two special cases of Problem 1.1, both in the case when eigenvalues of the matrix (1.2) belong to the field $\mathbb{F}$. In [11], he solves the Problem 1.1 in the case when $\operatorname{rank} B_2 = n_2$ and $\operatorname{rank} C_1 = m_1$. Moreover, in [12], he solves the Problem 1.1 in the case when the matrix X_1 is known.

This paper is a natural generalization of those results. As the main result (Theorem 3.1), we give a complete solution of Problem 1.1 in the case when the eigenvalues of the matrix (1.2) belong to $\mathbb{F}$, and $\mathbb{F}$ is an infinite field. In particular, this gives the complete solution of Problem 1.1 over algebraically closed fields. Moreover, in Theorem 4.2, we study the possible eigenvalues of the matrix (1.1) in the case when $\operatorname{rank} C_1 = n_1$ and $\operatorname{rank} C_2 = n_2$ while $\operatorname{rank} B_1 = \operatorname{rank} B_2 = 1$. In this special case, we give necessary and sufficient conditions for the existence of matrices $\bar{X}_1$ and $\bar{X}_2$ such that the matrix (1.1) has prescribed eigenvalues, over algebraically closed fields.

Since the proof of the main result strongly uses previous results from [11] and [12], we cite here the main result of [12], written in its transposed form, as it will be used later in the proof of Theorem 3.1:

Theorem 1.2. *Let $\mathbb{F}$ be a field. Let $c_1, \ldots, c_{m+n} \in \mathbb{F}$, $A_{11} \in \mathbb{F}^{m \times m}$, $A_{21} \in \mathbb{F}^{n \times m}$, and $A_{22} \in \mathbb{F}^{n \times n}$. Let $f_1(\lambda) | \cdots | f_m(\lambda)$ be the invariant factors of $\begin{bmatrix} \lambda I_m - A_{11} \\ -A_{21} \end{bmatrix}$, and let $g_1(\lambda) | \cdots | g_n(\lambda)$ be the invariant factors of $\begin{bmatrix} \lambda I_n - A_{22} & -A_{21} \end{bmatrix}$.*

There exists $A_{12} \in \mathbb{F}^{n \times m}$ such that the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{1.3}$$

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has eigenvalues $c_1, \ldots, c_{m+n}$ *if and only if the following conditions hold:*

$$(a) \quad c_1 + \cdots + c_{n+m} = \operatorname{tr} A_{11} + \operatorname{tr} A_{22}$$

$$(b) \quad f_1(\lambda) \cdots f_m(\lambda) g_1(\lambda) \cdots g_n(\lambda) | (\lambda - c_1) \cdots (\lambda - c_{n+m})$$

(c) *One of the following conditions is satisfied:*

(c_1) *For every* $\nu \in \mathbb{F}, A_{21}A_{11} + A_{22}A_{21} \neq \nu A_{21}$

(c_2) $A_{21}A_{11} + A_{22}A_{21} = \nu A_{21}$

with $\nu \in \mathbb{F}$, *and there exists a permutation*

$\pi : \{1, \ldots, m+n\} \to \{1, \ldots, m+n\}$ *such that*

$$c_{\pi(2i-1)} + c_{\pi(2i)} = \nu$$

for every $i = 1, \ldots, l$, *where* $l = \operatorname{rank} A_{21}$, *and*

$$c_{\pi(2l+1)}, \ldots, c_{\pi(m+n)}$$

are the roots of $f_1(\lambda) \cdots f_m(\lambda) g_1(\lambda) \cdots g_n(\lambda)$.

2. Notation and technical results.

Let $\mathbb{F}$ be a field. All the polynomials in this paper are considered to be monic. If f is a polynomial, $d(f)$ denotes its degree. If $\psi_1 | \cdots | \psi_n$ are invariant factors of a polynomial matrix $A(\lambda)$ over $\mathbb{F}[\lambda]$, $\operatorname{rank} A(\lambda) = n$, then we assume $\psi_i = 1$, for any $i \leq 0$, and $\psi_i = 0$, for any $i \geq n+1$.

Definition 2.1. *Let* $A, A' \in \mathbb{F}^{n \times n}$, $B, B' \in \mathbb{F}^{n \times l}$. *Two matrices*

$$K = \begin{bmatrix} A & B \end{bmatrix}, \qquad K' = \begin{bmatrix} A' & B' \end{bmatrix} \tag{2.1}$$

are said to be feedback equivalent if there exists a nonsingular matrix

$$P = \begin{bmatrix} N & 0 \\ V & T \end{bmatrix}$$

where $N \in \mathbb{F}^{n \times n}$, $V \in \mathbb{F}^{l \times n}$, $T \in \mathbb{F}^{l \times l}$, *such that* $K' = N^{-1}KP$.

It is easy to verify that two matrices of the form (2.1) are feedback equivalent if and only if the corresponding matrix pencils

$$R = \begin{bmatrix} \lambda I - A & -B \end{bmatrix} \text{ and } R' = \begin{bmatrix} \lambda I - A' & -B' \end{bmatrix} \tag{2.2}$$

are strictly equivalent, i.e., if there exist invertible matrices $D \in \mathbb{F}^{n \times n}$ and $T \in \mathbb{F}^{(n+l) \times (n+l)}$ such that $R = DR'T$; for details see [5].

By invariant polynomials of the matrix K from (2.1), we mean invariant factors of the corresponding matrix pencil R from (2.2).

