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ADDITIVITY OF MAPS ON TRIANGULAR ALGEBRAS*

XUEHAN CHENG[†] AND WU JING[‡]

Abstract. In this paper, it is proven that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map, and elementary surjective map on triangular algebras is automatically additive.

Key words. Multiplicative maps, Jordan maps, Jordan triple maps, Elementary maps, Triangular algebras, Standard operator algebras, Additivity.

AMS subject classifications. 16W99, 47B49, 47L10.

1. Introduction and Preliminaries. If a ring $\mathcal{R}$ contains a nontrivial idempotent, it is kind of surprising that every multiplicative bijective map from $\mathcal{R}$ onto an arbitrary ring is automatically additive. This result was given by Martindale III in his excellent paper [10]. More precisely, he proved:

THEOREM 1.1. ([10]) *Let $\mathcal{R}$ be a ring containing a family $\{e_\alpha : \alpha \in \Lambda\}$ of idempotents which satisfies:*

(i) $x\mathcal{R} = \{0\}$ *implies* $x = 0$,
(ii) *If $e_\alpha \mathcal{R} x = \{0\}$ for each $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathcal{R}x = \{0\}$ implies $x = 0$),*
(iii) *For each $\alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{R}(1 - e_\alpha) = \{0\}$ implies $e_\alpha x e_\alpha = 0$.*

Then any multiplicative bijective map from $\mathcal{R}$ onto an arbitrary ring $\mathcal{R}'$ is additive.

Note that the proof of [10] has become a standard argument and been applied widely by several authors in investigating the additivity of maps on rings as well as on operator algebras (see [4]-[9]). Following this standard argument (see [10]), in this paper we continue to study the additivity of maps on triangular algebras. We will show that every multiplicative bijective map, Jordan bijective map, Jordan triple bijective map, and elementary surjective map on triangular algebras is additive.

We now introduce some definitions and results.

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[†]College of Mathematics & Information, Ludong University, Yantai, 264025, P. R. China. Supported by NNSF of China (No. 10671086) and NSF of Ludong University (No. LY20062704).

[‡]Department of Mathematics & Computer Science, Fayetteville State University, Fayetteville, NC 28301, USA (wjing@uncfsu.edu).

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Definition 1.2. Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings, and ϕ be a map from $\mathcal{R}$ to $\mathcal{R}'$. Suppose that a, b, and c are arbitrary elements of $\mathcal{R}$.

(i) ϕ is said to be *multiplicative* if

$$\phi(ab) = \phi(a)\phi(b).$$

(ii) ϕ is called a *Jordan map* if

$$\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a).$$

(iii) ϕ is called a *Jordan triple map* if

$$\phi(abc + cba) = \phi(a)\phi(b)\phi(c) + \phi(c)\phi(b)\phi(a).$$

It was proved in [9] that if $\mathcal{A}$ is a unital prime algebra containing a nontrivial idempotent, or $\mathcal{A}$ is a unital algebra which has a system of matrix units, or $\mathcal{A}$ is a standard operator algebra on a Banach space, then every bijective Jordan map on $\mathcal{A}$ is additive. Lu also showed in [8] that each bijective Jordan triple map on a standard operator in a Banach space is additive.

Definition 1.3. ([2]) Let $\mathcal{R}$ and $\mathcal{R}'$ be two rings. Suppose that $M: \mathcal{R} \to \mathcal{R}'$ and $M^*: \mathcal{R}' \to \mathcal{R}$ are two maps. Call the ordered pair (M, M^*) an *elementary map* of $\mathcal{R} \times \mathcal{R}'$ if

$$\begin{cases} M(aM^*(x)b) = M(a)xM(b), \\ M^*(xM(a)y) = M^*(x)aM^*(y) \end{cases}$$

for all $a, b \in \mathcal{R}$ and $x, y \in \mathcal{R}'$.

Elementary maps were originally introduced by Brešar and Šerml in their nice paper [2]. There are many examples of elementary maps. It is obvious that, if $\phi: \mathcal{R} \to \mathcal{R}'$ is an isomorphism, then (ϕ, ϕ^{-1}) is an elementary map on $\mathcal{R} \times \mathcal{R}'$. For $a, b \in \mathcal{R}$, let $M_{a,b}(x) = axb$ for $x \in \mathcal{R}$. Then one can verify that $(M_{a,b}, M_{b,a})$ is an elementary map on $\mathcal{R} \times \mathcal{R}$. The additivity of elementary maps on operator algebras were studied in [2], [1], [6], and [11]. Li and Lu ([6]) also studied the additivity of elementary maps on prime rings. It was proved in [5] that if (M, M^*) is a Jordan elementary map of $\mathcal{R} \times \mathcal{R}'$, where $\mathcal{R}$ is a 2-torsion free prime ring containing a nontrivial idempotent and $\mathcal{R}'$ is an arbitrary ring, then both M and M^* are additive.

Recall that a *triangular algebra* $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is an algebra of the form

$$Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}$$

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under the usual matrix operations, where $\mathcal{A}$ and $\mathcal{B}$ are two algebras over a commutative ring $\mathcal{R}$, and $\mathcal{M}$ is an $(\mathcal{A},\mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module (see [3]).

Let $\mathcal{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra. Throughout this paper, we set

$$\mathcal{T}_{11} = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathcal{A}\},$$

$$\mathcal{T}_{12} = \{\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : m \in \mathcal{M}\},$$

and

$$\mathcal{T}_{22} = \{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in \mathcal{B}\}.$$

Then we may write $\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}$, and every element $a \in \mathcal{T}$ can be written as $a = a_{11} + a_{12} + a_{22}$. Note that notation a_{ij} denotes an arbitrary element of $\mathcal{T}_{ij}$. It should be mentioned here that this special structure of triangular algebras enables us to borrow the idea of [10] while we do not require the existence of nontrivial idempotents.

