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LINEAR ALGEBRA AND THE SUMS OF POWERS OF INTEGERS*

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Abstract. A general framework based on linear algebra is presented to obtain old and new polynomial expressions for the sums of powers of integers. This framework uses changes of polynomial basis, infinite lower triangular matrices and finite differences.

Key words. Finite differences, Polynomial space, Polynomial basis, Change of basis, Sum of powers of integers, Infinite lower triangular matrix.

AMS subject classifications. 11B83, 11B37, 15A03, 15A09.

1. Introduction.

Polynomial formulas to evaluate the sums of powers of integers

$$\sum_{k=1}^{K} k^n = 1^n + 2^n + 3^n + \cdots + K^n$$

has a long history. In the antic Greece, Archimedes ($287BC - 212BC$) obtained expressions for $n = 1$ and $n = 2$, and during the apex of Arab mathematical science, in the eleven century, Al-Haytham (965-1038), known in the West has Alhazen, obtained formulas for $n = 3$ and $n = 4$ [4, 8]. Later several mathematicians considered this problem: Faulhaber (1631), Pascal (1636), Fermat (1654), Bernoulli (1713), Euler (1755), Jacobi (1824), etc. The most celebrated results were obtained by J. Faulhaber [10, 12, 20, 7, 14, 13] and J. Bernoulli [19, 14]. The reader interested by the history of this problem could look at the preceding references and the following [2, 6, 9, 15, 18, 19, 21].

In this paper we present a unified approach for obtaining polynomial formulas for the sums of powers of integers. The method is based on finite differences and changes of basis for polynomial subspaces. The main result is the following theorem proved in Section 2.

THEOREM 1.1. *Let* $\mathcal{B}_p = \left\{p_i(x)\right\}_{i=0}^{+\infty}$ *be any family of polynomials such that the*

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polynomial $p_i(x)$ *is of degree* i*. For any fixed real number* τ *and any integer* $n \geq 0$ *there exist constants* $\left\{\alpha_{n,j}(\tau)\right\}_{j=-1}^{n}$ *such that*

$$\sum_{k=1}^{K} k^n = \sum_{j=-1}^{n} \alpha_{n,j}(\tau) p_{j+1}(K+\tau) \tag{1.1}$$

where

$$\alpha_{n,-1}(\tau) = -\sum_{j=0}^{n} \alpha_{n,j}(\tau) p_{j+1}(\tau) = -\sum_{j=0}^{n} \alpha_{n,j}(\tau) p_{j+1}(\tau-1). \ \Box \tag{1.2}$$

Well known examples of formulas that we can obtain from this result are the Bernoulli's polynomial expression [19, 14] (see Section 3), and the Faulhaber's polynomial expressions [10] (see Section 4.3).

In the last section the method is extended to obtain polynomial expressions for the l-fold

$$\Sigma_{K_0}^{(l)} K_1^n = \underbrace{\sum_{k_l=K_0}^{K_1} \sum_{k_{l-1}=K_0}^{k_l} \cdots \sum_{k_2=K_0}^{k_3} \sum_{k_1=K_0}^{k_2}}_{l \quad \text{summations}} k_1^n \tag{1.3}$$

Finally, let us observe that the method worked out in this paper is also suitable for implementation in a computer algebra system.

2. A general method for $\sum_{k=1}^{K} k^n$. Let $\mathcal{B}_p = \left\{p_i(x)\right\}_{i=0}^{+\infty}$ be a family of polynomials where $p_i(x)$ is of degree i for $i \geq 0$. Let Δ_τ be the finite difference operator defined for any fixed $\tau \in \mathbb{R}$ and for any function $F(x)$ by

$$\Delta_\tau F(x) = F(x+\tau) - F(x+\tau-1).$$

The method is based on the following two elementary results.

Lemma 2.1. *For any integer* $K_0 \leq K_1$ *we have*

$$\sum_{k=K_0}^{K_1} \Delta_\tau F(x+k) = F(x+K_1+\tau) - F(x+K_0+\tau-1). \ \Box \tag{2.1}$$

Lemma 2.2. *For any integer* $i \geq 0$*, if* $p_i(x)$ *is a polynomial of degree* i*, then* $q_i(x) = \Delta_\tau p_{i+1}(x)$ *is a polynomial of degree* i*.* $\Box$

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As a consequence of Lemma 2.2, for any integer $n \geq 0$ the sets $\mathcal{B}_s^n = \{e_i(x) = x^i\}_{i=0}^n$, and $\mathcal{B}_q^n = \{q_i(x) = \Delta_\tau p_{i+1}(x)\}_{i=0}^n$ are bases for the set $\mathcal{P}_n$ of polynomials of degree at most n, and

$$\mathcal{P}_n = \mathrm{Lin}\{e_i(x) | i = 0, ..., n\} = \mathrm{Lin}\{q_i(x) | i = 0, ..., n\}. \tag{2.2}$$

Let $\overrightarrow{E}(x) = (e_0(x), e_1(x), e_2(x), \cdots)^t$ and $\overrightarrow{Q}(x) = (q_0(x), q_1(x), q_2(x), \cdots)^t$ then from (2.2) we obtain

$$\overrightarrow{E}(x) = M\overrightarrow{Q}(x) \quad \text{and} \quad \overrightarrow{Q}(x) = N\overrightarrow{E}(x) \tag{2.3}$$

where M and N are two infinite lower triangular matrices given by

$$M = \Big(\alpha_{i,j}(\tau)\Big)_{\substack{i=0,1,...\\ j=0,1,...}} = \begin{pmatrix} \alpha_{0,0}(\tau) & 0 & \cdots & & \\ \alpha_{1,0}(\tau) & \alpha_{1,1}(\tau) & 0 & \cdots & \\ \alpha_{2,0}(\tau) & \alpha_{2,1}(\tau) & \alpha_{2,2}(\tau) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

and

$$N = \Big(\beta_{i,j}(\tau)\Big)_{\substack{i=0,1,...\\ j=0,1,...}} = \begin{pmatrix} \beta_{0,0}(\tau) & 0 & \cdots & & \\ \beta_{1,0}(\tau) & \beta_{1,1}(\tau) & 0 & \cdots & \\ \beta_{2,0}(\tau) & \beta_{2,1}(\tau) & \beta_{2,2}(\tau) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

These matrices are invertible and $MN = I = NM$.

