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FURTHER RESULTS ON THE CRAIG-SAKAMOTO EQUATION*

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Abstract. In this paper, necessary and sufficient conditions are stated for the Craig-Sakamoto equation $det(I - sA - tB) = det(I - sA)\, det(I - tB)$ to hold for all scalars $s,\, t \in \mathbb{C}$. Moreover, spectral properties for matrices A and B that satisfy this equation are investigated.

Key words. Determinant, Characteristic polynomial, Craig-Sakamoto equation, Bilinear form.

AMS subject classifications. 15A15, 15A18, 15A63.

1. Introduction. Let $M_n(\mathbb{C})$ be the set of $n \times n$ matrices with elements in $\mathbb{C}$. For A and $B \in M_n(\mathbb{C})$, the equation

$$det(I - sA - tB) = det(I - sA)\, det(I - tB) \tag{1.1}$$

for all scalars $s,\, t \in \mathbb{C}$ is known as the *Craig-Sakamoto* (CS) *equation*. Matrices A and B satisfying (1.1) are said to have the *CS property*. The CS equation is encountered in multivariate statisticts [1] and has drawn the interest of several researchers. Specifically, O. Taussky proved in [6] that the CS equation is equivalent to $AB = O$ when $A,\, B$ are normal matrices. Several proofs of this result in [1] are known, most recently by Olkin in [5] and by Li in [2]. Moreover, Matsuura in [4] refined Olkin's method using another type of determinantal result. The present author, together with M. Tsatsomeros and P. Psarrakos investigated in [3] the CS equation for general matrices and in relation to the eigenspaces of A, B and $sA + tB$. Being more specific, if $\sigma(X)$ denotes the spectrum for a matrix X, $m_X(\lambda)$ the algebraic multiplicity of $\lambda \in \sigma(X)$, and $E_X(\lambda) = Nul\left((X - \lambda I)^{\mu}\right)$, where $\mu = ind_{\lambda}(X)$ is the size of the largest Jordan block associated with λ in the Jordan canonical form of X, the following three propositions were shown in [3]:

PROPOSITION 1.1. *For $n \times n$ matrices A and B, the following statements are equivalent:*

I. *The CS equation holds.*

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II. *For every* $s, t \in \mathbb{C}$, $\sigma(sA \oplus tB) = \sigma\left((sA + tB) \oplus O_n\right)$, *where* O_n *denotes the* $n \times n$ *zero matrix.*

III. $\sigma(sA + tB) = \{\, s\mu_i + t\nu_i \,:\, \mu_i \in \sigma(A), \ \ \nu_i \in \sigma(B)\,\}$, *where the pairing of eigenvalues requires either* $\mu_i = 0$ *or* $\nu_i = 0$.

PROPOSITION 1.2. *Let* $n{\times}n$ *matrices* A, B *satisfy the Craig-Sakamoto equation. Then,*

I. $m_A(0) + m_B(0) \geq n$.

II. *If* A *is nonsingular, then* B *must be nilpotent.*

III. *If* $\lambda = 0$ *is a semisimple eigenvalue of* A *and* B, *then* $rank(A){+}rank(B) \leq n$.

PROPOSITION 1.3. *Let* $\lambda = 0$ *be a semisimple eigenvalue of* $n \times n$ *matrices* A *and* B *such that* $BE_A(0) \subset E_A(0)$. *Then the following are equivalent.*

I. *The CS equation holds.*

II. $\mathbb{C}^n = E_A(0) + E_B(0)$.

III. $AB = O$.

The remaining results in [3] are based on the basic assumption that $\lambda = 0$ is a semisimple eigenvalue of A and B. Relaxing this restriction, we shall attempt here to investigate the CS equation by focusing on the factorization of the two variable polynomial $f(s, t) = det(I - sA - tB)$.