Let X be a Banach space. We denote by $B(X)$ the algebra of all bounded linear operators on X. A subalgebra $\mathcal{A}$ of $B(X)$ is called a *standard operator algebra* if $\mathcal{A}$ contains all finite rank operators. Note that if $A \in \mathcal{A}$ and $A\mathcal{A} = \{0\}$ (or $\mathcal{A}A = \{0\}$), then $A = 0$.

2. Additivity of Multiplicative Maps. The aim of this section is to study the additivity of multiplicative maps on triangular algebras. We now state our first main result.

THEOREM 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a commutative ring $\mathcal{R}$, $\mathcal{M}$ be a faithful $(\mathcal{A},\mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$. Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ satisfy:*

(i) *For $a \in \mathcal{A}$, if $a\mathcal{A} = \{0\}$, or $\mathcal{A}a = \{0\}$, then $a = 0$,*
(ii) *For $b \in \mathcal{B}$, if $b\mathcal{B} = \{0\}$, or $\mathcal{B}b = \{0\}$, then $b = 0$,*
(iii) *For $m \in \mathcal{M}$, if $\mathcal{A}m = \{0\}$, or $m\mathcal{B} = \{0\}$, then $m = 0$.*

Then any multiplicative bijective map from $\mathcal{T}$ onto an arbitrary ring $\mathcal{R}'$ is additive.

The proof of this theorem is organized into a series of lemmas. In what follows, ϕ will be a multiplicative bijective map from $\mathcal{T}$ onto an arbitrary ring $\mathcal{R}'$.

LEMMA 2.2. $\phi(0) = 0$.

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Proof. Since ϕ is surjective, there exists $a \in \mathcal{T}$ such that $\phi(a) = 0$. Then $\phi(0) = \phi(0 \cdot a) = \phi(0)\phi(a) = \phi(0) \cdot 0 = 0$. □

LEMMA 2.3. *For any $a_{11} \in \mathcal{T}_{11}$ and $b_{12} \in \mathcal{T}_{12}$, we have*

$$\phi(a_{11} + b_{12}) = \phi(a_{11}) + \phi(b_{12}).$$

Proof. Let $c \in \mathcal{T}$ be chosen such that $\phi(c) = \phi(a_{11}) + \phi(b_{12})$.

For arbitrary $t_{11} \in \mathcal{T}_{11}$, we consider

$$\begin{aligned}\phi(ct_{11}) &= \phi(c)\phi(t_{11}) = (\phi(a_{11}) + \phi(b_{12}))\phi(t_{11}) \\ &= \phi(a_{11})\phi(t_{11}) + \phi(b_{12})\phi(t_{11}) = \phi(a_{11}t_{11}).\end{aligned}$$

Hence, $ct_{11} = a_{11}t_{11}$, and so $c_{11} = a_{11}$.

Similarly, we can get $c_{22} = 0$.

We now show that $c_{12} = b_{12}$. For any $t_{11} \in \mathcal{T}_{11}$ and $s_{22} \in \mathcal{T}_{22}$, we obtain

$$\phi(t_{11}cs_{22}) = \phi(t_{11})\phi(c)\phi(s_{22}) = \phi(t_{11})(\phi(a_{11}) + \phi(b_{12}))\phi(s_{22}) = \phi(t_{11}b_{12}s_{22}).$$

It follows that $t_{11}cs_{22} = t_{11}b_{12}s_{22}$, which gives us that $c_{12} = b_{12}$. □

Similarly, we have the following lemma.

LEMMA 2.4. *For arbitrary $a_{22} \in \mathcal{T}_{22}$ and $b_{12} \in \mathcal{T}_{12}$, the following holds true.*

$$\phi(a_{22} + b_{12}) = \phi(a_{22}) + \phi(b_{12}).$$

LEMMA 2.5. *For any $a_{11} \in \mathcal{T}_{11}$, $b_{12}, c_{12} \in \mathcal{T}_{12}$, and $d_{22} \in \mathcal{T}_{22}$, we have*

$$\phi(a_{11}b_{12} + c_{12}d_{22}) = \phi(a_{11}b_{12}) + \phi(c_{12}d_{22}).$$

Proof. By Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}&\phi(a_{11}b_{12} + c_{12}d_{22}) = \phi((a_{11} + c_{12})(b_{12} + d_{22})) \\ &= \phi(a_{11} + c_{12})\phi(b_{12} + d_{22}) = (\phi(a_{11}) + \phi(c_{12}))(\phi(b_{12}) + \phi(d_{22})) \\ &= \phi(a_{11})\phi(b_{12}) + \phi(a_{11})\phi(d_{22}) + \phi(c_{12})\phi(b_{12}) + \phi(c_{12})\phi(d_{22}) \\ &= \phi(a_{11}b_{12}) + \phi(c_{12}d_{22}).\end{aligned}$$ □

LEMMA 2.6. *ϕ is additive on $\mathcal{T}_{12}$.*

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Proof. Suppose that a_{12} and b_{12} are two elements of $\mathcal{T}$. We pick $c \in \mathcal{T}$ such that $\phi(c) = \phi(a_{12}) + \phi(b_{12})$. For any $t_{11} \in \mathcal{T}_{11}$ and $s_{22} \in \mathcal{T}_{22}$, we compute

$$\begin{aligned}\phi(t_{11}cs_{22}) &= \phi(t_{11})\phi(c)\phi(s_{22}) = \phi(t_{11})(\phi(a_{12}) + \phi(b_{12}))\phi(s_{22})\\ &= \phi(t_{11})\phi(a_{12})\phi(s_{22}) + \phi(t_{11})\phi(b_{12})\phi(s_{22}) = \phi(t_{11}a_{12}s_{22} + t_{11}b_{12}s_{22}).\end{aligned}$$

Note that in the last equality we apply Lemma 2.5. It follows that $t_{11}cs_{22} = t_{11}a_{12}s_{22} + t_{11}b_{12}s_{22}$, and so $c_{12} = a_{12} + b_{12}$.