From (2.3) and (2.1), we obtain

$$\sum_{k=K_0}^{K_1} e_n(x+k) = \sum_{j=0}^{n} \alpha_{n,j}(\tau)\Big[p_{j+1}(x+K_1+\tau) - p_{j+1}(x+K_0+\tau-1)\Big], \tag{2.4}$$

and if we set $x = 0$, $K_0 \leq K_1 = K$, and $K_0 = 0$ or 1, then we have a proof of Theorem 1.1.

We suggest two methods for computing the scalars $\alpha_{i,j}(\tau)$'s. The first method is by direct inversion of the infinite lower triangular matrix N while the second method uses the derivative of $q_n(x)$.

For the first method, we consider the system $MN = I$ and we obtain

$$\alpha_{i,j}(\tau) = \begin{cases} \frac{1}{\beta_{i,i}(\tau)} & \text{for} \quad j = i, \\ \\ -\frac{1}{\beta_{j,j}(\tau)} \sum_{l=j+1}^{i} \alpha_{i,l}(\tau)\beta_{l,j}(\tau) & \text{for} \quad j = i-1, ..., 0, \end{cases}$$

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or we consider the system $NM = I$ and we have

$$\alpha_{i,j}(\tau) = \begin{cases} -\frac{1}{\beta_{i,i}(\tau)} \sum_{l=j}^{i-1} \beta_{i,l}(\tau)\alpha_{l,j}(\tau) & \text{for} \quad j = 0, ..., i-1, \\ \frac{1}{\beta_{i,i}(\tau)} & \text{for} \quad j = i. \end{cases}$$

The second method proceeds as follows. Since $p_{n+1}^{(1)}(x)$ is a polynomial of degree n, we can write

$$p_{n+1}^{(1)}(x) = \sum_{j=0}^{n} \gamma_{n,j-1} p_j(x),$$

and

$$q_n^{(1)}(x) = \Delta_\tau p_{n+1}^{(1)}(x) = \sum_{j=1}^{n} \gamma_{n,j-1} \Delta_\tau p_j(x) = \sum_{j=0}^{n-1} \gamma_{n,j} q_j(x).$$

Using the matrix notation, we have

$$\overrightarrow{Q}^{(1)}(x) = \Gamma \overrightarrow{Q}(x) \tag{2.5}$$

where Γ is the infinite lower triangular matrix

$$\Gamma = \left(\gamma_{i,j}\right)_{\substack{i=0,1,... \\ j=0,1,...}} = \begin{pmatrix} 0 & \cdots & & & \\ \gamma_{1,0} & 0 & \cdots & & \\ \gamma_{2,0} & \gamma_{2,1} & 0 & \cdots & \\ \gamma_{3,0} & \gamma_{3,1} & \gamma_{3,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Moreover, we also have

$$\overrightarrow{E}^{(1)}(x) = DP\overrightarrow{E}(x), \tag{2.6}$$

where the infinite diagonal matrices D and P are defined by

$$D = \begin{pmatrix} 0 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & 0 & 2 & 0 & & \\ & & 0 & 3 & 0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \end{pmatrix}.$$

Hence from $\overrightarrow{E}^{(1)}(x) = M\overrightarrow{Q}^{(1)}(x)$, (2.5) and (2.6), it follows that $DPM = M\Gamma$.

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We obtain recursively each line of M by solving the system

$$\left\{ \begin{array}{l} DPM = M\Gamma, \\ \overrightarrow{E}(\xi) = M\overrightarrow{Q}(\xi), \end{array} \right.$$

where ξ is any arbitrary value for x. This leads to : $\alpha_{0,0}(\tau) \neq 0$, and for $i \geq 1$ we have

$$\alpha_{i,j}(\tau) = \left\{ \begin{array}{ll} \frac{i}{\gamma_{i,i-1}}\alpha_{i-1,i-1}(\tau) & \text{for} \quad j = i, \\ \frac{1}{\gamma_{j,j-1}}\left[i\alpha_{i-1,j}(\tau) - \sum_{l=j+1}^{i}\alpha_{i,l}(\tau)\gamma_{l,j-1}\right] & \text{for} \quad j = i-1, ..., 1, \\ \frac{e_i(\xi) - \sum_{j=1}^{i}\alpha_{i,j}(\tau)q_j(\xi)}{q_0(\xi)} & \text{for} \quad j = 0. \end{array} \right.$$

3. Bernoulli's polynomial formula. Let $\mathcal{B}_p = \left\{p_i(x) = e_i(x)\right\}_{i=0}^{+\infty}$ and $\tau = 0$. We will use the notation $\alpha_{i,j}(\tau) = \alpha_{i,j}$ and $\beta_{i,j}(\tau) = \beta_{i,j}$. We have

$$q_n(x) = e_{n+1}(x) - e_{n+1}(x-1) = \sum_{j=0}^{n} \binom{n+1}{j} (-1)^{n-j} e_j(x), \tag{3.1}$$

hence $\beta_{n,j} = \binom{n+1}{j}(-1)^{n-j}$ for $j = 0, ..., n$, and

$$N = \begin{pmatrix} 1 & 0 & \cdots & & & \\ -1 & 2 & 0 & \cdots & & \\ 1 & -3 & 3 & 0 & \cdots & \\ -1 & 4 & -6 & 4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

To compute M we use (2.5), and since $q_n^{(1)}(x) = (n+1)q_{n-1}(x)$ for $n \geq 1$, we have

$$\Gamma = \begin{pmatrix} 0 & & & & \\ 2 & 0 & & & \\ 0 & 3 & 0 & & \\ & 0 & 4 & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = (I+D)P.$$

Then solving the system

$$\left\{ \begin{array}{l} DPM = M(I+D)P, \\ \overrightarrow{E}(0) = M\overrightarrow{Q}(0) \quad \big(\text{or} \quad \overrightarrow{E}(1) = M\overrightarrow{Q}(1)\big), \end{array} \right.$$

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leads to : $\alpha_{0,0} = 1$, and for $i \geq 1$

$$\alpha_{i,j} = \frac{i}{j+1}\alpha_{i-1,j-1}$$

for $j = 1, ..., i$, and

$$\alpha_{i,0} = \sum_{j=1}^{i}(-1)^{j+1}\alpha_{i,j} = 1 - \sum_{j=1}^{i}\alpha_{i,j}. \tag{3.2}$$

From these relations we obtain

$$\alpha_{i,j} = \frac{1}{i+1}\begin{pmatrix} i+1 \\ j+1 \end{pmatrix}\alpha_{i-j,0} \tag{3.3}$$

for $j = 0, ..., i$.