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Let $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{n\times m}$. Denote by $S(A,B)$ the controllability matrix of the pair (A,B), i.e.,

$$S(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{F}^{n\times nm}.$$

If $\operatorname{rank} S(A,B) = n$, then we say that the pair (A,B) is controllable.

Lemma 2.2. ([1, Lemma 3.4], [2]) *Let $(A,B) \in \mathbb{F}^{n\times n} \times \mathbb{F}^{n\times m}$ be such that $\operatorname{rank} S(A,B) = r$. Then there exists a nonsingular matrix $P \in \mathbb{F}^{n\times n}$ such that*

$$PAP^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \qquad PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{2.3}$$

where $(A_1,B_1) \in \mathbb{F}^{r\times r} \times \mathbb{F}^{r\times m}$ is a controllable pair. The pair (PAP^{-1}, PB) is called the Kalman decomposition of the pair (A,B).

Moreover, the matrix A_1 from (2.3), is called *the restriction of the matrix A to the controllable space of the pair (A,B)*. Also, recall that the nontrivial invariant polynomials of the matrix A_3 from (2.3), coincide with the nontrivial invariant factors of the matrix pencil $\begin{bmatrix} \lambda I - A & -B \end{bmatrix}$. By trivial polynomials we mean polynomials equal to 1.

Analogously to [3], we introduce the following definition:

Definition 2.3. *Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ and $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ are SP-equivalent if there exist an invertible matrix $P \in \mathbb{F}^{n\times n}$ and an invertible polynomial matrix $Q(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that*

$$PA(\lambda)Q(\lambda) = B(\lambda).$$

Also, we give the following proposition which follows from Proposition 2 in [3]:

Proposition 2.4. *Let $\mathbb{F}$ be an infinite field and let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ with $\det A(\lambda) \neq 0$. Then $A(\lambda)$ is SP-equivalent to a lower triangular matrix $S(\lambda) = (s_{ij}(\lambda))$ with the following properties:*

1. $s_{ii}(\lambda) = s_i(\lambda),\ i = 1,\dots,n,$ *where* $s_1(\lambda)|\cdots|s_n(\lambda)$ *are the invariant factors of* $A(\lambda)$
2. $s_{ii}(\lambda)|s_{ji}(\lambda)$ *for all integers* i,j *with* $1 \le i \le j \le n$
3. *if* $i < j$ *and* $s_{ji}(\lambda) \neq 0$ *then* $s_{ji}(\lambda)$ *is monic and* $d(s_{ii}(\lambda)) < d(s_{ji}(\lambda)) < d(s_{jj}(\lambda))$.

The matrix $S(\lambda)$ is called *the SP-canonical form* of the matrix $A(\lambda)$.

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3. Main result. In the following theorem we give a solution to Problem 1.1, over infinite fields.

THEOREM 3.1. *Let $\mathbb{F}$ be an infinite field. Let $A_1 \in \mathbb{F}^{n_1\times n_1}$, $A_2 \in \mathbb{F}^{n_2\times n_2}$, $B_2 \in \mathbb{F}^{n_2\times m_2}$ and $C_1 \in \mathbb{F}^{p_1\times n_1}$. Let $\rho_1 = \operatorname{rank} B_2$ and $\rho_2 = \operatorname{rank} C_1$. Let $c_1,\dots,c_{n_1+n_2} \in \mathbb{F}$. Let $\phi(\lambda) = (\lambda - c_1)\cdots(\lambda - c_{n_1+n_2})$. Let $\alpha_1|\cdots|\alpha_{n_1}$ be the invariant factors of*

$$\begin{bmatrix} \lambda I - A_1 \\ -C_1 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n_1+p_1)\times n_1},$$

and $\beta_1|\cdots|\beta_{n_2}$ be the invariant factors of

$$\begin{bmatrix} \lambda I - A_2 & -B_2 \end{bmatrix} \in \mathbb{F}[\lambda]^{n_2\times(n_2+m_2)}.$$

Let $\gamma_1|\cdots|\gamma_x$, $x = n_2 - d(\prod_{i=1}^{n_2}\beta_i)$, be the invariant polynomials of the restriction of the matrix A_2 to the controllable space of the pair (A_2, B_2). Let $\delta_1|\cdots|\delta_y$, $y = n_1 - d(\prod_{i=1}^{n_1}\alpha_i)$, be the invariant polynomials of the restriction of the matrix A_1 to the controllable space of the pair (A_1^T, C_1^T).

There exist matrices $X_1 \in \mathbb{F}^{m_2\times p_1}$ and $X_2 \in \mathbb{F}^{n_1\times n_2}$ such that the matrix

$$\begin{bmatrix} A_1 & X_2 \\ B_2X_1C_1 & A_2 \end{bmatrix} \tag{3.1}$$

has $c_1,\dots,c_{n_1+n_2}$ as eigenvalues if and only if the following conditions are valid:

(*i*) $\operatorname{tr} A_1 + \operatorname{tr} A_2 = \sum_{i=1}^{n_1+n_2} c_i$,

(*ii*) $\alpha_1\cdots\alpha_{n_1}\beta_1\cdots\beta_{n_2}\gamma_1\cdots\gamma_{x-\rho_2}\delta_1\cdots\delta_{y-\rho_1}|\phi(\lambda)$,

(*iii*) *If $\rho_1 = x$, $\rho_2 = y$, $\gamma_i = \lambda - b$, $i = 1,\dots,x$, and $\delta_i = \lambda - a$, $i = 1,\dots,y$, for some $a, b \in \mathbb{F}$, then there exists a permutation $\pi : \{1,\dots,n_1+n_2\} \to \{1,\dots,n_1+n_2\}$ such that*

$$c_{\pi(2i-1)} + c_{\pi(2i)} = a + b$$

for every $i = 1,\dots,\min\{\rho_1,\rho_2\}$ and

$$c_{\pi(2\min\{\rho_1,\rho_2\}+1)},\dots,c_{\pi(n_1+n_2)}$$

are the roots of $\alpha_1\cdots\alpha_{n_1}\beta_1\cdots\beta_{n_2}\gamma_1\cdots\gamma_{x-\rho_2}\delta_1\cdots\delta_{y-\rho_1}$.