We can get $c_{11} = c_{22} = 0$ by considering $\phi(ct_{11})$ and $\phi(t_{22}c)$ for arbitrary $t_{11} \in \mathcal{T}_{11}$ and $t_{22} \in \mathcal{T}_{22}$ respectively. □

Lemma 2.7. *ϕ is additive on $\mathcal{T}_{11}$.*

Proof. For $a_{11}, b_{11} \in \mathcal{T}_{11}$, let $c \in \mathcal{T}$ be chosen such that $\phi(c) = \phi(a_{11}) + \phi(b_{11})$. We only show that $c_{11} = a_{11} + b_{11}$. One can easily get $c_{22} = c_{12} = 0$ by considering $\phi(t_{22}c)$ and $\phi(t_{11}cs_{22})$ for any $t_{11} \in \mathcal{T}_{11}$ and $t_{22}, s_{22} \in \mathcal{T}_{22}$.

For any $t_{12} \in \mathcal{T}_{12}$, using Lemma 2.6, we get

$$\begin{aligned}\phi(ct_{12}) &= \phi(c)\phi(t_{12}) = (\phi(a_{11}) + \phi(b_{11}))\phi(t_{12})\\ &= \phi(a_{11}t_{12}) + \phi(b_{11}t_{12}) = \phi(a_{11}t_{12} + b_{11}t_{12}),\end{aligned}$$

which leads to $ct_{12} = a_{11}t_{12} + b_{11}t_{12}$. Accordingly, $c_{11} = a_{11} + b_{11}$. □

Proof of Theorem 2.1: Suppose that a and b are two arbitrary elements of $\mathcal{T}$. We choose an element $c \in \mathcal{T}$ such that $\phi(c) = \phi(a) + \phi(b)$. For any $t_{11} \in \mathcal{T}_{11}$ and $s_{22} \in \mathcal{T}_{22}$, using Lemma 2.6, we obtain

$$\begin{aligned}\phi(t_{11}cs_{22}) &= \phi(t_{11})\phi(c)\phi(s_{22}) = \phi(t_{11})(\phi(a) + \phi(b))\phi(s_{22})\\ &= \phi(t_{11})\phi(a)\phi(s_{22}) + \phi(t_{11})\phi(b)\phi(s_{22})\\ &= \phi(t_{11}as_{22}) + \phi(t_{11}bs_{22}) = \phi(t_{11}as_{22} + t_{11}bs_{22}).\end{aligned}$$

Consequently, $c_{12} = a_{12} + b_{12}$.

Since ϕ is additive on $\mathcal{T}_{11}$, we can get $c_{11} = a_{11} + b_{11}$ from $\phi(ct_{11}) = \phi(at_{11}) + \phi(bt_{11}) = \phi(at_{11} + bt_{11})$.

In the similar manner, one can get $c_{22} = a_{22} + b_{22}$. The proof is complete. □

If algebras $\mathcal{A}$ and $\mathcal{B}$ contain identities, then we have the following result.

Corollary 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras over a commutative ring $\mathcal{R}$, $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Then any multiplicative bijective map from $\mathcal{R}$ onto an arbitrary ring $\mathcal{R}'$ is additive.*

We end this section with the case when $\mathcal{A}$ and $\mathcal{B}$ are standard operator algebras.

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Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras over a Banach space X, $\mathcal{M}$ be a faithful $(\mathcal{A},\mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$. Then any multiplicative bijective map from $\mathcal{R}$ onto an arbitrary ring $\mathcal{R}'$ is additive.*

3. Additivity of Jordan (Triple) Maps. In this section we deal with Jordan maps and Jordan triple maps on triangular algebras.

Throughout this section, $\mathcal{T} = Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ will be a triangular algebra, where $\mathcal{A}$, $\mathcal{B}$ are two algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule satisfying:

(i) If $a \in \mathcal{A}$ and $ax + xa = 0$ for all $x \in \mathcal{A}$, then $a = 0$,
(ii) If $b \in \mathcal{B}$ and $by + yb = 0$ for all $y \in \mathcal{B}$, then $b = 0$,
(iii) For $m \in \mathcal{M}$, if $\mathcal{A}m = \{0\}$, or $m\mathcal{B} = \{0\}$, then $m = 0$.

Map ϕ is a Jordan bijective map from $\mathcal{T}$ onto an arbitrary ring $\mathcal{R}'$.

We begin with the following lemma.

Lemma 3.1. $\phi(0) = 0$

Proof. Pick $a \in \mathcal{T}$ such that $\phi(a) = 0$. Then $\phi(0) = \phi(a \cdot 0 + 0 \cdot a) = \phi(a)\phi(0) + \phi(0)\phi(a) = 0$. □

Lemma 3.2. *Suppose that $a, b, c \in \mathcal{T}$ satisfying $\phi(c) = \phi(a) + \phi(b)$, then for any $t \in \mathcal{T}$*

$$\phi(tc + ct) = \phi(ta + at) + \phi(tb + bt).$$

Proof. Multiplying $\phi(c) = \phi(a) + \phi(b)$ by $\phi(t)$ from the left and the right respectively and adding them together, one can easily get $\phi(tc+ct) = \phi(ta+at)+\phi(tb+bt)$. □

Lemma 3.3. *For any $a_{11} \in \mathcal{T}_{11}$ and $b_{12} \in \mathcal{T}_{12}$, we have*

$$\phi(a_{11} + b_{12}) = \phi(a_{11}) + \phi(b_{12}).$$

Proof. Let $c \in \mathcal{T}$ be chosen such that $\phi(c) = \phi(a_{11}) + \phi(b_{12})$. Now for any $t_{22} \in \mathcal{T}_{22}$, by Lemma 3.2, we have

$$\phi(t_{22}c + ct_{22}) = \phi(t_{22}a_{11} + a_{11}t_{22}) + \phi(t_{22}b_{12} + b_{12}t_{22}) = \phi(b_{12}t_{22}).$$

It follows that $t_{22}c + ct_{22} = b_{12}t_{22}$, i.e., $t_{22}c_{22} + c_{12}t_{22} + c_{22}t_{22} = b_{12}t_{22}$. This implies that $c_{12}t_{22} = b_{12}t_{22}$ and $t_{22}c_{22} + c_{22}t_{22} = 0$, and so $c_{12} = b_{12}$ and $c_{22} = 0$.