Remark 3.1. From (3.2) and (3.3), if we set $\alpha_{i-j,0} = B_i$, the B_i's are the Bernoulli's numbers generated by : $B_0 = 1$, and for $i \geq 1$

$$\sum_{j=0}^{i}(-1)^j \begin{pmatrix} i+1 \\ j+1 \end{pmatrix} B_{i-j} = 0 \quad \text{or} \quad \frac{1}{i+1}\sum_{j=0}^{i}\begin{pmatrix} i+1 \\ j+1 \end{pmatrix} B_{i-j} = 1. \tag{3.4}$$

For $i \geq 1$, from (3.4) we get

$$\frac{2}{i+1}\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor}\begin{pmatrix} i+1 \\ 2j \end{pmatrix} B_{2j} = 1, \tag{3.5}$$

and

$$\frac{2}{i+1}\sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor}\begin{pmatrix} i+1 \\ 2j+1 \end{pmatrix} B_{2j+1} = 1. \tag{3.6}$$

Since $B_0 = 1$, from (3.5) we obtain the B_{2j}'s for $j \geq 1$. From (3.6) we have $B_1 = \frac{1}{2}$ and for $i \geq 3$

$$\sum_{j=1}^{\lfloor \frac{i-1}{2} \rfloor}\begin{pmatrix} i+1 \\ 2j+1 \end{pmatrix} B_{2j+1} = 0,$$

which implies that $B_{2j+1} = 0$ for $j \geq 1$. □

In this case (1.1) leads to the following celebrated Bernoulli's polynomial formula

$$\sum_{k=1}^{K} k^n = \frac{1}{n+1}\sum_{i=0}^{n}\begin{pmatrix} n+1 \\ i \end{pmatrix} B_i K^{n+1-i}.$$

Expressions (3.3) and (3.2), used to compute recursively the coefficients, have already been presented in [5, 9, 17, 3]. Several other proofs of this well known formula for the sum of powers of integers appeared elsewhere [1, 6, 11, 12, 15, 16, 18, 21].

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$$\sum K = \tfrac{1}{2}K^2 + \tfrac{1}{2}K$$

$$\sum K^2 = \tfrac{1}{3}K^3 + \tfrac{1}{2}K^2 + \tfrac{1}{6}K$$

$$\sum K^3 = \tfrac{1}{4}K^4 + \tfrac{1}{2}K^3 + \tfrac{1}{4}K^2$$

$$\sum K^4 = \tfrac{1}{5}K^5 + \tfrac{1}{2}K^4 + \tfrac{1}{3}K^3 - \tfrac{1}{30}K$$

$$\sum K^5 = \tfrac{1}{6}K^6 + \tfrac{1}{2}K^5 + \tfrac{5}{12}K^4 - \tfrac{1}{12}K^2$$

$$\sum K^6 = \tfrac{1}{7}K^7 + \tfrac{1}{2}K^6 + \tfrac{1}{2}K^5 - \tfrac{1}{6}K^3 + \tfrac{1}{42}K$$

$$\sum K^7 = \tfrac{1}{8}K^8 + \tfrac{1}{2}K^7 + \tfrac{7}{12}K^6 - \tfrac{7}{24}K^4 + \tfrac{1}{12}K^2$$

$$\sum K^8 = \tfrac{1}{9}K^9 + \tfrac{1}{2}K^8 + \tfrac{2}{3}K^7 - \tfrac{7}{15}K^5 + \tfrac{2}{9}K^3 - \tfrac{1}{30}K$$

$$\sum K^9 = \tfrac{1}{10}K^{10} + \tfrac{1}{2}K^9 + \tfrac{3}{4}K^8 - \tfrac{7}{10}K^6 + \tfrac{1}{2}K^4 - \tfrac{3}{20}K^2$$

$$\sum K^{10} = \tfrac{1}{11}K^{11} + \tfrac{1}{2}K^{10} + \tfrac{5}{6}K^9 - 1K^7 + 1K^5 - \tfrac{1}{2}K^3 + \tfrac{5}{66}K$$

Table 3.1
First 10 *Bernoulli's polynomial expressions for the sums of powers of integers.*

4. Towards the Faulhaber's polynomial formula. In this section we present three formulas for the sums of powers of integers related by the bases we use to obtain them. The last one is the Faulhaber's polynomial formula. Throughout this section $\tau = \frac{1}{2}$.

4.1. A first intermediate polynomial formula. The first formula of this section is very similar to the Bernoulli's polynomial formula. Let $\mathcal{B}_p = \left\{p_i(x) = e_i(x)\right\}_{i=0}^{+\infty}$, and let us use the notation $u_i(x) = \Delta_{1/2}p_{i+1}(x)$, $a_{i,j} = \alpha_{i,j}(\frac{1}{2})$ and $b_{i,j} = \beta_{i,j}(\frac{1}{2})$.

Let $\overrightarrow{U}(x) = (u_0(x), u_1(x), u_2(x), \cdots)^t$, and set $\overrightarrow{E}(x) = M_1\overrightarrow{U}(x)$ and $\overrightarrow{U}(x) =$

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$N_1\overrightarrow{E}(x)$ where

$$M_1 = \left(a_{i,j}\right)_{\substack{i=0,1,...\\ j=0,1,...}} = \begin{pmatrix} a_{0,0} & 0 & \cdots & & \\ a_{1,0} & a_{1,1} & 0 & \cdots & \\ a_{2,0} & a_{2,1} & a_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

and

$$N_1 = \left(b_{i,j}\right)_{\substack{i=0,1,...\\ j=0,1,...}} = \begin{pmatrix} b_{0,0} & 0 & \cdots & & \\ b_{1,0} & b_{1,1} & 0 & \cdots & \\ b_{2,0} & b_{2,1} & b_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