REMARK 3.2. *Before proceeding, we shall give the equivalent form of Theorem 3.1 that we will actually prove.*

Note that by Lemma 2.2, there exist invertible matrices $P_i \in \mathbb{F}^{n_i\times n_i}$, $i = 1, 2$,

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A publication of the International Linear Algebra Society
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$Q \in \mathbb{F}^{m_2\times m_2}$ and $R \in \mathbb{F}^{p_1\times p_1}$ such that

$$\left[\begin{array}{c} P_1^{-1}A_1P_1 \\ RC_1P_1 \end{array}\right] = \left[\begin{array}{c|c|c} T' & P' & S' \\ \hline 0 & H' & N' \\ 0 & E' & M' \\ \hline 0 & 0 & I_{\rho_2} \\ 0 & 0 & 0 \end{array}\right], \tag{3.2}$$

and

$$\left[\begin{array}{cc} P_2B_2Q & P_2A_2P_2^{-1} \end{array}\right] = \left[\begin{array}{cc|cc|c} 0 & I_{\rho_1} & M & N & S \\ \hline 0 & 0 & E & H & P \\ \hline 0 & 0 & 0 & 0 & T \end{array}\right], \tag{3.3}$$

where $T' \in \mathbb{F}^{(n_1-y)\times(n_1-y)}$, $H' \in \mathbb{F}^{(y-\rho_2)\times(y-\rho_2)}$, $M' \in \mathbb{F}^{\rho_2\times\rho_2}$, $T \in \mathbb{F}^{(n_2-x)\times(n_2-x)}$, $H \in \mathbb{F}^{(x-\rho_1)\times(x-\rho_1)}$, $M \in \mathbb{F}^{\rho_1\times\rho_1}$, and the pairs (H'^T, E'^T) and (H, E) are controllable.

Thus, the nontrivial among the polynomials $\alpha_1, \ldots, \alpha_{n_1}$, and $\beta_1, \ldots, \beta_{n_2}$, coincide with the nontrivial invariant polynomials of the matrices T' and T, respectively.

Since the matrix

$$\left[\begin{array}{cc} M & N \\ E & H \end{array}\right] \in \mathbb{F}^{x\times x}$$

is the restriction of the matrix A_2 to the controllable space of the pair (A_2, B_2), its invariant polynomials are $\gamma_1|\cdots|\gamma_x$. And, analogously, $\delta_1|\cdots|\delta_y$ are the invariant polynomials of the matrix

$$\left[\begin{array}{cc} H' & N' \\ E' & M' \end{array}\right] \in \mathbb{F}^{y\times y}.$$

Finally, in this way we have concluded that the matrix (3.1) is similar to the following one

$$\left[\begin{array}{c|c|c|c} T' & P' & S' & \\ 0 & H' & N' & Z_2 \\ 0 & E' & M' & \\ \hline \multicolumn{2}{c|}{0} & Z_1 & \begin{array}{ccc} M & N & S \end{array} \\ \hline \multicolumn{2}{c|}{0} & 0 & \begin{array}{ccc} E & H & P \\ \hline 0 & 0 & T \end{array} \end{array}\right], \tag{3.4}$$

where $Z_2 = P_1^{-1}X_2P_2^{-1}$ and

$$\left[\begin{array}{cc} 0 & Z_1 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{cc} 0 & I_{\rho_1} \\ 0 & 0 \end{array}\right] Q^{-1}X_1R^{-1} \left[\begin{array}{cc} 0 & I_{\rho_2} \\ 0 & 0 \end{array}\right], \quad Z_1 \in \mathbb{F}^{\rho_1\times\rho_2}.$$

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In this notation, the theorem becomes:

There exist matrices $Z_1 \in \mathbb{F}^{\rho_1 \times \rho_2}$ *and* $Z_2 \in \mathbb{F}^{n_1 \times n_2}$ *such that the matrix (3.4) has* $c_1, \ldots, c_{n_1+n_2} \in \mathbb{F}$ *as eigenvalues, if and only if the following conditions are valid:*

$(i)'$ $\operatorname{tr} T' + \operatorname{tr} H' + \operatorname{tr} M' + \operatorname{tr} T + \operatorname{tr} H + \operatorname{tr} M = \sum_{i=1}^{n_1+n_2} c_i,$

$(ii)'$ $\alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2} \gamma_1 \cdots \gamma_{x-\rho_2} \delta_1 \cdots \delta_{y-\rho_1} | \phi(\lambda),$

$(iii)'$ *One of the following statements is true:*

(a) *at least one of the matrices* M *or* M' *is not of the form* $\nu I, \quad \nu \in \mathbb{F}$, *or at least one of the matrices* E *or* E' *is nonzero.*

(b) $M' = aI_{\rho_2}, \quad M = bI_{\rho_1}, \quad E = 0, \quad E' = 0,$ *with* $a, b \in \mathbb{F}$, *and there exists a permutation* $\pi : \{1, \ldots, n_1+n_2\} \to \{1, \ldots, n_1+n_2\}$ *such that*

$c_{\pi(2i-1)} + c_{\pi(2i)} = a + b,$ *for every* $i = 1, \ldots, y$, *and*

$c_{\pi(2y+1)}, \ldots, c_{\pi(n_1+n_2)}$ *are the roots of* $\alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2} \gamma_1 \cdots \gamma_{x-y}$.