In section 2, considering the determinants in (1.1), new necessary and sufficient conditions for CS to hold are stated. The first criterion refers to the coefficients of the polynomials in (1.1). The second criterion refers to certain determinants defined via the rows of A and B. In section 3, the main result is related to the algebraic multiplicity of the eigenvalue $\lambda = 0$ of A and B and sufficient conditions such that $m_A(0) + m_B(0) = n$ are presented

2. Criteria for CS property. In this section we consider the polynomial in two variables

$$f(s,t) = det(I - sA - tB) = \sum_{\substack{p,\, q\, =\, 0 \\ p+q \leq n}}^{n} m_{pq}\, s^p\, t^q. \tag{2.1}$$

By denoting $x = \begin{bmatrix} 1 & s & s^2 & \cdots & s^n \end{bmatrix}^T$ and $y = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix}^T$, (2.1) can be written as

$$f(s,t) = x^T M y,$$

where $M = [m_{pq}]_{p,q=0}^{n}$ is an $(n+1) \times (n+1)$ matrix, with $m_{00} = 1$.

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PROPOSITION 2.1. *Let* $A, B \in M_n(\mathbb{C})$. *The CS equation holds for the pair of matrices* A *and* B *if and only if* $rank M = 1$.

Proof. Let A and B have the CS property. Then the equation (1.1) can be formulated as

$$x^T M y = x^T a\, b^T y, \tag{2.2}$$

where

$$a = \begin{bmatrix} 1 & a_{n-1} & \cdots & a_0 \end{bmatrix}^T, \qquad b = \begin{bmatrix} 1 & b_{n-1} & \cdots & b_0 \end{bmatrix}^T$$

and a_i, b_i are the coefficients of the characteristic polynomials

$$det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0, \quad det(\lambda I - B) = \lambda^n + b_{n-1}\lambda^{n-1} + \ldots + b_0.$$

Hence, by (2.2), for all distinct $s_1, s_2, \ldots, s_{n+1}$ and all distinct $t_1, t_2, \ldots, t_{n+1}$ we have that

$$V^T \left(M - a\, b^T\right) W = O, \tag{2.3}$$

where

$$V = \begin{bmatrix} 1 & \cdots & 1 \\ s_1 & \cdots & s_{n+1} \\ \vdots & & \vdots \\ s_1^n & \cdots & s_{n+1}^n \end{bmatrix}, \qquad W = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_{n+1} \\ \vdots & & \vdots \\ t_1^n & \cdots & t_{n+1}^n \end{bmatrix}.$$

By (2.3), due to the invertibility of V and W, we have that $M = a\, b^T$, i.e., $rank M = 1$.

Conversely, if $rank M = 1$, then $M = k\, \ell^T$, where the vectors $k, \ell \in \mathbb{C}^{n+1}$. Therefore,

$$f(s,t) = x^T M y = x^T k\, \ell^T y = k(s)\ell(t),$$

where $k(s)$ and $\ell(t)$ are polynomials. Since, $f(0,0) = 1 = k(0)\ell(0)$, and

$$\begin{aligned} det(I - sA) &= f(s,0) = k(s)\ell(0), \\ det(I - tB) &= f(0,t) = k(0)\ell(t), \end{aligned}$$

we have

$$f(s,t) = k(s)\ell(0)k(0)\ell(t) = det(I - sA)\, det(I - tB). \qquad \square$$

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Example 2.2. Consider the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\gamma & 1 \\ 0 & 0 & 1-\gamma \end{bmatrix}, \quad B = \begin{bmatrix} 0 & \gamma & 0 \\ 1/\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have

$$\begin{aligned} f(s,t) = det(I - sA - tB) &= 1 + 2(\gamma-1)s + (\gamma-1)^2 s^2 - t^2 + (1-\gamma)t^2 s \\ &= x^T \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2(\gamma-1) & 0 & 1-\gamma & 0 \\ (\gamma-1)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} y \end{aligned}$$

and

$$det(I - sA) = (1 + (\gamma-1)s)^2, \quad det(I - tB) = 1 - t^2.$$

By Proposition 2.1 we recognize that A, B have the CS property if and only if $\gamma = 1$.