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From

$$\phi(t_{12}c + ct_{12}) = \phi(t_{12}a_{11} + a_{11}t_{12}) + \phi(t_{12}b_{12} + b_{12}t_{12}) = \phi(a_{11}t_{12}),$$

one can get $c_{11} = a_{11}$. □

Similarly, we have the following lemma.

LEMMA 3.4. *For arbitrary $a_{12} \in \mathcal{T}_{12}$ and $b_{22} \in \mathcal{T}_{22}$, the following is true.*

$$\phi(a_{12} + b_{22}) = \phi(a_{12}) + \phi(b_{22}).$$

LEMMA 3.5. $\phi(a_{11}b_{12} + c_{12}d_{22}) = \phi(a_{11}b_{12}) + \phi(c_{12}d_{22})$ *holds true for any $a_{11} \in \mathcal{T}_{11}$, $b_{12}, c_{12} \in \mathcal{T}_{12}$, and $d_{22} \in \mathcal{T}_{22}$.*

Proof. By Lemma 3.3 and Lemma 3.4, we compute

$$\begin{aligned}
&\phi(a_{11}b_{12} + c_{12}d_{22})\\
&= \phi((a_{11} + c_{12})(b_{12} + d_{22}) + (b_{12} + d_{22})(a_{11} + c_{12}))\\
&= \phi(a_{11} + c_{12})\phi(b_{12} + d_{22}) + \phi(b_{12} + d_{22})\phi(a_{11} + c_{12})\\
&= (\phi(a_{11}) + \phi(c_{12}))(\phi(b_{12}) + \phi(d_{22})) + (\phi(b_{12}) + \phi(d_{22}))(\phi(a_{11}) + \phi(c_{12}))\\
&= \phi(a_{11}b_{12} + b_{12}a_{11}) + \phi(a_{11}d_{22} + d_{22}a_{11}) + \phi(c_{12}b_{12} + b_{12}c_{12}) + \phi(c_{12}d_{22} + d_{22}c_{12})\\
&= \phi(a_{11}b_{12}) + \phi(c_{12}d_{22}).
\end{aligned}$$ □

LEMMA 3.6. *ϕ is additive on $\mathcal{T}_{12}$.*

Proof. Let a_{12} and b_{12} be any two elements of $\mathcal{T}_{12}$. Since ϕ is surjective, there exists a $c \in \mathcal{T}$ such that $\phi(c) = \phi(a_{12}) + \phi(b_{12})$.

Now for any $t_{22} \in \mathcal{T}_{22}$, by Lemma 3.2, we obtain

$$\phi(t_{22}c + ct_{22}) = \phi(t_{22}a_{12} + a_{12}t_{22}) + \phi(t_{22}b_{12} + b_{12}t_{22}) = \phi(a_{12}t_{22}) + \phi(b_{12}t_{22}).$$

Again, using Lemma 3.2, for any $s_{11} \in \mathcal{T}_{11}$, we have

$$\begin{aligned}
&\phi(s_{11}(t_{22}c + ct_{22}) + (t_{22}c + ct_{22})s_{11})\\
&= \phi(s_{11}a_{12}t_{22} + a_{12}t_{22}s_{11}) + \phi(s_{11}b_{12}t_{22} + b_{12}t_{22}s_{11})\\
&= \phi(s_{11}a_{12}t_{22}) + \phi(s_{11}b_{12}t_{22}) = \phi(s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}).
\end{aligned}$$

In the last equality we apply Lemma 3.5. It follows that

$$s_{11}c_{12}t_{22} = s_{11}a_{12}t_{22} + s_{11}b_{12}t_{22}.$$

This gives us $c_{12} = a_{12} + b_{12}$.

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To show $c_{11} = 0$, we first consider $\phi(t_{11}c + ct_{11})$ for any $t_{11} \in \mathcal{T}_{11}$. We have

$$\phi(t_{11}c + ct_{11}) = \phi(t_{11}a_{12} + a_{12}t_{11}) + \phi(t_{11}b_{12} + b_{12}t_{11}) = \phi(t_{11}a_{12}) + \phi(t_{11}b_{12}).$$

Furthermore, for arbitrary $s_{12} \in \mathcal{T}_{12}$,

$$\begin{aligned} &\phi(s_{12}(t_{11}c + ct_{11}) + (t_{11}c + ct_{11})s_{12}) \\ &= \phi(s_{12}t_{11}a_{12} + t_{11}a_{12}s_{12}) + \phi(s_{12}t_{11}b_{12} + t_{11}b_{12}s_{12}) = 0. \end{aligned}$$

This implies that $t_{11}c_{11}s_{12} + c_{11}t_{11}s_{12} = 0$, and so $c_{11} = 0$.

Note that $\phi(t_{12}c+ct_{12}) = \phi(t_{12}a_{12}+a_{12}t_{12})+\phi(t_{12}b_{12}+b_{12}t_{12}) = 0$. Now, $c_{22} = 0$ follows easily. □

Lemma 3.7. *ϕ is additive on $\mathcal{T}_{11}$.*

Proof. Suppose that a_{11} and b_{11} are two arbitrary elements of $\mathcal{T}_{11}$. Let $c \in \mathcal{T}$ be an element of $\mathcal{T}$ such that $\phi(c) = \phi(a_{11}) + \phi(b_{11})$.

