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A NOTE ON THE LARGEST EIGENVALUE OF NON-REGULAR GRAPHS*

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Abstract. The spectral radius of connected non-regular graphs is considered. Let λ_1 be the largest eigenvalue of the adjacency matrix of a graph G on n vertices with maximum degree Δ. By studying the λ_1-extremal graphs, it is proved that if G is non-regular and connected, then $\Delta - \lambda_1 > \dfrac{\Delta+1}{n(3n+\Delta-8)}$. This improves the recent results by B.L. Liu et al.

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Key words. Spectral radius, Non-regular graph, λ_1-extremal graph, Maximum degree.

1. Introduction. Let $G = (V, E)$ be a simple graph on vertex set V and edge set E, where $|V| = n$. The eigenvalues of the adjacency matrix of G are called the eigenvalues of G. The largest eigenvalue of G, denoted by $\lambda_1(G)$, is called the spectral radius of G. Let D denote the diameter of G. We suppose throughout the paper that G is a simple graph. For any vertex u, let $\Gamma(u)$ be the set of all neighbors of u and $d(u) = |\Gamma(u)|$ be the degree of u. A nonincreasing sequence $\pi = (d_1, d_2, ..., d_n)$ of non-negative integers is called (*connected*) *graphic* if there exits a (connected) simple graph on n vertices, for which $d_1, d_2, ..., d_n$ are the degrees of its vertices. Let Δ and δ be the maximum and minimum degree of vertices of G, respectively. A graph is called regular if $d(u) = \Delta$ for any $u \in V$. It is easy to see that the spectral radius of a regular graph is Δ with $(1, 1, ..., 1)^T$ as a corresponding eigenvector. We will use $G - e$ $(G + e)$ to denote the graph obtained from G by deleting (adding) the edge e. For other notations in graph theory, we follow from [2].

Stevanović [8] first found a lower bound of $\Delta - \lambda_1$ for the connected non-regular graphs. Then the results from [8] were improved in [9, 4, 7, 3]. In [4, 7], the authors showed that

$$\Delta - \lambda_1 \geq \frac{1}{n(D+1)} \qquad ([4,\ 7]) \tag{1.1}$$

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and

$$D \leq \frac{3n+\Delta-5}{\Delta+1} \qquad ([7]). \tag{1.2}$$

B.L. Liu et al. obtained

$$\Delta - \lambda_1 \geq \frac{\Delta+1}{n(3n+2\Delta-4)} \qquad ([7]). \tag{1.3}$$

Recently, S.M. Cioabă [3] improved (1.1) as follows:

$$\Delta - \lambda_1 > \frac{1}{nD} \qquad ([3]). \tag{1.4}$$

Thus combining (1.2) and (1.4), the inequality (1.3) can be improved as follows:

$$\Delta - \lambda_1 > \frac{\Delta+1}{n(3n+\Delta-5)}. \tag{1.5}$$

In this note we improve the inequality (1.2) on λ_1-extremal graphs. Furthermore, we obtain the following inequality which improves (1.5).

$$\Delta - \lambda_1 > \frac{\Delta+1}{n(3n+\Delta-8)}.$$

2. Preparation. Firstly, we state a well-known result which is just Frobenius's theorem applied to graphs.

LEMMA 2.1. *Let G be a connected graph and $\lambda_1(G)$ be its spectral radius. Then $\lambda_1(G+uv) > \lambda_1(G)$ for any $uv \notin E$.*

DEFINITION 2.2. [7] Let G be a connected non-regular graph. Then the graph G is called λ_1-*extremal* if $\lambda_1(G) \geq \lambda_1(G')$ for any other connected non-regular graph G' with the same number of vertices and maximum degree as G.

THEOREM 2.3. *Let G be a λ_1-extremal graph on n vertices with maximum degree Δ. Define*

$$V_{<\Delta} = \{u : u \in V \text{ and } d(u) < \Delta\}.$$

Then G must have one of the following properties:
(1) $|V_{<\Delta}| \geq 2$ *and* $V_{<\Delta}$ *induces a complete graph.*
(2) $|V_{<\Delta}| = 1$.
(3) $V_{<\Delta} = \{u, v\}$, $uv \notin E(G)$ *and* $d(u) = d(v) = \Delta - 1$.

Proof. By contradiction, suppose that G is a λ_1-extremal graph without properties (1), (2) and (3). It follows that $|V_{<\Delta}| \geq 2$. Then there are two cases.

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Case 1: $V_{<\Delta} = \{u, v\}$, $uv \notin E(G)$, $d(u) < \Delta - 1$ and $d(v) \leq \Delta - 1$. Then the graph $G + uv$ has the same maximum degree as G. By Lemma 2.1, we obtain $\lambda_1(G + uv) > \lambda_1(G)$, contradicting the choice of G.