We have

$$\begin{aligned} u_n(x) &= e_{n+1}(x+\frac{1}{2}) - e_{n+1}(x-\frac{1}{2}) \\ &= \sum_{j=0}^{n} \binom{n+1}{j} (\frac{1}{2})^{n+1-j}[1+(-1)^{n-j}]e_j(x), \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} (\frac{1}{2})^{2j} e_{n-2j}(x), \end{aligned} \tag{4.1}$$

and hence

$$N_1 = \begin{pmatrix} 1 & 0 & \cdots & & & & \\ 0 & \binom{2}{1} & 0 & \cdots & & & \\ \frac{1}{2^2}\binom{3}{3} & 0 & \binom{3}{1} & 0 & \cdots & & \\ 0 & \frac{1}{2^2}\binom{4}{3} & 0 & \binom{4}{1} & 0 & \cdots & \\ \frac{1}{2^4}\binom{5}{5} & 0 & \frac{1}{2^2}\binom{5}{3} & 0 & \binom{5}{1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

From (4.1), we not only have (2.2) but also

$$\mathrm{Lin}\{e_{n-2i}(x) | i = 0, ..., \lfloor \frac{n}{2} \rfloor\} = \mathrm{Lin}\{u_{n-2i}(x) | i = 0, ..., \lfloor \frac{n}{2} \rfloor\},$$

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and this observation implies that $a_{n,n-(2j+1)} = 0 = b_{n,n-(2j+1)}$ for $j = 0, ..., \lfloor \frac{n-1}{2} \rfloor$. Therefore the sum of powers of integers (1.1) is given by

$$\sum_{k=1}^{K} k^n = a_{n,-1} + \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,n-2j}(K + \frac{1}{2})^{n+1-2j} \tag{4.2}$$

with (1.2) as

$$a_{n,-1} = -\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (\frac{1}{2})^{n+1-2j} a_{n,n-2j} = (-1)^n \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (\frac{1}{2})^{n+1-2j} a_{n,n-2j}. \tag{4.3}$$

It follows that $a_{n,-1} = 0$ for n even.

Using (2.5), and since $q_n^{(1)}(x) = (n+1)q_{n-1}(x)$ for $n \geq 1$, we have

$$\Gamma = (I + D)P = \begin{pmatrix} 0 & & & & \\ 2 & 0 & & & \\ 0 & 3 & 0 & & \\ & 0 & 4 & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Solving the system

$$\begin{cases} DPM_1 = M_1(I + D)P, \\ \overrightarrow{E}(0) = M_1 \overrightarrow{U}(0) \quad \big(\text{or} \quad \overrightarrow{E}(1) = M_1 \overrightarrow{U}(1)\big), \end{cases}$$

leads to : $a_{0,0} = 1$, and for $i \geq 1$

$$a_{i,j} = \frac{i}{j+1} a_{i-1,j-1}$$

for $j = 1, ..., i$, and

$$a_{i,0} = -\sum_{j=1}^{i} (\frac{1}{2})^{j+1} [1 + (-1)^j] a_{i,j} = -\frac{[1 + (-1)^i]}{2} \sum_{j=0}^{\frac{i-1}{2}} (\frac{1}{2})^{i-2j} a_{i,i-2j}. \tag{4.4}$$

Hence $a_{i,0} = 0$ for odd i. From these relations we obtain

$$a_{i,j} = \frac{1}{i+1} \begin{pmatrix} i+1 \\ j+1 \end{pmatrix} a_{i-j,0} \tag{4.5}$$

for $j = 0, ..., i$.

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REMARK 4.1. As for the Bernoulli's numbers, let us set $a_{i,0} = A_i$. Then from (4.4) and (4.5), the A_i's are generated by : $A_0 = 1$ and for $i \geq 1$

$$\sum_{j=0}^{\frac{i}{2}} (\frac{1}{2})^{2j} \begin{pmatrix} i+1 \\ 2j+1 \end{pmatrix} A_{i-2j} = 0.$$

Using $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ in $\overrightarrow{E}(x) = M_1 \overrightarrow{U}(x)$, we have

$$\Big(\frac{1}{2}\Big)^i = \sum_{j=0}^{i} a_{i,j} \quad \text{and} \quad \Big(-\frac{1}{2}\Big)^i = \sum_{j=0}^{i} (-1)^j a_{i,j}$$

which leads to

$$\Big(\frac{1}{2}\Big)^i = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} a_{i,i-2j} \quad \text{and} \quad 0 = \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} a_{i,i-(2j+1)}.$$

We also conclude from these relations that $a_{i,i-(2j+1)} = 0$ and $A_{2j+1} = 0$ for any $j \geq 0$. □

Finally, (4.2) becomes

$$\sum_{k=1}^{K} k^n = \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} a_{n,n-2j} (K + \frac{1}{2})^{n+1-2j}.$$

where the $a_{n,n-2j}$'s are given by (4.3), (4.4) and (4.5).

4.2. A second intermediate polynomial formula. Let $\mathcal{B}_p = \{p_i(x)\}_{i=0}^{+\infty}$ where

$$p_i(x) = \begin{cases} e_i(x) & \text{for} \quad i = 0, 1, \\ \\ e_i(x) - \frac{1}{4} e_{i-2}(x) & \text{for} \quad i \geq 2. \end{cases}$$

Let us use the notation $v_i(x) = \Delta_{1/2} p_{i+1}(x)$, $c_{i,j} = \alpha_{i,j}(\frac{1}{2})$ and $d_{i,j} = \beta_{i,j}(\frac{1}{2})$.