For any $t_{22} \in \mathcal{T}_{22}$, we get

$$\phi(t_{22}c + ct_{22}) = \phi(t_{22}a_{11} + a_{11}t_{22}) + \phi(t_{22}b_{11} + b_{11}t_{22}) = 0.$$

Therefore, $t_{22}c + ct_{22} = 0$, which leads to $c_{12} = c_{22} = 0$.

Similarly, we can get $c_{11} = a_{11} + b_{11}$ from

$$\begin{aligned} \phi(t_{12}c + ct_{12}) &= \phi(t_{12}a_{11} + a_{11}t_{12}) + \phi(t_{12}b_{11} + b_{11}t_{12}) \\ &= \phi(a_{11}t_{12}) + \phi(b_{11}t_{12}) = \phi(a_{11}t_{12} + b_{11}t_{12}). \end{aligned}$$ □

Lemma 3.8. *ϕ is additive on $\mathcal{T}_{22}$.*

Proof. For any $a_{22}, b_{22} \in \mathcal{T}_{22}$, by the surjectivity of ϕ, there is $c \in \mathcal{T}$ satisfying $\phi(c) = \phi(a_{22}) + \phi(b_{22})$.

Now, for any $t_{11} \in \mathcal{T}_{11}$, by Lemma 3.2, we have

$$\phi(t_{11}c + ct_{11}) = \phi(t_{11}a_{22} + a_{22}t_{11}) + \phi(t_{11}b_{22} + b_{22}c_{11}) = 0.$$

This implies that $t_{11}c_{11} + t_{11}c_{12} + c_{11}t_{11} = 0$. And so $c_{11} = 0$ and $c_{12} = 0$.

Similarly, we can get $c_{22} = a_{22} + b_{22}$ by considering $\phi(t_{12}c + ct_{12})$ for any $t_{12} \in \mathcal{T}_{12}$. □

Lemma 3.9. *For each $a_{11} \in \mathcal{T}_{11}$ and $b_{22} \in \mathcal{T}_{22}$, we have*

$$\phi(a_{11} + b_{22}) = \phi(a_{11}) + \phi(b_{22}).$$

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Proof. Since ϕ is surjective, we can pick $c \in \mathcal{T}$ such that $\phi(c) = \phi(a_{11}) + \phi(b_{22})$.

Considering $\phi(t_{11}c+ct_{11})$ and $\phi(t_{22}c+ct_{22})$ for arbitrary $t_{11} \in \mathcal{T}_{11}$ and $t_{22} \in \mathcal{T}_{22}$, one can infer that $c_{11} = a_{11}$, $c_{12} = 0$, and $c_{22} = b_{22}$. □

LEMMA 3.10. *For any $a_{11} \in \mathcal{T}_{11}$, $b_{12} \in \mathcal{T}_{12}$, and $c_{22} \in \mathcal{T}_{22}$,*

$$\phi(a_{11} + b_{12} + c_{22}) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{22}).$$

Proof. Let $d \in \mathcal{T}$ be chosen such that $\phi(d) = \phi(a_{11}) + \phi(b_{12}) + \phi(c_{22})$. On one hand, by Lemma 3.3, we have

$$\phi(d) = \phi(a_{11} + b_{12}) + \phi(c_{22}).$$

Now for any $t_{11} \in \mathcal{T}_{11}$, we obtain

$$\begin{aligned}\phi(t_{11}d + dt_{11}) &= \phi(t_{11}(a_{11} + b_{12}) + (a_{11} + b_{12})t_{11}) + \phi(t_{11}c_{22} + c_{22}t_{11}) \\ &= \phi(t_{11}a_{11} + t_{11}b_{12} + a_{11}t_{11}),\end{aligned}$$

which gives us

$$t_{11}d_{11} + t_{11}d_{12} + d_{11}t_{11} = t_{11}a_{11} + t_{11}b_{12} + a_{11}t_{11}.$$

Hence, $d_{11} = a_{11}$ and $d_{12} = b_{12}$.

On the other hand, by Lemma 3.9, we see that

$$\phi(d) = \phi(a_{11} + c_{22}) + \phi(b_{12}).$$

For any $t_{12} \in \mathcal{T}_{12}$, we have

$$\begin{aligned}\phi(t_{12}d + dt_{12}) &= \phi(t_{12}(a_{11} + c_{22}) + (a_{11} + c_{22})t_{12}) + \phi(t_{12}b_{12} + b_{12}t_{12}) \\ &= \phi(t_{12}c_{22} + a_{11}t_{12}).\end{aligned}$$

We can infer $d_{22} = c_{22}$ from the fact that $t_{12}d + dt_{12} = t_{12}c_{22} + a_{11}t_{12}$. □

We are in a position to prove the main result of this section.

THEOREM 3.11. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra, where $\mathcal{A}$, $\mathcal{B}$ are two algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ satisfy:*

(i) *If $a \in \mathcal{A}$ and $ax + xa = 0$ for all $x \in \mathcal{A}$, then $a = 0$,*

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(ii) *If $b \in \mathcal{B}$ and $by + yb = 0$ for all $y \in \mathcal{B}$, then $b = 0$,*
(iii) *For $m \in \mathcal{M}$, if $\mathcal{A}m = \{0\}$, or $m\mathcal{B} = \{0\}$, then $m = 0$.*

Let ϕ be a Jordan map from $\mathcal{T}$ to an arbitrary ring $\mathcal{R}'$, i.e., for any $s, t \in \mathcal{T}$,

$$\phi(st + ts) = \phi(s)\phi(t) + \phi(t)\phi(s).$$

If ϕ is bijective, then ϕ is additive.