In the following we denote by $C\begin{pmatrix} a_{i_1,\dots,i_p} \\ b_{j_1,\dots,j_q} \end{pmatrix}$ the determinant of order $p+q$ ($\leq n$) defined by the $i_1,\dots,i_p$ rows of A and $j_1,\dots,j_q$ rows of B; all indices are assume to be in increasing order. For example, when $i_1 < i_2 < j_1 < i_3 < \cdots < j_q < \cdots < i_p$, then

$$C\begin{pmatrix} a_{i_1,\dots,i_p} \\ b_{j_1,\dots,j_q} \end{pmatrix} = det \begin{bmatrix} a_{i_1i_1} & a_{i_1i_2} & a_{i_1j_1} & a_{i_1i_3} & \cdots & a_{i_1j_q} & \cdots & a_{i_1i_p} \\ a_{i_2i_1} & a_{i_2i_2} & a_{i_2j_1} & a_{i_2i_3} & \cdots & a_{i_2j_q} & \cdots & a_{i_2i_p} \\ b_{j_1i_1} & b_{j_1i_2} & b_{j_1j_1} & b_{j_1i_3} & \cdots & b_{j_1j_q} & \cdots & b_{j_1i_p} \\ a_{i_3i_1} & a_{i_3i_2} & a_{i_3j_1} & a_{i_3i_3} & & & & \\ \vdots & \vdots & & & \ddots & & & \vdots \\ b_{j_qi_1} & b_{j_qi_2} & & & & b_{j_qj_q} & & \vdots \\ \vdots & \vdots & & & & & \ddots & \\ a_{i_pi_1} & a_{i_pi_2} & & \cdots & & & & a_{i_pi_p} \end{bmatrix}.$$

Recall that the coefficient of $\lambda^{n-\rho}$ in the characteristic polynomial $det(\lambda I - sA - tB)$ is equal to $(-1)^\rho$ times the sum of all principal minors of order ρ of $sA + tB$. Using the multilinearity of the determinant, we can thus deduce that for $\rho = p + q$, this coefficient is

$$(-1)^{p+q} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_{p+q} \leq n} det\left[s\, a_{i_\ell i_h} + t\, b_{i_\ell i_h}\right]_{\ell,h=1}^{p+q} = \tag{2.4}$$

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$$= (-1)^{p+q} \sum_{1\le i_1\le i_2\le\cdots\le i_{p+q}\le n} s^{p+q}\, det\,[\,a_{i_\ell i_h}]_{\ell,h=1}^{p+q} + s^{p+q-1}\, t \sum_{k=1}^{p+q} C\begin{pmatrix} a_{i_1,\dots,\,i_{k-1},i_{k+1},\dots,i_{p+q}} \\ b_{i_k} \end{pmatrix} + s^{p+q-2}\, t^2 \sum_{\substack{k,\tau=1\\ k\neq\tau}}^{p+q} C\begin{pmatrix} a_{i_1,\dots,\,i_{k-1},i_{k+1},\dots,i_{\tau-1},i_{\tau+1},\dots,i_{p+q}} \\ b_{i_k,i_\tau} \end{pmatrix} \cdots + t^{p+q} det\,[\,b_{i_\ell i_h}]_{\ell,h=1}^{p+q}).$$

Hence, we have established that for $\lambda = 1,$ the coefficient m_{pq} of the monomial $s^p t^q$ in (2.1) is given by

$$m_{pq} = (-1)^{p+q} \sum_{1\le i_1\le i_2\le\ldots\le i_{p+q}\le n}\; \sum_{k_1,\dots,k_q=1}^{p+q} C\begin{pmatrix} a_{i_1,\dots,i_{p+q}} \\ b_{i_{k_1},\dots,\,i_{k_q}} \end{pmatrix}$$

$$= (-1)^{p+q} \sum_{1\le\, i_1,\dots,\, i_p,\, j_1,\dots,\, j_q\,\le\, n} C\begin{pmatrix} a_{i_1,\dots,\, i_p} \\ b_{j_1,\dots,\, j_q} \end{pmatrix}, \qquad m_{00} = 1. \tag{2.5}$$

Note that in (2.5) the summands are constructed for all ordered subsets of $p+q$ indices of i's and j's from $\{1,\cdots,n\}$, corresponding to p rows of A and q rows of $B,$ respectively.