(Note that $E = 0$ *implies* $x = \rho_1$ *and* $E' = 0$ *implies* $y = \rho_2$*).*

Proof.

Necessity:

Without loss of generality, we assume that $\rho_1 \geq \rho_2 > 0$. Thus, condition $(ii)'$ becomes

$$\alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2} \gamma_1 \cdots \gamma_{x-\rho_2} | \phi(\lambda).$$

Suppose that there exist matrices Z_1 and Z_2 such that the matrix (3.4) has prescribed eigenvalues from the field $\mathbb{F}$. Then condition $(i)'$ is trivially satisfied.

Let

$$Z_2 = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \in \mathbb{F}^{n_1 \times n_2}, \text{ where } Z_{21} \in \mathbb{F}^{y \times x}.$$

Denote by $\xi(\lambda)$, the product of the invariant polynomials of the matrix

$$\left[\begin{array}{cc|cc} H' & N' & \multicolumn{2}{c}{\multirow{2}{*}{Z_{21}}} \\ E' & M' & & \\ \hline 0 & Z_1 & M & N \\ 0 & 0 & E & H \end{array}\right] \in \mathbb{F}^{(x+y)\times(x+y)}. \tag{3.5}$$

Then we have

$$\alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2} \xi(\lambda) = \phi(\lambda). \tag{3.6}$$

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Let $\mu_1 | \cdots | \mu_x$ be the invariant factors of

$$\left[\begin{array}{cc|cc} 0 & Z_1 & \lambda I - M & N \\ 0 & 0 & E & \lambda I - H \end{array}\right].$$

Using the classical Sá-Thompson result (see [10, 13]) we obtain

$$\mu_i \mid \gamma_i \mid \mu_{i+\rho_2}, \quad i = 1, \ldots, x, \quad \text{and} \quad \prod_{i=1}^{x} \mu_i | \xi(\lambda).$$

Thus, $\gamma_1 \cdots \gamma_{x-\rho_2} | \xi(\lambda)$, which together with (3.6) gives the condition $(ii)'$.

In order to prove the necessity of condition $(iii)'$, we shall use the result from Theorem 1 in [11].

In fact, if both matrices E and E' are zero, and if $M' = aI_{\rho_2}$ and $M = bI_{\rho_1}$, $a, b \in \mathbb{F}$, then the matrices (3.2) and (3.3) are of the forms

$$\left[\begin{array}{c|c} T' & S' \\ \hline 0 & M' \\ \hline 0 & I_{\rho_2} \\ 0 & 0 \end{array}\right] \tag{3.7}$$

and

$$\left[\begin{array}{c|c|c|c} 0 & I_{\rho_1} & M & S \\ \hline 0 & 0 & 0 & T \end{array}\right], \tag{3.8}$$

respectively. Thus, in this case, the matrix (3.4) becomes

$$\left[\begin{array}{c|c|cc} T' & S' & Z_{11} & Z_{12} \\ 0 & M' & Z_{21} & Z_{22} \\ \hline 0 & Z_1 & M & S \\ \hline 0 & 0 & 0 & T \end{array}\right]. \tag{3.9}$$

Since $\alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2} | \phi(\lambda)$, and $d(\prod_{i=1}^{n_1} \alpha_i) = n_1 - y$, $d(\prod_{i=1}^{n_2} \beta_i) = n_2 - x$, then $n_1 + n_2 - y - x$ of the eigenvalues $c_1, \ldots, c_{n_1+n_2}$ are the roots of the polynomial $\alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2}$. We shall assume, without loss of generality, that those eigenvalues are $c_{x+y+1}, \ldots, c_{n_1+n_2}$.

Let

$$\bar{\phi}(\lambda) := (\lambda - c_1) \cdots (\lambda - c_{x+y}).$$

Furthermore, from the existence of matrices Z_1 and Z_2 such that the matrix (3.9) has prescribed eigenvalues $c_1, \ldots, c_{n_1+n_2}$, there exist matrices Z_1 and Z_{21} such that the matrix

$$\left[\begin{array}{c|c} M' & Z_{21} \\ \hline Z_1 & M \end{array}\right]$$

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has prescribed eigenvalues $c_1, \dots, c_{x+y}$. Finally, since $M' = aI$ and $M = bI$, for some $a, b \in \mathbb{F}$, by applying Theorem 1 from [11], there exists a permutation $\pi : \{1, \dots, x+y\} \to \{1, \dots, x+y\}$ such that

$$c_{\pi(2i-1)} + c_{\pi(2i)} = a + b$$

for every $i = 1, \dots, y$, and $c_{\pi(j)} = b$, for $2y < j \leq x+y$. Putting everything together gives condition $(iii)'$, as wanted.

Sufficiency:

Consider the matrix (3.4). Suppose that conditions $(i)'$, $(ii)'$ and $(iii)'$ are valid.