Let $\overrightarrow{V}(x) = (v_0(x), v_1(x), v_2(x), \cdots)^t$, and set $\overrightarrow{E}(x) = M_2 \overrightarrow{V}(x)$ and $\overrightarrow{V}(x) = N_2 \overrightarrow{E}(x)$ where the two lower triangular matrices are defined by

$$M_2 = \Big(c_{i,j}\Big)_{\substack{i=0,1,... \\ j=0,1,...}} = \begin{pmatrix} c_{0,0} & 0 & \cdots & & \\ c_{1,0} & c_{1,1} & 0 & \cdots & \\ c_{2,0} & c_{2,1} & c_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

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$$\sum K = \tfrac{1}{2}U^2 - \tfrac{1}{8}$$

$$\sum K^2 = \tfrac{1}{3}U^3 - \tfrac{1}{12}U$$

$$\sum K^3 = \tfrac{1}{4}U^4 - \tfrac{1}{8}U^2 + \tfrac{1}{64}$$

$$\sum K^4 = \tfrac{1}{5}U^5 - \tfrac{1}{6}U^3 + \tfrac{7}{240}U$$

$$\sum K^5 = \tfrac{1}{6}U^6 - \tfrac{5}{24}U^4 + \tfrac{7}{96}U^2 - \tfrac{1}{128}$$

$$\sum K^6 = \tfrac{1}{7}U^7 - \tfrac{1}{4}U^5 + \tfrac{7}{48}U^3 - \tfrac{31}{1344}U$$

$$\sum K^7 = \tfrac{1}{8}U^8 - \tfrac{7}{24}U^6 + \tfrac{49}{192}U^4 - \tfrac{31}{384}U^2 + \tfrac{17}{2048}$$

$$\sum K^8 = \tfrac{1}{9}U^9 - \tfrac{1}{3}U^7 + \tfrac{49}{120}U^5 - \tfrac{31}{144}U^3 + \tfrac{127}{3840}U$$

$$\sum K^9 = \tfrac{1}{10}U^{10} - \tfrac{3}{8}U^8 + \tfrac{49}{80}U^6 - \tfrac{31}{64}U^4 + \tfrac{381}{2560}U^2 - \tfrac{31}{2048}$$

$$\sum K^{10} = \tfrac{1}{11}U^{11} - \tfrac{5}{12}U^9 + \tfrac{7}{8}U^7 - \tfrac{31}{32}U^5 + \tfrac{127}{256}U^3 - \tfrac{2555}{33792}U$$

Table 4.1
First 10 Bernoulli's like polynomial expressions for the sums of powers of integers, $U = K + \frac{1}{2}$.

and

$$N_2 = \left(d_{i,j}\right)_{\substack{i=0,1,\ldots\\ j=0,1,\ldots}} = \begin{pmatrix} d_{0,0} & 0 & \cdots & & \\ d_{1,0} & d_{1,1} & 0 & \cdots & \\ d_{2,0} & d_{2,1} & d_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

We have $v_0(x) = e_0(x)$, $v_1(x) = 2e_1(x)$, and for $i \geq 2$

$$\begin{aligned} v_i(x) &= u_i(x) - \frac{1}{4}u_{i-2}(x) \\ &= \sum_{j=1}^{i} \left[\binom{i}{j-1} + \binom{i-1}{j-1} \right] (\frac{1}{2})^{i-j+1} \left[1 + (-1)^{i-j}\right] e_j(x) \\ &= \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} \left[\binom{i}{2j+1} + \binom{i-1}{2j} \right] (\frac{1}{2})^{2j} e_{i-2j}(x). \end{aligned} \tag{4.6}$$

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Hence

$$N_2 = \begin{pmatrix} 1 & 0 & \cdots & & & & & \\ 0 & 2 & 0 & \cdots & & & & \\ 0 & 0 & 3 & 0 & \cdots & & & \\ 0 & \frac{1}{2} & 0 & 4 & 0 & \cdots & & \\ 0 & 0 & \frac{7}{4} & 0 & 5 & 0 & \cdots & \\ 0 & \frac{1}{8} & 0 & 4 & 0 & 6 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then for $n \geq 1$

$$\mathrm{Lin}\{e_{n-2i}(x)|i = 0, ..., \lfloor \frac{n-1}{2} \rfloor\} = \mathrm{Lin}\{v_{n-2i}(x)|i = 0, ..., \lfloor \frac{n-1}{2} \rfloor\},$$

and $c_{n,0} = 0 = d_{n,0}$ for $n \geq 1$, and $c_{n,n-(2j+1)} = 0 = d_{n,n-(2j+1)}c_{n,n-(2j+1)} = 0 = d_{n,n-(2j+1)}$for any $j = 0, ..., \lfloor \frac{n-1}{2} \rfloor$. It follows that (1.1) becomes

$$\sum_{k=1}^{K} k^n = \begin{cases} K(K+1)\sum_{j=0}^{\frac{n-1}{2}} c_{n,n-2j}(K+\frac{1}{2})^{n-1-2j} & \text{for } n \text{ odd,} \\ \\ K(K+1)(K+\frac{1}{2})\sum_{j=0}^{\frac{n}{2}-1} c_{n,n-2j}(K+\frac{1}{2})^{n-2-2j} & \text{for } n \text{ even.} \end{cases}$$

REMARK 4.2. From (4.6) we have $\overrightarrow{V}(x) = (I - \frac{1}{4}P^2)\overrightarrow{U}(x)$. Then $\overrightarrow{E}(x) = M_2(I - \frac{1}{4}P^2)\overrightarrow{U}(x)$. We also have $\overrightarrow{E}(x) = M_1\overrightarrow{U}(x)$. It follows that $M_2(I - \frac{1}{4}P^2) = M_1$. But $(I - \frac{1}{4}P^2)^{-1} = \sum_{l=0}^{+\infty}(\frac{1}{4})^l P^{2l}$, then $M_2 = M_1(I - \frac{1}{4}P^2)^{-1} = M_1\sum_{l=0}^{+\infty}(\frac{1}{4})^l P^{2l}$, and

$$c_{i,i-2j} = \sum_{l=0}^{j}(\frac{1}{4})^{j-l}a_{i,i-2l}$$

for $j = 0, ..., \lfloor \frac{i}{2} \rfloor$. $\square$

4.3. Faulhaber's polynomial formula. Let $\mathcal{B}_p = \{p_i(x)\}_{i=0}^{+\infty}$ where

$$p_i(x) = x^{i-2\lfloor \frac{i}{2} \rfloor}\left[(x - \frac{1}{2})(x + \frac{1}{2})\right]^{\lfloor \frac{i}{2} \rfloor}.$$

Let us use the notation $w_i(x) = \Delta_{1/2}p_{i+1}(x)$, $f_{i,j} = \alpha_{i,j}(\frac{1}{2})$, and $g_{i,j} = \beta_{i,j}(\frac{1}{2})$.