For example, for $n\times n$ matrices A and B the coefficients of t, st, s^2 and s^2t are, respectively, equal to

$$m_{01} = -\sum_{1\le j\le n} C(b_j) = -\,(b_{11}+b_{22}+\cdots+b_{nn}\,) = -trB$$

$$m_{11} = \sum_{1\le i<j\le\, n} C\begin{pmatrix} a_i \\ b_j \end{pmatrix} = \sum_{\substack{i,j=1\\ i<j}}^{n} \left(det\begin{bmatrix} a_{ii} & a_{ij} \\ b_{ji} & b_{jj} \end{bmatrix} + det\begin{bmatrix} b_{ii} & b_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}\right)$$

$$m_{20} = \sum_{1\le i,j\le n} C(a_{ij}) = \sum_{\substack{i,j=1\\ i<j}}^{n} det\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

and

$$m_{21} = -\sum_{1\le i\le\, j\le\, k\le\, n} C\begin{pmatrix} a_{i,j} \\ b_k \end{pmatrix} =$$

$$= -\sum_{1\le i\le\, j\le\, k\le\, n} \left(det\begin{bmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ b_{ki} & b_{kj} & b_{kk} \end{bmatrix} + det\begin{bmatrix} a_{ii} & a_{ij} & a_{ik} \\ b_{ji} & b_{jj} & b_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{bmatrix} + det\begin{bmatrix} b_{ii} & b_{ij} & b_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{bmatrix}\right).$$

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Hence, the matrix M defined in (2.1) looks like

$$\left[\begin{array}{cccc}
1 & -\sum C(b_j) & \sum C(b_{j_1,j_2}) & \cdots \\
-\sum C(a_i) & \sum C\begin{pmatrix} a_i \\ b_j \end{pmatrix} & -\sum C\begin{pmatrix} a_i \\ b_{j_1,j_2} \end{pmatrix} & \cdots \\
\sum C\left(a_{i_1,i_2}\right) & -\sum C\begin{pmatrix} a_{i_1,i_2} \\ b_j \end{pmatrix} & \vdots & \\
\vdots & \vdots & & \\
\vdots & (-1)^n \sum C\begin{pmatrix} a_{i_1,\ldots,i_{n-1}} \\ b_j \end{pmatrix} & 0 & \cdots \\
(-1)^n detA & 0 & 0 & \cdots
\end{array}\right.$$

$$\left.\begin{array}{ccc}
\cdots & (-1)^{n-1}\sum C(b_{j_1,\ldots,j_{n-1}}) & (-1)^n detB \\
\cdots & (-1)^n \sum C\begin{pmatrix} a_i \\ b_{j_1,\ldots,j_{n-1}} \end{pmatrix} & 0 \\
\vdots & \vdots & \vdots \\
 & & \\
\cdots & \cdots & 0 \\
\cdots & \cdots & 0
\end{array}\right].$$

The indicated zero entries in M correspond to the coefficients of monomials of $f(s,t)$ with degree $\geq n+1$. These terms are not present in $det(I-sA-tB)$, since by (2.5) the order of the corresponding determinant is greater than n. The above formulation of M provides a way of finding $rankM$ without computing $det(I-sA-tB)$ explicitly. Therefore, using the criterion in Proposition 2.1, we induce the following necessary and sufficient conditions.

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Proposition 2.3. *The $n \times n$ matrices A and B have the CS property if and only if*

$$\sum C(a_{i_1,\ldots,i_p}) \sum C(b_{j_1,\ldots,j_q}) = \sum C\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix} \quad \textit{when} \quad p+q \le n,$$

(2.6)

$$\textit{and} \quad \sum C(a_{i_1,\ldots,i_p}) \sum C(b_{j_1,\ldots,j_q}) = 0 \qquad \textit{when} \quad p+q > n.$$

Example 2.4. Let A in (1.1) be a nilpotent matrix. Then,

$$\sum C(a_i) = \sum C(a_{i,j}) = \cdots = detA = 0$$

and by Proposition 2.3,

$$\sum C\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix} = 0 \quad ; \quad p,q = 1,2,\ldots,n-1.$$

In this case, $M = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & b_{n-1} & \cdots & b_1 & b_0 \end{bmatrix}$.

We conclude this section with some remarks:

(1) Note that Proposition 2.3 gives an answer to the following problem:
For a given $n \times n$ matrix A identify the set

$$CS(A) = \{B : A \textit{ and } B \textit{ have the CS property}\};$$

see [3, Theorem 2.1]) and the discussion therein of matrix pairs with Property L.