Let

$$Z_2 = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \in \mathbb{F}^{n_1 \times n_2}, \text{ where } Z_{21} \in \mathbb{F}^{y \times x}.$$

Our aim is to define matrices Z_1 and Z_2 (i.e. Z_{11}, Z_{12}, Z_{21} and Z_{22}) such that the matrix (3.4) has $c_1, \dots, c_{n_1+n_2}$ as eigenvalues.

Since the pair (H, E) is controllable, the matrix

$$\left[\begin{array}{c|cc} Z_1 & \lambda I - M & -N \\ 0 & -E & \lambda I - H \end{array}\right],$$

is equivalent to

$$\left[\begin{array}{c|c|c} Z_1 & K(\lambda) & S(\lambda) \\ \hline 0 & 0 & I_{x-\rho_1} \end{array}\right], \tag{3.10}$$

for some matrices $K(\lambda) \in \mathbb{F}[\lambda]^{\rho_1 \times \rho_1}$ and $S(\lambda) \in \mathbb{F}[\lambda]^{\rho_1 \times (x-\rho_1)}$.

By Proposition 2.4, there exist invertible matrices $Q \in \mathbb{F}^{\rho_1 \times \rho_1}$ and $Q(\lambda) \in \mathbb{F}[\lambda]^{\rho_1 \times \rho_1}$ such that the matrix $QK(\lambda)Q(\lambda)$ is the SP-canonical form of the matrix $K(\lambda)$, with polynomials $\gamma_x, \dots, \gamma_{x-\rho_2}, \dots, \gamma_{x-\rho_1+1}$ on the main diagonal:

$$\bar{K}(\lambda) = QK(\lambda)Q(\lambda) = \begin{bmatrix} \gamma_{x-\rho_1+1} & & & & \\ * & \ddots & & & \\ * & * & \gamma_{x-\rho_2} & & \\ * & * & * & \ddots & \\ * & * & * & * & \gamma_x \end{bmatrix},$$

where nonmarked entries are equal to zero and $*$ denote unimportant entries. This last statement is true since there are at least $x - \rho_1$ trivial polynomials among $\gamma_1 | \cdots | \gamma_x$,

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and the nontrivial polynomials among $\gamma_1|\cdots|\gamma_x$ coincide with the nontrivial invariant factors of $K(\lambda)$.

Let

$$Z_1 = Q^{-1}\begin{bmatrix} 0 \\ I_{\rho_2} \end{bmatrix} = Q^{-1}L. \tag{3.11}$$

Now, the matrix (3.5) becomes

$$\left[\begin{array}{cc|cc} H' & N' & \multicolumn{2}{c}{\multirow{2}{*}{Z_{21}}} \\ E' & M' & & \\ \hline 0 & Q^{-1}L & M & N \\ 0 & 0 & E & H \end{array}\right]. \tag{3.12}$$

Since the product of the invariant polynomials of the matrices T and T' divides $\phi(\lambda)$, put $Z_{11} = 0$, $Z_{12} = 0$ and $Z_{22} = 0$. Also, as in the necessity part of the proof, we shall assume, without loss of generality, that $c_{x+y+1}, \ldots, c_{n_1+n_2}$, are the zeros of the polynomial $\alpha_1 \cdots \alpha_{n_1}\beta_1 \cdots \beta_{n_2}$. Now, the problem reduces to defining the matrix Z_{21} such that the matrix (3.12) has $c_1, \ldots, c_{x+y}$ as eigenvalues.

The product of the invariant factors of the matrix

$$\left[\begin{array}{c|c} \lambda I - H' & -N' \\ -E' & \lambda I - M' \\ \hline 0 & -Q^{-1}L \\ 0 & 0 \end{array}\right] \tag{3.13}$$

is equal to the product of the invariant factors of the matrix

$$\left[\begin{array}{c|c} \lambda I - H' & -N' \\ -E' & \lambda I - M' \\ \hline 0 & -I_{\rho_2} \end{array}\right] \in \mathbb{F}^{(y+\rho_2)\times y},$$

which is equal to 1. Also, the product of the invariant factors of the matrix

$$\left[\begin{array}{cc|cc} \lambda I - M & -N & 0 & -Q^{-1}L \\ -E & \lambda I - H & 0 & 0 \end{array}\right] \tag{3.14}$$

is equal to the product of $\gamma_1, \ldots, \gamma_{x-\rho_2}$. Therefore, from condition $(ii)'$, we have that the product of the invariant factors of the matrices (3.13) and (3.14), divide $\bar{\phi}(\lambda) := (\lambda - c_1)\cdots(\lambda - c_{x+y})$. So, in order to apply Theorem 1.2, and thus to conclude the existence of the matrix Z_{21} with the wanted properties, we need to prove that condition (c) from Theorem 1.2 is valid. In our case, condition (c) from Theorem 1.2 becomes:

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(*c*) *If*

$$\left[\begin{array}{c|c} 0 & Q^{-1}L \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} H' & N' \\ \hline E' & M' \end{array}\right]+\left[\begin{array}{c|c} M & N \\ \hline E & H \end{array}\right]\left[\begin{array}{c|c} 0 & Q^{-1}L \\ \hline 0 & 0 \end{array}\right]=\left[\begin{array}{c|c} 0 & \nu Q^{-1}L \\ \hline 0 & 0 \end{array}\right], \tag{3.15}$$

for some $\nu \in \mathbb{F}$*, then there exists a permutation* $\pi : \{1,\dots,x+y\} \to \{1,\dots,x+y\}$ *such that*

$$c_{\pi(2i-1)} + c_{\pi(2i)} = \nu$$

for every $i = 1,\dots,\rho_2$*, where* $c_{\pi(2y+1)},\dots,c_{\pi(x+y)}$ *are the roots of* $\gamma_1 \cdots \gamma_{x-\rho_2}$*.*

From conditions $(i)'$, $(ii)'$ and $(iii)'$, in order to prove condition (c), it is enough to prove that (3.15) implies $M' = aI_{\rho_2}$, $M = bI_{\rho_1}$, $E = 0$ and $E' = 0$, for some $a, b \in \mathbb{F}$.