Let $\overrightarrow{W}(x) = (w_0(x), w_1(x), w_2(x), \cdots)^t$, then $\overrightarrow{E}(x) = M_F\overrightarrow{W}(x)$ and $\overrightarrow{W}(x) = N_F\overrightarrow{E}(x)$ where the two lower triangular matrices are defined by

$$M_F = \left(f_{i,j}\right)_{\substack{i=0,1,... \\ j=0,1,...}} = \begin{pmatrix} f_{0,0} & 0 & \cdots & & \\ f_{1,0} & f_{1,1} & 0 & \cdots & \\ f_{2,0} & f_{2,1} & f_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

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$$\sum K = \tfrac{1}{2}V$$

$$\sum K^2 = \tfrac{1}{3}UV$$

$$\sum K^3 = V[\tfrac{1}{4}U^2 - \tfrac{1}{16}]$$

$$\sum K^4 = UV[\tfrac{1}{5}U^2 - \tfrac{7}{60}]$$

$$\sum K^5 = V[\tfrac{1}{6}U^4 - \tfrac{1}{6}U^2 + \tfrac{1}{32}]$$

$$\sum K^6 = UV[\tfrac{1}{7}U^4 - \tfrac{3}{14}U^2 + \tfrac{31}{336}]$$

$$\sum K^7 = V[\tfrac{1}{8}U^6 - \tfrac{25}{96}U^4 + \tfrac{73}{384}U^2 - \tfrac{17}{512}]$$

$$\sum K^8 = UV[\tfrac{1}{9}U^6 - \tfrac{11}{36}U^4 + \tfrac{239}{720}U^2 - \tfrac{127}{960}]$$

$$\sum K^9 = V[\tfrac{1}{10}U^8 - \tfrac{7}{20}U^6 + \tfrac{21}{40}U^4 - \tfrac{113}{320}U^2 + \tfrac{31}{512}]$$

$$\sum K^{10} = UV[\tfrac{1}{11}U^8 - \tfrac{13}{33}U^6 + \tfrac{205}{264}U^4 - \tfrac{409}{528}U^2 + \tfrac{2555}{8448}]$$

Table 4.2
First 10 intermediate polynomial expressions for the sums of powers of integers, $V = K(K+1)$.

and

$$N_F = \left(g_{i,j}\right)_{\substack{i=0,1,\ldots\\ j=0,1,\ldots}} = \begin{pmatrix} g_{0,0} & 0 & \cdots & & \\ g_{1,0} & g_{1,1} & 0 & \cdots & \\ g_{2,0} & g_{2,1} & g_{2,2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

We have

$$w_{2k}(x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \left[\binom{k+1}{2j+1} + \binom{k}{2j+1} \right] e_{2k-2j}$$

and

$$w_{2k+1}(x) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} 2 \binom{k+1}{2j+1} e_{2k+1-2j}.$$

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The matrix N_F is

$$N_F = \begin{pmatrix} 1 & 0 & \cdots & & & & & \\ 0 & 2 & 0 & \cdots & & & & \\ 0 & 0 & 3 & 0 & \cdots & & & \\ 0 & 0 & 0 & 4 & 0 & \cdots & & \\ 0 & 0 & 1 & 0 & 5 & 0 & \cdots & \\ 0 & 0 & 0 & 2 & 0 & 6 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Consequently for $i \geq 2$

$$\mathrm{Lin}\{w_{i-2j}(x)|j = 0, ..., \lfloor \frac{i}{2} \rfloor - 1\} = \mathrm{Lin}\{e_{i-2j}(x)|j = 0, ..., \lfloor \frac{i}{2} \rfloor - 1\},$$

and also

$$e_i(x) = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor - 1} f_{i,i-2j} w_{i-2j}(x). \tag{4.7}$$

Hence, for $n \geq 2$, (1.1) becomes

$$\sum_{k=1}^{K} k^n = \begin{cases} \big[K(K+1)\big]^2 \sum_{j=0}^{\frac{n-3}{2}} f_{n,n-2j} \big[K(K+1)\big]^{\frac{n-3}{2}-j} & \text{for } n \text{ odd}, \\ (K+\frac{1}{2})K(K+1) \sum_{j=0}^{\frac{n}{2}-1} f_{n,n-2j} \big[K(K+1)\big]^{\frac{n}{2}-1-j} & \text{for } n \text{ even}, \end{cases}$$

which is the Faulhaber's polynomial expression for the sums of powers of integers.

To compute recursively the coefficients $f_{i,i-2j}$, we observe that

$$w^{(1)}_{2l+2}(x) = (1 + 2(l + 1))w_{2l+1}(x) + \frac{l+1}{2} w_{2l-1}(x), \tag{4.8}$$

and

$$w^{(1)}_{2l+1}(x) = 2(l + 1)w_{2l}(x). \tag{4.9}$$

Then, from (4.7)

$$(2l + 2)x^{2l+1} = \sum_{j=0}^{l} f_{2l+2,2l+2-2j} \Big[(1 + 2(l - j + 1))w_{2(l-j)+1}(x) \tag{4.10}$$

$$+ \frac{(l - j) + 1}{2} w_{2(l-j)-1}(x)\Big], \tag{4.11}$$

and

$$(2l + 1)x^{2l} = \sum_{j=0}^{l-1} f_{2l+1,2l+1-2j} 2(l - j + 1))w_{2(l-j)}(x). \tag{4.12}$$

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Then again using (4.7), (4.8) and (4.9), we obtain the recursive method : assume $f_{2,0} = 0$ and $f_{2,2} = \frac{1}{3}$, for $l \geq 1$:
(a) to compute $f_{2l+1,2l+1-2j}$ from $f_{2l,2l-2j}$ we use the relation

$$f_{2l+1,2l+1-2j} = \frac{(2l+1)}{2(l+1-j)} f_{2l,2l-2j} \tag{4.13}$$

for $j = 0, ..., l-1$;
(b) to compute $f_{2l+2,2l+2-2j}$ from $f_{2l+1,2l+1-2j}$, we set $f_{2l+2,2l+4} = 0$ and $f_{2l+1,1} = 0$ and we use

$$f_{2l+2,2l+2-2j} = \frac{2(l+1)}{(2(l-j)+3)} \Big[f_{2l+1,2l+1-2j} - \frac{(l+2-j)}{4(l+1)} f_{2l+2,2l+2-2(j-1)} \Big] \tag{4.14}$$

for $j = 0, ..., l$.