(2) If $a(s) = det(I - sA)$ and $b(t) = det(I - tB)$, the higher order derivatives of these polynomials at the origin are

$$\frac{1}{p!}\, a^{(p)}(0) = \sum C(a_{i_1,\ldots,i_p})\,, \qquad \frac{1}{q!}\, b^{(q)}(0) = \sum C(b_{j_1,\ldots,j_q}),$$

and

$$\frac{1}{p!q!}\,\frac{\partial^{\,p+q} f(0,0)}{\partial s^p\,\partial t^q} = \sum C\begin{pmatrix} a_{i_1,\ldots,i_p} \\ b_{j_1,\ldots,j_q} \end{pmatrix}.$$

Thus, considering Taylor series expansions for the polynomials in (1.1) and since

$$a^{(p)}(0)\, b^{(q)}(0) = \frac{\partial^{p+q} f(0,0)}{\partial s^p\,\partial t^q}, \quad \text{for} \quad p+q \le n,$$

$$a^{(p)}(0)\, b^{(q)}(0) = 0, \quad \text{for} \quad p+q > n,$$

we observe that the equations in (2.6) arise once more.

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3. Spectral results. In this section we will first obtain a result on the CS property using basic polynomial theory. Recall that, by Proposition 1.2 **II**, the CS equation holds only when at least one of the matrices A or B is singular.

Definition. The pair of matrices $A, B \in M_n(\mathbb{C})$ is called *r-complementary in rows* if the matrix $N(i_1, i_2, \cdots, i_r) \in M_n(\mathbb{C})$ obtained from A by substituting r rows $a_{i_1}, a_{i_2}, \ldots, a_{i_r}$ of A by the corresponding rows $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$ of B, is nonsingular.

Note that when A, B are r-complementary and $ImA \cap ImB \neq \emptyset$, then $n - r \leq rank(B)$.

To illustrate the above definition, the pair of matrices

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is 1-complementary in rows but not 2-complementary in rows, since $detN(1) = det\begin{bmatrix} b_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$ and $detN(1,2) = det\begin{bmatrix} b_1 \\ b_2 \\ a_3 \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.$
The pair $\hat{A}$, B, where

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is neither 1 nor 2−complementary in rows, on behalf of the fact that $rank\begin{bmatrix} \hat{A} \\ B \end{bmatrix} = 3.$
Clearly, the 3rd row of A or $\hat{A}$ can not be substituted.

Proposition 3.1. *Let the pair of $n \times n$ singular matrices A, B be $[n - m_B(0)]$-complementary in rows and suppose that the number*

$$\theta = \sum_{i_1,\ldots,i_{n-m_B(0)}} detN(i_1, i_2, \ldots, i_{n-m_B(0)}) \tag{3.1}$$

is nonzero; the sum is taken over all possible combinations $i_1, \ldots, i_{n-m_B(0)}$ of $n - m_B(0)$ of the indices $1, 2, \ldots, n$. If A and B satisfy the CS equation, then

$$m_A(0) + m_B(0) = n.$$

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Proof. Let $rankB = b\,(< n)$. Then $\lambda = 0$ is an eigenvalue of B with algebraic multiplicity $m_B(0) = m \geq n - b$. Denote

$$\beta(t) \doteq det(tI - B) = t^n + \beta_1 t^{n-1} + \cdots + \beta_{n-m} t^m,$$

where $\beta_k = (-1)^k \sum B_k$, the summation being over all $k \times k$ principal minors B_k of B. Then

$$\begin{aligned} det(tB - I) &= (-1)^n\, t^n\, det(t^{-1} I - B) \\ &= (-1)^n \left(1 + \beta_1\, t + \cdots + \beta_{n-m}\, t^{n-m}\right). \end{aligned}$$