The equation (3.15) is equivalent to

$$\left[\begin{array}{c|c} Q^{-1}LE' & Q^{-1}LM' + MQ^{-1}L \\ \hline 0 & EQ^{-1}L \end{array}\right]=\left[\begin{array}{c|c} 0 & \nu Q^{-1}L \\ \hline 0 & 0 \end{array}\right]. \tag{3.16}$$

Now, let

$$Y := \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}\left[\begin{array}{c|c} M & N \\ \hline E & H \end{array}\right]\begin{bmatrix} Q^{-1} & 0 \\ 0 & I \end{bmatrix}=\left[\begin{array}{c|c|c} A & B & C \\ \hline D & G & F \\ \hline S & W & H \end{array}\right]\in \mathbb{F}^{x\times x}, \tag{3.17}$$

where $A \in \mathbb{F}^{(\rho_1-\rho_2)\times(\rho_1-\rho_2)}$, $G \in \mathbb{F}^{\rho_2\times\rho_2}$.

With this notation, the equation (3.16) is equivalent to the following ones:

$$E' = 0, \tag{3.18}$$

$$M' + G = \nu I, \tag{3.19}$$

$$W = 0, \tag{3.20}$$

$$B = 0. \tag{3.21}$$

From (3.10), and by definition of $\bar{K}(\lambda)$, the matrix $\lambda I - Y$ is equivalent to

$$\left[\begin{array}{c|c} \bar{K}(\lambda) & QS(\lambda) \\ \hline 0 & I_{x-\rho_1} \end{array}\right]. \tag{3.22}$$

Thus, the submatrix of $\lambda I - Y$ formed by the rows $1,\dots,\rho_1-\rho_2,\rho_1+1,\dots,x$, has the same invariant factors as the submatrix of (3.22) formed by the same rows. In fact, by using the form (3.17), we obtain that the matrix

$$\lambda I - Z := \begin{bmatrix} \lambda I - A & -C \\ -S & \lambda I - H \end{bmatrix}, \tag{3.23}$$

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has the same invariant factors as the submatrix of (3.22) formed by the rows $1, \dots, \rho_1 - \rho_2, \rho_1 + 1, \dots, x$. In particular, the degree of the product of the invariant factors of this submatrix is equal to the degree of the product of the invariant factors of (3.23).

The dimension of the matrix (3.23) is equal to the product of its invariant factors, i.e.,

$$\dim Z = d(\gamma_1 \cdots \gamma_{x-\rho_2}).$$

Now, we have two cases:

$$\dim Z > 0, \tag{3.24}$$

$$\dim Z = 0. \tag{3.25}$$

In the first case, we have that $d(\gamma_{x-\rho_2}) \geq 1$. From $\dim Z = d(\gamma_1 \cdots \gamma_{x-\rho_2})$, and since $\gamma_1 | \cdots | \gamma_x$ are the invariant polynomials of Y, we have

$$\rho_2 = d(\gamma_{x-\rho_2+1} \cdots \gamma_x).$$

Thus, the degrees of all $\gamma_1 | \cdots | \gamma_x$ are equal to one, i.e., the matrices

$$\begin{bmatrix} M & N \\ E & H \end{bmatrix} \quad \text{and} \quad Y$$

are of the form kI, for some $k \in \mathbb{F}$.

Hence, the equation (3.15) implies that $E = 0$, $E' = 0$, $M = aI$, $M' = bI$, $a, b \in \mathbb{F}$, i.e., condition (c) from Theorem 1.2 is satisfied. Thus, there exists a matrix Z_2 with the wanted properties.

The case (3.25) can occur only if $\rho_1 = \rho_2 = x = y$. In this particular case the matrix (3.5) becomes:

$$\left[\begin{array}{c|c} M' & Z_{21} \\ \hline Z_1 & M \end{array}\right] \in \mathbb{F}^{2x \times 2x}.$$

Furthermore, in this case, condition $(iii)'$ becomes:

If $M' = aI_x$ and $M = bI_x$, with $a, b \in \mathbb{F}$, then there exists a permutation $\pi : \{1, \dots, 2x\} \to \{1, \dots, 2x\}$, such that

$$c_{\pi(2i-1)} + c_{\pi(2i)} = a + b \quad \textit{for every} \quad i = 1, \dots, x.$$

Now, we can apply the result of Theorem 1 from [11], and thus we finish the proof. □

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A publication of the International Linear Algebra Society
Volume 17, pp. 316-332, July 2008

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4. Special case. Consider the matrix (1.1). Let $\operatorname{rank} B_2 = \operatorname{rank} B_1 = 1$ and let $\operatorname{rank} C_1 = n_1$ and $\operatorname{rank} C_2 = n_2$. Then we have the following matrix completion problem:

PROBLEM 4.1. *Let $\mathbb{F}$ be a field. Let $\operatorname{rank} B_2 = \operatorname{rank} B_1 = 1$. Determine the possible eigenvalues of the matrix*

$$\left[\begin{array}{c|c} A_1 & B_1X_2 \\ \hline B_2X_1 & A_2 \end{array}\right] \tag{4.1}$$

when the matrices $X_1 \in \mathbb{F}^{m_2\times n_1}$ and $X_2 \in \mathbb{F}^{m_1\times n_2}$ vary.