Case 2: $|V_{<\Delta}| > 2$ and $V_{<\Delta}$ does not induce a complete graph. Then there exist two vertices $u, v \in V_{<\Delta}$ and $uv \notin E(G)$. Similarly arguing to case 1, we obtain $\lambda_1(G + uv) > \lambda_1(G)$, contradicting the choice of G.

Combining the above two cases, the proof follows. ☐

Using the properties mentioned in Theorem 2.3, we give the following definition.

Definition 2.4. Let G be a connected non-regular graph on n vertices with maximum degree Δ. Then

the graph G is called *type*-I if it has property (1),
the graph G is called *type*-II if it has property (2),
the graph G is called *type*-III if it has property (3).

Lemma 2.5. [6] *Let G be a simple connected graph with n vertices, m edges and spectral radius $\lambda_1(G)$. Then*

$$\lambda_1(G) \leq \frac{\delta - 1 + \sqrt{(\delta + 1)^2 + 4(2m - \delta n)}}{2}$$

and equality holds if and only if G is either a regular graph or a graph in which each vertex has degree either δ or $n - 1$.

We first consider the λ_1-extremal graphs with $\Delta = 2$ or $\Delta = n - 1$. When $\Delta = 2$, the λ_1-extremal graph is the path with $\lambda_1(P_n) = 2\cos(\frac{\pi}{n+1})$. When $\Delta = n - 1$, similarly arguing to Theorem 2.3, we know that the λ_1-extremal graph is $K_n - e$. By Lemma 2.5, we obtain

$$\lambda_1(K_n - e) = \frac{n - 3 + \sqrt{(n + 1)^2 - 8}}{2}. \tag{2.1}$$

Theorem 2.3 shows that the λ_1-extremal graphs must be type-I, type-II or type-III, but in what follows, we will prove that when $2 < \Delta < n - 1$, any type-III graph is not λ_1-extremal.

Lemma 2.6. [5] *Let $\pi = (d_1, d_2, ..., d_n)$ be a nonincreasing sequence of non-negative integers. Then π is graphic if and only if*

$$\sum_{i=1}^{n} d_i \text{ is even and } \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k,\ d_i\},\ \text{for all } k = 1, 2, ..., n-1. \tag{2.2}$$

Lemma 2.7. *Let $\pi = (d_1, d_2, ..., d_n)$ be a nonincreasing sequence of positive*

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integers and $d_{n-1} \geq 2$*,* $d_n \geq 1$*. Then* π *is graphic if and only if it is connected graphic.*

Proof. If π is connected graphic, then it is obviously graphic. Conversely, suppose that π is graphic and G is a disconnected graph with the degree sequence π. Without loss of generality, suppose that G has two components G_1 and G_2. Noticing $d_{n-1} \geq 2$ and $d_n \geq 1$, we suppose that any vertex in G_1 has degree at least two and any edge $u_2v_2 \in E(G_2)$. Then it follows that there exists one edge u_1v_1 in G_1 which is not the cut edge, i.e. $G_1 - u_1v_1$ is still connected. Otherwise, G_1 is a tree, a contradiction. Consider $G' = G - u_1v_1 - u_2v_2 + u_1u_2 + v_1v_2$. It is easy to see that G' is a connected graph with the degree sequence π. ☐

LEMMA 2.8. *Let* $\pi = (d_1, d_2, ..., d_n) = (\Delta, \Delta, ..., \Delta, \Delta - 1, \Delta - 1)$ *and* $\pi' = (d'_1, d'_2, ..., d'_n) = (\Delta, \Delta, ..., \Delta, \Delta - 2)$ *with* $2 < \Delta < n - 1$*. If* π *is connected graphic, then* π' *is connected graphic.*

Proof. Since $2 < \Delta < n - 1$, we obtain $d'_{n-1} \geq 3$ and $d'_n \geq 1$. Then by Lemma 2.7, we need only to prove that π' is graphic. Let G be a connected graph with the degree sequence π. Since π is graphic, by Lemma 2.6, we obtain

$$\sum_{i=1}^{n} d_i \text{ is even and } \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d_i\}, \text{ for all } k = 1, 2, ..., n-1.$$

For π' we will prove that (2.2) is still true. Obviously, $\sum_{i=1}^{n} d'_i = \sum_{i=1}^{n} d_i$ is even. Then we need only to prove that the inequality is true. We split our proof into four cases.