The polynomial $\widetilde{\beta}(t) = 1 + \beta_1\, t + \cdots + \beta_{n-m}\, t^{n-m}$ has precisely $n - m$ nonzero roots, say $t_1, t_2, \cdots, t_{n-m}$, since $\widetilde{\beta}(0) = 1 \neq 0$. Moreover, by multilinearity of determinants as functions of the rows, we have

$$\begin{aligned} & det(sA + tB - I) = \\ &\quad = det \begin{bmatrix} sa_{11} & sa_{12} & \dots & sa_{1n} \\ sa_{21} & sa_{22} & \dots & sa_{2n} \\ \vdots & & & \vdots \\ sa_{n1} & sa_{n2} & \dots & sa_{nn} \end{bmatrix} \\ & + det \begin{bmatrix} tb_{11} - 1 & tb_{12} & \dots & tb_{1n} \\ sa_{21} & sa_{22} & \dots & sa_{2n} \\ \vdots & & & \vdots \\ sa_{n1} & sa_{n2} & \dots & sa_{nn} \end{bmatrix} + \cdots + det \begin{bmatrix} sa_{11} & \dots & sa_{1n} \\ \vdots & & \vdots \\ sa_{n-1,1} & \dots & sa_{n-1,n} \\ tb_{n1} & \dots & tb_{n,n-1} - 1 \end{bmatrix} \\ & + det \begin{bmatrix} tb_{11} - 1 & tb_{12} & \dots & tb_{1n} \\ tb_{21} & tb_{22} - 1 & \dots & tb_{2n} \\ sa_{31} & \dots & & sa_{3n} \\ \vdots & & & \vdots \\ sa_{n1} & sa_{n2} & \dots & sa_{nn} \end{bmatrix} + det \begin{bmatrix} tb_{11} - 1 & tb_{12} & \dots & & tb_{1n} \\ sa_{21} & sa_{22} & \dots & & sa_{2n} \\ tb_{31} & tb_{32} & tb_{33} - 1 & \dots & tb_{3n} \\ sa_{41} & \dots & & & sa_{4n} \\ \vdots & & & & \vdots \\ sa_{n1} & \dots & & & sa_{nn} \end{bmatrix} \\ &\quad + \cdots + det \begin{bmatrix} tb_{11} - 1 & tb_{12} & \dots & tb_{1n} \\ tb_{21} & tb_{22} - 1 & \dots & tb_{2n} \\ \vdots & & & \vdots \\ tb_{n1} & tb_{n2} & \dots & tb_{nn} - 1 \end{bmatrix} \end{aligned}$$

$$= (detA)s^n + f_1(t)s^{n-1} + \cdots + f_{n-1}(t)s + det(tB - I), \tag{3.2}$$

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where

$$f_1(t)=\sum_i det\hat{A}_i,\quad \text{with}\quad \hat{A}_i=\begin{bmatrix} a_{11} & & \cdots & & a_{1n}\\ \vdots & & & & \vdots\\ tb_{i1} & \cdots & tb_{ii}-1 & \cdots & tb_{in}\\ \vdots & & & & \vdots\\ a_{n1} & & \cdots & & a_{nn}\end{bmatrix}.$$

Note that $\widehat{A}_i$ arises from A when the $i-$row of A is substituted by the $i-$row of $tB-I$. Also, in (3.2),

$$f_2(t)=\sum_{i,j} det\hat{A}_{ij}\quad \text{where}\quad \hat{A}_{ij}=\begin{bmatrix} a_{11} & & \cdots & & & a_{1n}\\ \vdots & & & & & \vdots\\ tb_{i1} & \cdots & tb_{ii}-1 & \cdots & & tb_{in}\\ \vdots & & \ddots & & & \vdots\\ tb_{j1} & \cdots & & tb_{jj}-1 & \cdots & tb_{jn}\\ \vdots & & & & & \vdots\\ a_{n1} & & \cdots & & & a_{nn}\end{bmatrix};$$

i.e., $\hat{A}_{ij}$ is obtained from A, substituting rows i and j by the corresponding rows of $tB-I$. The summation in $f_2(t)$ is taken over all pairs of distinct indices i,j in $\{1,2,\ldots,n\}$. Hence, by (3.2), the CS equation

$$(-1)^n\, det(sA+tB-I)=det(sA-I)\, det(tB-I),\qquad \forall\ s,\, t$$

and for $t=t_1,\, t_2,\, \cdots, t_{n-m}$, we obtain

$$(detA)s^n+f_1(t_i)s^{n-1}+\cdots+f_{n-1}(t_i)s=0,\qquad \forall\ \ s,\ \ i=1,2,\ldots,n-m.$$