Remark 4.3. Using the matrix notation, (4.8) and (4.9) lead to the lower triangular matrix Γ such that $\overrightarrow{W}^{(1)}(x) = \Gamma \overrightarrow{W}(x)$ and its nonzero elements are $\gamma_{i,i-1} = i+1$ for $i \geq 1$, and $\gamma_{2l,2l-3} = \frac{l}{4}$ for $l \geq 2$. Solving the system $DPM_F = M_F \Gamma$, we obtain (4.13) and (4.14). □

The formula (4.13) was known by Faulhaber [10, 14] and appears also in [12, 20, 7, 15].

4.4. Other polynomial formulas. We could find other formulas trying with other sets $\mathcal{B}$. For example, take any integer $m \geq 2$ and set $p_i(x) = e_i(x)$ for $i \leq 0$ and

$$p_i(x) = x^{i-2\lfloor \frac{i}{m} \rfloor} \Big[(x - \frac{1}{2})(x + \frac{1}{2}) \Big]^{\lfloor \frac{i}{m} \rfloor}$$

for $i \geq 1$. The value $m = 2$ corresponds to the Faulhaber's case.

5. An extension of the method. In this section by repeating the application of the difference operator we obtain expressions for l-fold summations of powers of integers. Let us use the following notation for the l-fold summations

$$\Sigma_{K_0}^{(l)} \Delta_\tau^\sigma F(x+K_1) = \begin{cases} \Delta_\tau^\sigma F(x+K_1) & \text{for} \quad l = 0, \\ \Sigma_{k_l=K_0}^{K_1} \cdots \Sigma_{k_1=K_0}^{k_2} \Delta_\tau^\sigma F(x+k_1) & \text{for} \quad l \geq 1, \end{cases}$$

and

$$\Sigma_{K_0}^{(l)} K_1^0 = \begin{cases} 1 & \text{for} \quad l = 0, \\ \Sigma_{k_l=K_0}^{K_1} \cdots \Sigma_{k_1=K_0}^{k_2} 1 & \text{for} \quad l \geq 1, \end{cases}$$

for any nonnegative integer l.

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$\sum K$	$=$	$\frac{1}{2}V$
$\sum K^2$	$=$	$\frac{1}{3}UV$
$\sum K^3$	$=$	$\frac{1}{4}V^2$
$\sum K^4$	$=$	$UV[\frac{1}{5}V - \frac{1}{15}]$
$\sum K^5$	$=$	$V^2[\frac{1}{6}V - \frac{1}{12}]$
$\sum K^6$	$=$	$UV[\frac{1}{7}V^2 - \frac{1}{7}V + \frac{1}{21}]$
$\sum K^7$	$=$	$V^2[\frac{1}{8}V^2 - \frac{1}{6}V + \frac{1}{12}]$
$\sum K^8$	$=$	$UV[\frac{1}{9}V^3 - \frac{2}{9}V^2 + \frac{1}{5}V - \frac{1}{15}]$
$\sum K^9$	$=$	$V^2[\frac{1}{10}V^3 - \frac{1}{4}V^2 + \frac{3}{10}V - \frac{3}{20}]$
$\sum K^{10}$	$=$	$UV[\frac{1}{11}V^4 - \frac{10}{33}V^3 + \frac{17}{33}V^2 - \frac{5}{11}V + \frac{5}{33}]$

Table 4.3
First 10 Faulhaber's polynomial expressions for the sums of powers of integers.

Lemma 5.1. *For any integers $K_0 \leq K_1$ and $\sigma \geq 1$ we have*

$$\Sigma_{k=K_0}^{K_1} \Delta_\tau^\sigma F(x+k) = \Delta_\tau^{\sigma-1} F(x+K_1+\tau) - \Delta_\tau^{\sigma-1} F(x+K_0+\tau-1). \square$$

More generally we have

Lemma 5.2. *For any integers $K_0 \leq K_1$ and $0 \leq l \leq \sigma$ we have*

$$\Sigma_{K_0}^{(l)} \Delta_\tau^\sigma F(x+K_1) = \Delta_\tau^{\sigma-l} F(x+K_1+l\tau) - \Sigma_{j=0}^{l-1} \left[\Sigma_{K_0}^{(j)} K_1^0\right] \Delta_\tau^{\sigma-(l-j)} F(x+K_0+(l-j)\tau-1). \square$$

Lemma 5.3. *For any integer $i \geq 0$, if $p_i(x)$ is a polynomial of degree i, then $q_i(x) = \Delta_\tau^\sigma p_{i+\sigma}(x)$ is a polynomial of degree i.* $\square$

From Lemma 5.3 we have

$$\mathcal{P}_n = \text{Lin}\{e_i(x) | i = 0, ..., n\} = \text{Lin}\{q_i(x) | i = 0, ..., n\}.$$

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Let $\overrightarrow{E}(x) = M\overrightarrow{Q}(x)$ and $\overrightarrow{Q}(x) = N\overrightarrow{E}(x)$ where M and N are two lower triangular matrices

$$M = \Big(\alpha_{i,j}(\tau,\sigma)\Big)_{\substack{i=0,1,...\\ j=0,1,...}} = \begin{pmatrix} \alpha_{0,0}(\tau,\sigma) & 0 & \cdots & & & \\ \alpha_{1,0}(\tau,\sigma) & \alpha_{1,1}(\tau,\sigma) & 0 & \cdots & & \\ \alpha_{2,0}(\tau,\sigma) & \alpha_{2,1}(\tau,\sigma) & \alpha_{2,2}(\tau,\sigma) & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \end{pmatrix}$$

and

$$N = \Big(\beta_{i,j}(\tau,\sigma)\Big)_{\substack{i=0,1,...\\ j=0,1,...}} = \begin{pmatrix} \beta_{0,0}(\tau,\sigma)) & 0 & \cdots & & & \\ \beta_{1,0}(\tau,\sigma)) & \beta_{1,1}(\tau,\sigma)) & 0 & \cdots & & \\ \beta_{2,0}(\tau,\sigma)) & \beta_{2,1}(\tau,\sigma)) & \beta_{2,2}(\tau,\sigma)) & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \end{pmatrix},$$

such that $MN = I = NM$. It follows that

$$e_i(x) = \sum_{j=0}^{i} \alpha_{i,j}(\tau,\sigma) q_j(x) = \sum_{j=0}^{i} \alpha_{i,j}(\tau,\sigma)\Delta_\tau^\sigma p_{j+\sigma}(x), \tag{5.1}$$

and

$$q_i(x) = \sum_{j=0}^{i} \beta_{i,j}(\tau,\sigma) e_j(x).$$

From (5.1) and the Lemma 5.2 we have

$$\Sigma_{K_0}^{(l)}(x+K_1)^n = \sum_{j=0}^{n} \alpha_{n,j}(\tau,\sigma)\Sigma_{K_0}^{(l)}\Delta_\tau^\sigma p_{j+\sigma}(x+K_1).$$