Proof. For arbitrary s and t in $\mathcal{T}$. We write $s = s_{11}+s_{12}+s_{22}$ and $t = t_{11}+t_{12}+t_{22}$. We compute

$$\begin{aligned}
\phi(s+t) &= \phi((s_{11} + s_{12} + s_{22}) + (t_{11} + t_{12} + t_{22}))\\
&= \phi((s_{11} + t_{11}) + (s_{12} + t_{12}) + (s_{22} + t_{22}))\\
&= \phi(s_{11} + t_{11}) + \phi(s_{12} + t_{12}) + \phi(s_{22} + t_{22})\\
&= \phi(s_{11}) + \phi(t_{11}) + \phi(s_{12}) + \phi(t_{12}) + \phi(s_{22}) + \phi(t_{22})\\
&= (\phi(s_{11}) + \phi(s_{12}) + \phi(s_{22})) + (\phi(t_{11}) + \phi(t_{12}) + \phi(t_{22}))\\
&= \phi(s) + \phi(t).
\end{aligned}$$

The proof is complete. □

We only outline the proof of the following result as it is a modification of that of the related results for the case of Jordan mappings. We also want to mention here there the assumptions on algebras $\mathcal{A}$ and $\mathcal{B}$ in the following theorem are different from these in Theorem 3.11.

THEOREM 3.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a commutative ring $\mathcal{R}$, $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ satisfy:*

(i) *For $a \in \mathcal{A}$, if $a\mathcal{A} = \{0\}$, or $\mathcal{A}a = \{0\}$, then $a = 0$,*
(ii) *For $b \in \mathcal{B}$, if $b\mathcal{B} = \{0\}$, or $\mathcal{B}b = \{0\}$, then $b = 0$,*
(iii) *For $m \in \mathcal{M}$, if $\mathcal{A}m = \{0\}$, or $m\mathcal{B} = \{0\}$, then $m = 0$.*

Let ψ be a Jordan triple map from $\mathcal{T}$ to an arbitrary ring $\mathcal{R}'$, i.e., for any $r, s, t \in \mathcal{T}$,

$$\psi(rst + tsr) = \psi(r)\psi(s)\psi(t) + \psi(t)\psi(s)\psi(r).$$

If ψ is bijective, then ψ is additive.

Proof. We divide the proof into a series of steps.

Step 1. $\psi(0) = 0$.
We find $a \in \mathcal{T}$ such that $\psi(a) = 0$. Then $\psi(0) = \psi(0 \cdot a \cdot 0) + \psi(0 \cdot a \cdot 0) = \psi(0)\psi(a)\psi(0) + \psi(0)\psi(a)\psi(0) = 0$.

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Step 2. If $\psi(c) = \psi(a) + \psi(b)$, then

$$\psi(stc + cts) = \psi(sta + ats) + \psi(stb + bts)$$

holds true for all $s, t \in \mathcal{T}$.

One can get this easily by modifying the proof of Lemma 3.2.

Step 3. $\psi(a_{11} + b_{12}) = \psi(a_{11}) + \psi(b_{12})$ and $\psi(a_{12} + b_{22}) = \psi(a_{12}) + \psi(b_{22})$.

We only show $\psi(a_{11} + b_{12}) = \psi(a_{11}) + \psi(b_{12})$. Similarly we can get $\psi(a_{12} + b_{22}) = \psi(a_{12}) + \psi(b_{22})$.

Let $c \in \mathcal{T}$ be chosen such that $\psi(c) = \psi(a_{11}) + \psi(b_{12})$. Then by Step 2, for any $s, t \in \mathcal{T}$, we have

$$\psi(stc + cts) = \psi(sta_{11} + a_{11}ts) + \psi(stb_{12} + b_{12}ts). \tag{3.1}$$

Let $s = s_{11} \in \mathcal{T}_{11}$ and $t = t_{12} \in \mathcal{T}_{12}$ in the above equality, we have

$$\psi(s_{11}t_{12}c + ct_{12}s_{11}) = \psi(s_{11}t_{12}a_{11} + a_{11}t_{12}s_{11}) + \psi(s_{11}t_{12}b_{12} + b_{12}t_{12}s_{11}).$$

Consequently, $\psi(s_{11}t_{12}c_{22}) = 0$. Therefore, $s_{11}t_{12}c_{22} = 0$. Since s_{11} and t_{12} are arbitrary, we get $c_{22} = 0$. Note that here we use the fact that $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule.

Let $s = s_{22} \in \mathcal{T}_{22}$ and $t = t_{12} \in \mathcal{T}_{12}$, equality (3.1) turns to be

$$\psi(s_{22}t_{12}c + ct_{12}s_{22}) = \psi(s_{22}t_{12}a_{11} + a_{11}t_{12}s_{22}) + \psi(s_{22}t_{12}b_{12} + b_{12}t_{12}s_{22}),$$

this leads to $\psi(c_{11}t_{12}s_{22}) = \psi(a_{11}t_{12}s_{22})$. Therefore, $c_{11}t_{12}s_{22} = a_{11}t_{12}s_{22}$, which implies that $c_{11} = a_{11}$.

Now let $s = s_{22} \in \mathcal{T}_{22}$ and $t = t_{22} \in \mathcal{T}_{22}$ in identity (3.1), we get

$$\psi(s_{22}t_{22}c + ct_{22}s_{22}) = \psi(s_{22}t_{22}a_{11} + a_{11}t_{22}s_{22}) + \psi(s_{22}t_{22}b_{12} + b_{12}t_{22}s_{22}).$$

This yields that $\psi(c_{12}t_{22}s_{22}) = \psi(b_{12}t_{22}s_{22})$. Accordingly, $c_{12} = b_{12}$.

Step 4. $\psi(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) = \psi(t_{11}a_{11}b_{12}) + \psi(t_{11}c_{12}d_{22})$.