Consequently,

$$\text{(3.3)}\quad detA=0,\ \ f_1(t_i)=f_2(t_i)=\cdots=f_{n-1}(t_i)=0,\ \text{ for all }\ i=1,2,\ldots,n-m\,.$$

Due to the pair A, B being $[n-m_B(0)]$-complementary and the leading coefficient of polynomial $f_{n-m}(t)$ being the nonzero θ, we have that $deg(f_{n-m}(t))=n-m$. Moreover $deg(f_k(t))\le n-m$, for $k=1,2,\ldots,n-m-1$, and due to (3.3) we have

$$f_1(t)=f_2(t)=\cdots=f_{n-m-1}(t)=0,\quad \forall\ \ t.$$

Recalling that A_k denotes a typical $k\times k$ principal submatrix of A, since $f_1(t)=0$, we clearly have that

$$f_1(0)=\sum detA_{n-1}=0.$$

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Similarly, since $f_2(t) = 0, \cdots, f_{n-m-1}(t) = 0$ for all t, it follows, respectively, that

$$f_2(0) = \sum detA_{n-2} = 0, \; \cdots, \; f_{n-m-1}(0) = \sum detA_{m+1} = 0.$$

Consequently,

$$\begin{aligned} \delta_A(\lambda) &= det(\lambda I - A) = \lambda^n - f_{n-1}(0)\lambda^{n-1} + f_{n-2}(0)\lambda^{n-2} + \cdots + (-1)^n detA \\ &= \lambda^n - f_{n-1}(0)\lambda^{n-1} + f_{n-2}(0)\lambda^{n-2} + \cdots + (-1)^m f_{n-m}(0)\lambda^{n-m}. \end{aligned} \tag{3.4}$$

In (3.4), $f_{n-m}(0) \neq 0$, since $(-1)^{n-m}c_m = \theta t_1 t_2 \cdots t_{n-m}$. Thus, $\lambda = 0$ is an eigenvalue of A with algebraic multiplicity $n - m_B(0)$, whereby we conclude

$$m_A(0) + m_B(0) = n. \quad \square$$

Corollary 3.2. *Let the pair of $n \times n$ singular and $[n - m_B(0)]$-complementary matrices A, B have the CS property. If the number θ in (3.1) is nonzero, the following hold:*

I. *If $\lambda = 0$ is a semisimple eigenvalue of A and B, then $rankA + rankB = n$.*
II. *If $\lambda = 0$ is a semisimple eigenvalue of A, then $rankA = m_B(0)$.*

Proof. **I.** Clearly Proposition 3.1 holds and since

$$n - rankA \leq m_A(0) = n - m_B(0),$$

we have $rankA + rankB \geq m_B(0) + r \geq n$. Hence, by Proposition 1.2 **III**, we obtain

$$rankA + rankB = n.$$

II. By the assumption and Proposition 3.1, $rankA = n - m_A(0) = m_B(0)$. $\square$

To close this section, we present a property of the generalized eigenspaces of the nonzero eigenvalues of A and B.

Proposition 3.3. *Let zero be a semisimple eigenvalue of the $n \times n$ matrices A and B and assume that $E_A(0) + E_B(0) = \mathbb{C}^n$. If for some $\lambda \in \sigma(A)\backslash\{0\}$ (or $\mu \in \sigma(B)\backslash\{0\}$) we have that $E_A(\lambda) \subseteq E_B(0)$ (resp., $E_B(\mu) \subseteq E_A(0)$), then*

I. *A, B have the CS property.*
II. *$E_A(\lambda) = E_{I-sA-tB}(1 - s\lambda)$ and $E_B(\mu) = E_{I-sA-tB}(1 - t\mu)$.*

Proof. **I.** Since $E_A(\lambda) \subseteq E_B(0)$, for every $w = w_1 + w_2 \in \mathbb{C}^n$, where w_1 belongs to the direct sum $\bigoplus_\lambda E_A(\lambda)$, $w_2 \in E_A(0)$, we have $BAw = BA(w_1 + w_2) = BAw_1 = 0$. Thus, $BA = 0$ and consequently $AE_B(0) \subseteq E_B(0)$. The assumption $E_A(0) + E_B(0) = \mathbb{C}^n$, as well as Proposition 1.3, lead to the statement **I**.