If we set $x = 0$ and $1 = K_0 \leq K_1 = K$, the l-fold summation of powers of integers is

$$\Sigma_1^{(l)} K^n = \sum_{j=0}^{n} \alpha_{n,j}(\tau,\sigma)\Sigma_1^{(l)}\Delta_\tau^\sigma p_{j+\sigma}(K).$$

The scalars $\alpha_{i,j}(\tau,\sigma)$'s can be computed recursively by inversion of the lower triangular matrix N if this matrix is known or by the following procedure. Since

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$p^{(1)}_{n+\sigma}(x)$ is a polynomial of degree $n+\sigma-1$, we have

$$p^{(1)}_{n+\sigma}(x) = \sum_{j=0}^{n+\sigma-1} \gamma_{n,j-\sigma} p_j(x).$$

Then

$$q^{(1)}_n(x) = \Delta^\sigma_\tau p^{(1)}_{n+\sigma}(x) = \sum_{j=\sigma}^{n+\sigma-1} \gamma_{n,j-\sigma} \Delta^\sigma_\tau p_j(x) = \sum_{j=0}^{n-1} \gamma_{n,j} q_j(x),$$

and we write $\overrightarrow{Q}^{(1)}(x) = \Gamma \overrightarrow{Q}(x)$ where Γ is a lower triangular matrix with zero values on the diagonal. We also have $\overrightarrow{E}^{(1)}(x) = DP\overrightarrow{E}(x)$. Using these identities with $\overrightarrow{E}^{(1)}(x) = M\overrightarrow{Q}^{(1)}(x)$, it follows that

$$DPM = M\Gamma.$$

Adding

$$\overrightarrow{E}(\xi) = M\overrightarrow{Q}(\xi)$$

for any fixed $x = \xi$, we can solve for M.

6. Examples. We present two families of formulas based on the general approach. The details are left to the reader.

6.1. A Bernoulli's type example. Let $\mathcal{B}_p = \{p_i(x) = e_i(x)\}_{i=0}^{+\infty}$, and let us use the notation $u_i(x) = \Delta^\sigma_{1/2} p_{i+\sigma}(x)$, $a^{(\sigma)}_{i,j} = \alpha_{i,j}(\frac{1}{2},\sigma)$ and $b^{(\sigma)}_{i,j} = \beta_{i,j}(\frac{1}{2},\sigma)$. We have

$$u_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} b^{(\sigma)}_{n,n-2j} e_{n-2j}(x)$$

then

$$\mathrm{Lin}\{e_{n-2i}(x) | i = 0, ..., \lfloor \frac{n}{2} \rfloor\} = \mathrm{Lin}\{u_{n-2i}(x) | i = 0, ..., \lfloor \frac{n}{2} \rfloor\}.$$

It follows that

$$e_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} a^{(\sigma)}_{n,n-2j} u_{n-2j}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} a^{(\sigma)}_{n,n-2j} \Delta^\sigma_{1/2} p_{n+\sigma-2j}(x).$$

and

$$\Sigma^{(\sigma)}_1 K^n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} a^{(\sigma)}_{n,n-2j} \Sigma^{(\sigma)}_1 \Delta^\sigma_{1/2} p_{n+\sigma-2j}(K).$$

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For example, let $\sigma = 2$ we have

$$\Sigma_1^{(2)} K^n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,n-2j}^{(2)} \Big[(K+1)^{n+2-2j} - (K+1) \Big].$$

6.2. A Faulhaber's type example. Let $\mathcal{B}_p = \big\{ p_i(x) \big\}_{i=0}^{+\infty}$ with

$$p_i(x) = x^{i-2\lfloor \frac{i}{2} \rfloor} \big[(x - \sigma \frac{1}{2})(x + \sigma \frac{1}{2}) \big]^{\lfloor \frac{i}{2} \rfloor}$$

for $i \geq 1$. We use the notation $w_i(x) = \Delta_{1/2}^{\sigma} p_{i+\sigma}(x)$, $f_{i,j}^{(\sigma)} = \alpha_{i,j}(\frac{1}{2}, \sigma)$, and $g_{i,j}^{(\sigma)} = \beta_{i,j}(\frac{1}{2}, \sigma)$. It is possible to show that

$$w_i(x) = \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} g_{i,i-2j}^{(\sigma)} e_{i-2j}(x)$$

with $g_{i,0}^{(\sigma)} = 0$ and $f_{i,0}^{(\sigma)} = 0$. Then

$$\mathrm{Lin}\big\{ e_{i-2j}(x) | j = 0, ..., \lfloor \frac{i-1}{2} \rfloor \big\} = \mathrm{Lin}\big\{ w_{i-2j}(x) | j = 0, ..., \lfloor \frac{i-1}{2} \rfloor \big\},$$

and we can write

$$e_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} f_{n,n-2j}^{(\sigma)} w_{n-2j}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} f_{n,n-2j}^{(\sigma)} \Delta_{1/2}^{\sigma} p_{n+\sigma-2j}(x)$$

and obtain

$$\Sigma_1^{(\sigma)} K^n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} f_{n,n-2j}^{(\sigma)} \Sigma_1^{(\sigma)} \Delta_{1/2}^{\sigma} p_{n+\sigma-2j}(K).$$

For example, for $\sigma = 2$ we have

$$\begin{aligned} \Sigma_1^{(2)} K^n &= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} f_{n,n-2j}^{(2)} \Sigma_1^{(2)} \Delta_{1/2}^{2} p_{n+2-2j}(K) \\ &= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} f_{n,n-2j}^{(2)} (K+1)^{n-2\lfloor \frac{n}{2} \rfloor} \Big[K(K+2) \Big]^{\lfloor \frac{n}{2} \rfloor + 1 - j}. \end{aligned}$$

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