We compute

$$\begin{aligned}
&\psi(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22})\\
&= \psi(t_{11}(a_{11} + c_{12})(b_{12} + d_{22}) + (b_{12} + d_{22})(a_{11} + c_{12})t_{11})\\
&= \psi(t_{11})\psi(a_{11} + c_{12})\psi(b_{12} + d_{22}) + \psi(b_{12} + d_{22})\psi(a_{11} + c_{12})\psi(t_{11})\\
&= \psi(t_{11})(\psi(a_{11}) + \psi(c_{12}))(\psi(b_{12}) + \psi(d_{22}))\\
&\quad +(\psi(b_{12}) + \psi(d_{22}))(\psi(a_{11}) + \psi(c_{12}))\psi(t_{11})\\
&= \psi(t_{11}a_{11}b_{12} + b_{12}a_{11}t_{11}) + \psi(t_{11}a_{11}d_{22} + d_{22}a_{11}t_{11})\\
&\quad +\psi(t_{11}c_{12}b_{12} + b_{12}c_{12}t_{11}) + \psi(t_{11}c_{12}d_{22} + d_{22}c_{12}t_{11})\\
&= \psi(t_{11}a_{11}b_{12}) + \psi(t_{11}c_{12}d_{22}).
\end{aligned}$$

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Step 5. ψ is additive on $\mathcal{T}_{12}, \mathcal{T}_{11}$, and $\mathcal{T}_{22}$.

We only show that ψ is additive on $\mathcal{T}_{12}$. Choose $c \in \mathcal{T}$ such that $\psi(c) = \psi(a_{12}) + \psi(b_{12})$. For any $s_{11} \in \mathcal{T}_{11}$ and $t_{12} \in \mathcal{T}_{12}$, by Step 2, we have

$$\psi(s_{11}t_{12}c + ct_{12}s_{11}) = \psi(s_{11}t_{12}a_{12} + a_{12}t_{12}s_{11}) + \psi(s_{11}t_{12}b_{12} + b_{12}t_{12}s_{11}).$$

Then we get $\psi(s_{11}t_{12}c_{22}) = 0$, which leads to $s_{11}t_{12}c_{22} = 0$. Hence $c_{22} = 0$. Similarly, considering $s = s_{22} \in \mathcal{T}_{22}$ and $t = t_{12} \in \mathcal{T}_{12}$, we obtain $c_{11} = 0$. We now show that $c_{12} = a_{12} + b_{12}$.

For any $s = s_{22} \in \mathcal{T}_{22}$ and $t = t_{22} \in \mathcal{T}_{22}$, by Step 2, we have

$$\psi(c_{12}t_{22}s_{22}) = \psi(a_{12}t_{22}s_{22}) + \psi(b_{12}t_{22}s_{22}).$$

Applying Step 2 to the above identity for arbitrary $e_{11}, f_{11} \in \mathcal{T}_{11}$, we obtain

$$\begin{aligned} & \psi(e_{11}f_{11}c_{12}t_{22}s_{22} + c_{12}t_{22}s_{22}f_{11}e_{11}) \\ = & \psi(e_{11}f_{11}a_{12}t_{22}s_{22} + a_{12}t_{22}s_{22}f_{11}e_{11}) \\ & + \psi(e_{11}f_{11}b_{12}t_{22}s_{22} + b_{12}t_{22}s_{22}f_{11}e_{11}) \\ = & \psi(e_{11}f_{11}a_{12}t_{22}s_{22}) + \psi(e_{11}f_{11}b_{12}t_{22}s_{22}) \\ = & \psi(e_{11}f_{11}a_{12}t_{22}s_{22} + e_{11}f_{11}b_{12}t_{22}s_{22}). \end{aligned}$$

Note that in the last equality we apply Step 4. Now we obtain that

$$\psi(e_{11}f_{11}c_{12}t_{22}s_{22}) = \psi(e_{11}f_{11}a_{12}t_{22}s_{22} + e_{11}f_{11}b_{12}t_{22}s_{22}).$$

Therefore, we can infer that $c_{12} = a_{12} + b_{12}$.

Step 6. $\psi(a_{11} + b_{22}) = \psi(a_{11}) + \psi(b_{22})$.

Pick $c \in \mathcal{T}$ with $\psi(c) = \psi(a_{11}) + \psi(b_{22})$.

For any $s, t \in \mathcal{T}$, applying Step 2, we have

$$\psi(stc + cts) = \psi(sta_{11} + a_{11}ts) + \psi(stb_{22} + b_{22}ts). \tag{3.2}$$

Let $s = s_{22} \in \mathcal{T}_{22}$ and $t = t_{12} \in \mathcal{T}_{12}$, then above identity becomes

$$\psi(c_{11}t_{12}s_{22}) = \psi(a_{11}t_{12}s_{22}).$$

This yields that $c_{11} = a_{11}$.

Similarly, by letting $s = s_{11} \in \mathcal{T}_{11}$ and $t = t_{12} \in \mathcal{T}_{12}$, $s = s_{11} \in \mathcal{T}_{11}$ and $t = t_{11} \in \mathcal{T}_{11}$ in equality (3.2) respectively, we can get $c_{22} = b_{22}$ and $c_{12} = 0$.

Step 7. $\psi(a_{11} + b_{12} + c_{22}) = \psi(a_{11}) + \psi(b_{12}) + \psi(c_{22})$.

Similar to Step 6 and the proof of Lemma 3.10.

Step 8. ψ is additive.

The same as the proof of Theorem 3.11. □

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The following corollary follows directly if both $\mathcal{A}$ and $\mathcal{B}$ are unital.

Corollary 3.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras over a commutative ring $\mathcal{R}$, $\mathcal{M}$ be a be faithful $(\mathcal{A},\mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$. Suppose that ψ is a Jordan triple map from $\mathcal{T}$ to an arbitrary ring $\mathcal{R}'$. If ψ is bijective, then ψ is additive.*

Similar to Corollary 2.9, if both $\mathcal{A}$ and $\mathcal{B}$ are standard operator algebras, we have:

Corollary 3.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras over a Banach space X, $\mathcal{M}$ a be faithful $(\mathcal{A},\mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$. Then any Jordan triple bijective map from $\mathcal{R}$ onto an arbitrary ring $\mathcal{R}'$ is additive.*

4. Additivity of Elementary Maps.

In this section we will prove the following result about the additivity of elementary maps on triangular algebras.