Case 1: $1 \leq k \leq \Delta - 2$, then $k \leq n - 4$ and

$$\begin{aligned}\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i &\leq k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d_i\} \\ &= k(k-1) + k(n-k) \\ &= k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d'_i\}.\end{aligned}$$

Case 2: $1 < k = \Delta - 1$, then $k \leq n - 3$ and

$$k(k-1) + k(n-k) - \sum_{i=1}^{k} d_i = k(k-1) + k(n-k) - k\Delta = k(n-k-2) \geq k > 1.$$

Thus

$$\sum_{i=1}^{k} d'_i = \sum_{i=1}^{k} d_i < k(k-1) + k(n-k) - 1 = k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d'_i\}.$$

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Case 3: $\Delta \le k \le n-2$, then

$$\sum_{i=1}^{k} d_i' = \sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d_i\}$$
$$= k(k-1) + (n-k-2)\Delta + 2\Delta - 2$$
$$= k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d_i'\}.$$

Case 4: $k = n-1$, then

$$k(k-1) + \Delta - \sum_{i=1}^{k} d_i = (n-1)(n-2) + \Delta - [(n-1)\Delta - 1]$$
$$= (n-2)(n-1-\Delta) + 1 \ge 4,$$

where the last inequality holds since $2 < \Delta < n-1$. Hence

$$\sum_{i=1}^{k} d_i' = \sum_{i=1}^{k} d_i + 1 < k(k+1) + \Delta - 2$$
$$= k(k-1) + \sum_{i=k+1}^{n} \min\{k,\ d_i'\}.$$

Combining the above four cases, the inequality is true. Then by Lemma 2.6, the π' is graphic. This completes the proof. ☐

As we know, *majorization* on degree sequences is defined as follows: for two sequences $\pi = (d_1, d_2, ..., d_n)$, $\pi' = (d_1', d_2', ..., d_n')$ we write $\pi \trianglelefteq \pi'$ if and only if $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d_i'$ and $\sum_{i=1}^{j} d_i \le \sum_{i=1}^{j} d_i'$ for all $j = 1, 2, ..., n$. We claim that G has the *greatest maximum* eigenvalue if $\lambda_1(G) \ge \lambda_1(G')$ for any other graph G' in the class $\mathcal{C}_\pi$, where $\mathcal{C}_\pi$=$\{G : G$ is a connected graph with the degree sequence $\pi\}$.

Lemma 2.9. [1] *Let π and π' be two distinct degree sequences with $\pi \trianglelefteq \pi'$. Let G and G' be graphs with the greatest maximum eigenvalues in classes $\mathcal{C}_\pi$ and $\mathcal{C}_{\pi'}$, respectively. Then $\lambda_1(G) < \lambda_1(G')$.*

Theorem 2.10. *Let G be a connected graph with degree sequence*

$$\pi = (\Delta, \Delta, ..., \Delta, \Delta - 1, \Delta - 1)$$

and $2 < \Delta < n-1$. Then there exists a connected graph G' with degree sequence $\pi' = (\Delta, \Delta, ..., \Delta, \Delta - 2)$ such that $\lambda_1(G) < \lambda_1(G')$.

Proof. By Lemma 2.8, there exists a connected graph G' with degree sequence $\pi' = (\Delta, \Delta, ..., \Delta, \Delta - 2)$. We suppose that G (G') is the graph with the greatest

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maximum eigenvalue in $\mathcal{C}_\pi$ ($\mathcal{C}_{\pi'}$). It is obvious that $\pi \trianglelefteq \pi'$. By Lemma 2.9, we obtain $\lambda_1(G) < \lambda_1(G')$. □

THEOREM 2.11. *Let G be a λ_1-extremal graph on n vertices with the maximum degree Δ and $2 < \Delta < n-1$. Then G must be either type-I or type-II.*

Proof. Suppose that G is a type-III graph with the greatest maximum eigenvalue in class $\mathcal{C}_\pi$, where $\pi = (\Delta, \Delta, ..., \Delta, \Delta-1, \Delta-1)$. By Theorem 2.10, there exists a graph G' with degree sequence $\pi' = (\Delta, \Delta, ..., \Delta, \Delta-2)$ and greatest maximum eigenvalue in class $\mathcal{C}'_\pi$ such that $\lambda_1(G') > \lambda_1(G)$. It follows that G is not λ_1-extremal. □

REMARK. Although Theorem 2.11 shows that the λ_1-extremal graph with $2 < \Delta < n-1$ must be type-I or type-II, there exist some graphs with property (1) or (2) which are not λ_1-extremal. Let G_1, G_2 be connected graphs with degree sequences (3, 3, 3, 3, 2, 2), (5, 5, 5, 5, 5, 5, 2), respectively. Clearly G_1 (G_2) is a type-I (type-II) graph. However, by checking the Table 1 of [7], we know they are not the λ_1-extremal graphs. After some computer experiments, we give a conjecture about the λ_1-extremal graphs as follows:

CONJECTURE 2.12. *Let G be a connected non-regular graph on n vertices and $2 < \Delta < n-1$. Then G is λ_1-extremal if and only if G is a graph with the greatest maximum eigenvalue in classes $\mathcal{C}_\pi$ and $\pi = (\Delta, \Delta, ..., \Delta, \delta)$, where*

$$\delta = \begin{cases} (\Delta-1), & \text{when } n\Delta \text{ is odd,} \\ (\Delta-2), & \text{when } n\Delta \text{ is even.} \end{cases}$$

3. Main Results.

THEOREM 3.1. *Let G be a type-I or type-II graph on n vertices with diameter D. Then*

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1}. \tag{3.1}$$

Proof. Since G is a type-I or type-II graph, we have $\Delta \geq 3$. Let u, v be two vertices at distance D and $P : u = u_0 \leftrightarrow u_1 \leftrightarrow ... \leftrightarrow u_D = v$ be the shortest path connecting u and v. We first claim that $|V_{<\Delta} \cap V(P)| \leq 2$. Otherwise, G must be a type-I graph and suppose $\{u_p, u_q, u_r\} \subseteq V(P) \bigcap V_{<\Delta}$ with $p < q < r$. Then by definition of type-I graph, we obtain that u_pu_q, u_qu_r and $u_pu_r \in E(G)$. Therefore, P is not the shortest path connecting u and v, a contradiction.

Then there are two cases.

Case 1: $V_{<\Delta} \cap V(P) = \varnothing$. Define $T = \{i : i \equiv 0 \bmod 3 \text{ and } i \leq (D-3)\} \bigcup \{D\}$. Thus $|\ T\ | = \lceil \frac{D+1}{3} \rceil$. Let $d(u_i, u_j)$ denote the distance between u_i and u_j. Since P is

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the shortest path connecting u and v, we have $d(u_i, u_j) \geq 3$ and $\Gamma(u_i) \cap \Gamma(u_j) = \varnothing$ for any distinct $i, j \in T$. Notice that $u_i \in V(P)$ for any $i \in T$. We obtain

$$\mid \Gamma(u_i) - V(P) \mid = \begin{cases} \Delta - 1, & \text{if} \quad i \in \{0, \mathrm{D}\}, \\ \Delta - 2, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} n &\geq |V(P)| + \sum_{i \in T} |\Gamma(u_i) - V(P)| \\ &\geq D + 1 + (|T| - 2)(\Delta - 2) + 2(\Delta - 1) \\ &\geq D + 1 + (\frac{D+1}{3} - 2)(\Delta - 2) + 2(\Delta - 1). \end{aligned}$$

Thus

$$D \leq \frac{3n - \Delta - 7}{\Delta + 1}.$$

Case 2: Either $V_{<\Delta} \cap V(P) = \{u_p, u_q\}$ with $q = p + 1$ or $V_{<\Delta} \cap V(P) = \{u_p\}$. The proof is similar to the proof of [7]. We obtain the same result

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1}.$$

Combining the above two cases, the proof follows. □

Lemma 3.2. [3] *Let G be a connected non-regular graph on n vertices with maximum degree Δ and diameter D. Then*

$$\Delta - \lambda_1 > \frac{1}{nD}.$$

Theorem 3.3. *Let G be a connected non-regular graph on n vertices with maximum degree Δ. Then*

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

Proof. Without loss of generality, we suppose that G is a λ_1-extremal graph. Since G is connected and non-regular, then $n \geq 3$ and $\Delta \geq 2$. When $\Delta = n - 1$ and $n \geq 5$, the λ_1-extremal graph is $K_n - e$ with $D = 2$. Then by Lemma 3.2, we obtain

$$\lambda_1(K_n - e) < \Delta - \frac{1}{2n} < \Delta - \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

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When $\Delta = n-1$ and $n = 3$, the λ_1-extremal graph is P_3 with $\lambda_1(P_3)$=1.4142. When $\Delta = n-1$ and $n = 4$, the λ_1-extremal graph is $K_4 - e$ with $\lambda_1(K_4 - e)$=2.5616. By direct calculation, we know that the inequality is true. When $2 < \Delta < n-1$, applying Theorem 3.1 and Lemma 3.2, we obtain the result. When $\Delta = 2$ and $n > 3$, the λ_1-extremal graph is P_n. By adding some edges to P_n, we can attain $K_n - e$. Then following the Lemma 2.1, we obtain $\lambda_1(P_n) < \lambda_1(K_n - e)$. This completes the proof. □

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