In the following theorem we give a complete solution to Problem 4.1, in the case when $\mathbb{F}$ is an algebraically closed field:

THEOREM 4.2. *Let $\mathbb{F}$ be an algebraically closed field. Let $A_1 \in \mathbb{F}^{n_1\times n_1}$, $A_2 \in \mathbb{F}^{n_2\times n_2}$, $B_1 \in \mathbb{F}^{n_1\times m_1}$ and $B_2 \in \mathbb{F}^{n_2\times m_2}$ be such that $\operatorname{rank} B_1 = \operatorname{rank} B_2 = 1$. Let $c_1, \dots, c_{n_1+n_2} \in \mathbb{F}$. There exist matrices $X_1 \in \mathbb{F}^{m_2\times n_1}$ and $X_2 \in \mathbb{F}^{m_1\times n_2}$ such that the matrix (4.1) has $c_1, \dots, c_{n_1+n_2}$ as eigenvalues if and only if the following conditions are valid:*

$$(i)\operatorname{tr} A_1 + \operatorname{tr} A_2 = \sum_{i=1}^{n_1+n_2} c_i,$$
$$(ii)\alpha_1\cdots\alpha_{n_1}\beta_1\cdots\beta_{n_2}|\phi(\lambda).$$

Here $\phi(\lambda) = (\lambda - c_1)\cdots(\lambda - c_{n_1+n_2})$, while $\alpha_1|\cdots|\alpha_{n_1}$ are the invariant factors of

$$\begin{bmatrix} \lambda I - A_1 & -B_1 \end{bmatrix},$$

and $\beta_1|\cdots|\beta_{n_2}$ are the invariant factors of

$$\begin{bmatrix} \lambda I - A_2 & -B_2 \end{bmatrix}.$$

REMARK 4.3. *As in Theorem 3.1, before proceeding, we give the matrix similar to the matrix (4.1) that will be used in the proof.*

Let $\sum_{i=1}^{n_1} d(\alpha_i) = x$ and $\sum_{i=1}^{n_2} d(\beta_i) = y$. Since $\operatorname{rank} B_1 = \operatorname{rank} B_2 = 1$, by Lemma 2.2 there exist invertible matrices $P_i \in \mathbb{F}^{n_i\times n_i}$, $i = 1, 2$, $Q \in \mathbb{F}^{m_2\times m_2}$ and $R \in \mathbb{F}^{m_1\times m_1}$ such that

$$\begin{bmatrix} P_1A_1P_1^{-1} & P_1B_1R \end{bmatrix} = \left[\begin{array}{cc|c|cc} m' & n' & s' & 1 & 0 \\ \hline E' & H' & P' & 0 & 0 \\ \hline 0 & 0 & T' & 0 & 0 \end{array}\right], \tag{4.2}$$

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and

$$\begin{bmatrix} P_2B_2Q & P_2A_2P_2^{-1} \end{bmatrix} = \left[\begin{array}{cc|cc|c} 0 & 1 & m & n & s \\ \hline 0 & 0 & E & H & P \\ \hline 0 & 0 & 0 & 0 & T \end{array}\right], \tag{4.3}$$

where $m, m' \in \mathbb{F}$, $H \in \mathbb{F}^{(n_2-y-1)\times(n_2-y-1)}$, $H' \in \mathbb{F}^{(n_1-x-1)\times(n_1-x-1)}$, $T \in \mathbb{F}^{y\times y}$, $T' \in \mathbb{F}^{x\times x}$, *and* (H,E) *and* (H',E') *are controllable pairs of matrices.*

Hence, the matrix (4.1) is similar to the following one

$$\left[\begin{array}{ccc|ccc} m' & n' & s' & z_1^2 & \cdots & z_{n_2}^2 \\ \hline E' & H' & P' & 0 & 0 & 0 \\ 0 & 0 & T' & 0 & 0 & 0 \\ \hline z_1^1 & \cdots & z_{n_1}^1 & m & n & s \\ \hline 0 & 0 & 0 & E & H & P \\ 0 & 0 & 0 & 0 & 0 & T \end{array}\right], \tag{4.4}$$

where

$$\begin{bmatrix} z_1^2 & \cdots & z_{n_2}^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R^{-1}X_2P_2^{-1},$$

and

$$\begin{bmatrix} z_1^1 & \cdots & z_{n_1}^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Q^{-1}X_1P_2^{-1}.$$

Also, the nontrivial polynomials among $\alpha_1, \ldots, \alpha_{n_1}$ *and* $\beta_1, \ldots, \beta_{n_2}$*, coincide with the nontrivial invariant polynomials of the matrices* T' *and* T*, respectively.*

Proof.

Necessity:

The first condition is trivially satisfied. Furthermore, since the nontrivial invariant polynomials of the matrix T' coincide with the nontrivial polynomials among $\alpha_1, \ldots, \alpha_{n_1}$, and the nontrivial invariant polynomials of the matrix T coincide with the nontrivial polynomials among $\beta_1, \ldots, \beta_{n_2}$, from the form of the matrix (4.4), we have

$$\alpha_1 \cdots \alpha_{n_1}\beta_1 \cdots \beta_{n_2} | \phi(\lambda),$$

as wanted.