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II. Let $\lambda \in \sigma(A)\backslash\{0\}$ and $x_k \in E_A(\lambda)$ be a generalized eigenvector of A of order k, i.e., $(A-\lambda I)^k x_k = 0$. By assumption, $x_k \in E_B(0)$, and thus

$$\begin{aligned}(I - sA - tB)x_k = (I - sA)x_k &= x_k - s(\,\lambda x_k + x_{k-1}\,) \\ &= (1 - s\lambda)x_k - s x_{k-1}.\end{aligned}$$

Hence, for the whole Jordan chain $x_1, \ldots, x_k, \ldots, x_\tau$ of λ, we have

$$(I - sA - tB)\begin{bmatrix} x_1 & \ldots & x_\tau \end{bmatrix} =$$

$$= \begin{bmatrix} x_1 & \ldots & x_\tau \end{bmatrix} \begin{bmatrix} 1-s\lambda & -s & & & \\ 0 & 1-s\lambda & -s & O & \\ \vdots & & \ddots & \ddots & \\ & & & 1-s\lambda & -s \\ 0 & 0 & & & 1-s\lambda \end{bmatrix}_{\tau\times\tau}. \tag{3.5}$$

Moreover, by clause **III** in Proposition 1.1, $s\lambda$ and $t\mu \in \sigma(sA+tB)$. The equivalence of the CS equation and $\mathbb{C}^n = E_A(0) + E_B(0)$ in Proposition 1.3 and the assumption $E_A(\lambda) \subseteq E_B(0)$, lead to $E_B(\mu) \subseteq E_A(0)$. Hence, if $y_\ell \in E_B(\mu)$ is a generalized eigenvector of order ℓ, then $y_\ell \in E_A(0)$ and

$$\begin{aligned}(I - sA - tB)y_\ell = (I - tB)y_\ell &= y_\ell - t(\,\mu y_\ell + y_{\ell-1}\,) \\ &= (1 - t\mu)y_\ell - t y_{\ell-1}.\end{aligned}$$

Thus, for the whole Jordan chain $y_1, \ldots, y_\ell, \ldots, y_\sigma$ of μ, we obtain

$$(I - sA - tB)\begin{bmatrix} y_1 & \ldots & y_\sigma \end{bmatrix} =$$

$$= \begin{bmatrix} y_1 & \ldots & y_\sigma \end{bmatrix} \begin{bmatrix} 1-t\mu & -t & & & \\ 0 & 1-t\mu & -t & O & \\ \vdots & & \ddots & \ddots & \\ & & & 1-t\mu & -t \\ 0 & 0 & & & 1-t\mu \end{bmatrix}_{\sigma\times\sigma}. \tag{3.6}$$

The equaions in **II** for any s, t are now implied by (3.5) and (3.6), respectively. ☐

Remark 3.4. For $z \in E_A(0) \bigcap E_B(0)$, we have $(I - sA - tB)z = z$, $\forall\, s, t$. Therefore, by the above proposition, the Jordan canonical forms of $I - sA - tB$ and of the matrix

$$F = I_\nu \bigoplus_{\lambda_A \neq 0} \begin{bmatrix} 1-s\lambda_A & -s & & O \\ & 1-s\lambda_A & \ddots & \\ & & \ddots & -s \\ O & & & 1-s\lambda_A \end{bmatrix}$$

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$$\bigoplus_{\mu_B\neq 0}\begin{bmatrix} 1-t\mu_B & -t & & O \\ & 1-t\mu_B & \ddots & \\ & & \ddots & -t \\ O & & & 1-t\mu_B \end{bmatrix},$$

are similar.

Note that the order ν of the submatrix I_ν of F coincides with the number of linear independent eigenvectors corresponding to the eigenvalue $\lambda = 1$ of $I - sA - tB$. These eigenvectors belong to $E_B(0)\backslash E_A(\lambda)$, $E_A(0)\backslash E_B(\mu)$, and $E_A(0)\bigcap E_B(0)$, and ν is equal to

$$\nu = n - (rankA + rankB) = n - \left(dim \bigcup_{\lambda\neq 0} E_A(\lambda) + dim \bigcup_{\mu\neq 0} E_B(\mu) \right).$$

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