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Sufficiency:

Since $E \in \mathbb{F}^{(n_2-y-1)\times 1}$ and $E' \in \mathbb{F}^{(n_1-x-1)\times 1}$, the matrices $\left[\begin{array}{cc} E & H \end{array}\right]$ and $\left[\begin{array}{cc} E' & H' \end{array}\right]$ from (4.4) can be considered in the following feedback equivalent forms:

$$M = \left[\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{array}\right] \in \mathbb{F}^{(n_2-y-1)\times(n_2-y)}$$

and

$$M' = \left[\begin{array}{c|cccc} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{array}\right] \in \mathbb{F}^{(n_1-x-1)\times(n_1-x)},$$

respectively.

Thus, the matrix (4.4) is similar to the following one

$$\left[\begin{array}{cc|cc} w' & s' & \bar{x} & x' \\ \hline M' & K' & 0 & 0 \\ 0 & T' & 0 & 0 \\ \hline \bar{y} & y' & w & s \\ \hline 0 & 0 & M & K \\ 0 & 0 & 0 & T \end{array}\right], \tag{4.5}$$

for corresponding matrices $w \in \mathbb{F}^{1\times(n_2-y)}$, $w' \in \mathbb{F}^{1\times(n_1-x)}$, $K \in \mathbb{F}^{(n_2-y-1)\times y}$, $K' \in \mathbb{F}^{(n_1-x-1)\times x}$.

By applying the second condition, our problem reduces to proving the existence of row matrices

$$\bar{x} = \left[\begin{array}{ccc} x_1 & \cdots & x_{n_2-y} \end{array}\right] \in \mathbb{F}^{1\times(n_2-y)}$$

and

$$\bar{y} = \left[\begin{array}{ccc} y_1 & \cdots & y_{n_1-x} \end{array}\right] \in \mathbb{F}^{1\times(n_1-x)},$$

such that the product of the invariant polynomials of the matrix

$$\left[\begin{array}{c|c} w' & \bar{x} \\ \hline M' & 0 \\ \hline \bar{y} & w \\ \hline 0 & M \end{array}\right] := \left[\begin{array}{c|c} C & D \\ \hline E & F \end{array}\right] \in \mathbb{F}^{(n_1+n_2-x-y)\times(n_1+n_2-x-y)}, \tag{4.6}$$

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with $C \in \mathbb{F}^{(n_1-x)\times(n_1-x)}$, is equal to $\Delta = \phi(\lambda)/(\alpha_1 \dots \alpha_{n_1}\beta_1 \dots \beta_{n_2})$.

Let Δ_1 and Δ_2 be the determinants of the matrices $\lambda I - C$ and $\lambda I - F$, respectively.

Let

$$w' = \begin{bmatrix} a_1 & \cdots & a_{n_1-x} \end{bmatrix} \in \mathbb{F}^{1\times(n_1-x)} \text{ and } w = \begin{bmatrix} b_1 & \cdots & b_{n_2-y} \end{bmatrix} \in \mathbb{F}^{1\times(n_2-y)}.$$

Then we have

$$\Delta_1 = \lambda^{n_1-x} - a_1\lambda^{n_1-x-1} - \cdots - a_{n_1-x}$$

and

$$\Delta_2 = \lambda^{n_2-y} - b_1\lambda^{n_2-y-1} - \cdots - b_{n_2-y}.$$

From condition (i), the polynomial $\Delta_1\Delta_2 - \Delta$ has degree at most $n_1+n_2-x-y-2$. Since $\mathbb{F}$ is an algebraically closed field, there exist polynomials

$$x(\lambda) = -x_1\lambda^{n_2-y-1} - \cdots - x_{n_2-y-1}\lambda - x_{n_2-y}$$

and

$$y(\lambda) = -y_1\lambda^{n_1-x-1} - \cdots - y_{n_1-x-1}\lambda - y_{n_1-x},$$

of degrees at most $n_2 - y - 1$ and $n_1 - x - 1$, respectively, such that

$$x(\lambda)y(\lambda) = \Delta_1\Delta_2 - \Delta. \tag{4.7}$$

Now, define

$$\bar{x} := \begin{bmatrix} x_1 & \cdots & x_{n_2-y} \end{bmatrix} \in \mathbb{F}^{1\times(n_2-y)}$$

and

$$\bar{y} := \begin{bmatrix} y_1 & \cdots & y_{n_1-x} \end{bmatrix} \in \mathbb{F}^{1\times(n_1-x)}.$$

Then the matrix

$$\left[\begin{array}{c|c} \lambda I - C & -D \\ \hline -E & \lambda I - F \end{array}\right]$$

is equivalent to the following one

$$\left[\begin{array}{c|cc} I & \multicolumn{2}{c}{0} \\ \hline 0 & \Delta_1 & x(\lambda) \\ & y(\lambda) & \Delta_2 \end{array}\right]. \tag{4.8}$$

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A publication of the International Linear Algebra Society
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Obviously the determinant of the matrix (4.8) is equal to

$$\Delta_1\Delta_2 - x(\lambda)y(\lambda) = \Delta,$$

as wanted.□

Acknowledgment. The author is supported by *Fundação para a Ciência e a Tecnologia* / (FCT), grant no. SFRH/BPD/26607/2006. When this work was started, the author was supported by grant SFRH/BD/6726/2001.

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