Theorem 4.1. *Let $\mathcal{R}'$ be an arbitrary ring. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a commutative ring $\mathcal{R}$, $\mathcal{M}$ be a faithful $(\mathcal{A},\mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$. Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ satisfy:*

(i) *For $a\in\mathcal{A}$, if $a\mathcal{A}=\{0\}$, or $\mathcal{A}a=\{0\}$, then $a=0$,*
(ii) *For $b\in\mathcal{B}$, if $b\mathcal{B}=\{0\}$, or $\mathcal{B}b=\{0\}$, then $b=0$,*
(iii) *For $m\in\mathcal{M}$, if $\mathcal{A}m=\{0\}$, or $m\mathcal{B}=\{0\}$, then $m=0$.*

Suppose that (M,M^*) is an elementary map on $\mathcal{T}\times\mathcal{R}'$, and both M and M^* are surjective. Then both M and M^* are additive.

For the sake of clarity, we divide the proof into a series of lemmas. We begin with the following trivial one.

Lemma 4.2. *$M(0)=0$ and $M^*(0)=0$.*

Proof. We have $M(0)=M(0M^*(0)0)=M(0)0M(0)=0$.

Similarly, $M^*(0)=M^*(0M(0)0)=M^*(0)0M^*(0)=0$. ∎

The following result shows that both M and M^* are bijective.

Lemma 4.3. *Both M and M^* are injective.*

Proof. Suppose that $M(a)=M(b)$ for some a and b in $\mathcal{T}$. We write $a=a_{11}+a_{12}+a_{22}$ and $b=b_{11}+b_{12}+b_{22}$.

For arbitrary x and y in $\mathcal{R}'$, we have

$$M^*(x)aM^*(y)=M^*(xM(a)y)=M^*(xM(b)y)=M^*(x)bM^*(y).$$

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This, by the surjectivity of M^*, is equivalent to

$$sat = sbt \tag{4.1}$$

for arbitrary $s, t \in \mathcal{T}$.

In particular, letting $s = s_{11}, t = t_{11} \in \mathcal{T}_{11}$ in equality (4.1), we get $s_{11}a_{11}t_{11} = s_{11}b_{11}t_{11}$. And so, by condition (i) in Theorem 4.1, $a_{11} = b_{11}$.

Similarly, we can get $a_{22} = b_{22}$ by letting $s = s_{22}$ and $t = t_{22}$ in identity (4.1).

We now show that $a_{12} = b_{12}$. For any $s_{11} \in \mathcal{T}_{11}$ and $t_{22} \in \mathcal{T}_{22}$, then equality (4.1) becomes $s_{11}at_{22} = s_{11}bt_{22}$, i.e., $s_{11}a_{12}t_{22} = s_{11}b_{12}t_{22}$. Therefore $a_{12} = b_{12}$.

To complete the proof, it remains to show that M^* is injective. Let x and y be in $\mathcal{R}'$ such that $M^*(x) = M^*(y)$. Now for any $a, b \in \mathcal{T}$, we have

$$\begin{aligned}
&M^*M(a)M^{-1}(x)M^*M(b)\\
&= M^*(M(a)MM^{-1}(x)M(b)) = M^*(M(a)xM(b))\\
&= M^*M(aM^*(x)b) = M^*M(aM^*(y)b) = M^*(M(a)yM(b))\\
&= M^*(M(a)MM^{-1}(y)M(b)) = M^*M(a)M^{-1}(y)M^*M(b).
\end{aligned}$$

Thus

$$M^*M(a)M^{-1}(x)M^*M(b) = M^*M(a)M^{-1}(y)M^*M(b).$$

Equivalently,

$$sM^{-1}(x)t = sM^{-1}(y)t$$

for any $s, t \in \mathcal{T}$ since M^*M is surjective.

It follows from the same argument above that $M^{-1}(x) = M^{-1}(y)$, and so $x = y$, as desired. □

Lemma 4.4. *The pair $(M^{*^{-1}}, M^{-1})$ is an elementary map on $\mathcal{T} \times \mathcal{R}'$. That is,*

$$\begin{cases} M^{*^{-1}}(aM^{-1}(x)b) = M^{*^{-1}}(a)xM^{*^{-1}}(b),\\ M^{-1}(xM^{*^{-1}}(a)y) = M^{-1}(x)aM^{-1}(y) \end{cases}$$

for all $a, b \in \mathcal{T}$ and $x, y \in \mathcal{R}'$.

Proof. The first identity follows from the following observation.

$$M^*(M^{*^{-1}}(a)xM^{*^{-1}}(b)) = M^*(M^{*^{-1}}(a)MM^{-1}(x)M^{*^{-1}}(b)) = aM^{-1}(x)b.$$

The second one goes similarly. □

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The following result will be used frequently throughout this section.

Lemma 4.5. *Let $a, b, c \in \mathcal{T}$.*

(i) *If $M(c) = M(a) + M(b)$, then $M^{*^{-1}}(sct) = M^{*^{-1}}(sat) + M^{*^{-1}}(sbt)$ for all $s, t \in \mathcal{T}$,*

(ii) *If $M^{*^{-1}}(c) = M^{*^{-1}}(a) + M^{*^{-1}}(b)$, then $M(sct) = M(sat) + M(sbt)$ for all $s, t \in \mathcal{T}$.*

Proof. We only prove (i), and (ii) goes similarly.

By Lemma 4.4, we have

$$\begin{aligned} M^{*^{-1}}(sct) &= M^{*^{-1}}(sM^{-1}M(c)t) = M^{*^{-1}}(s)M(c)M^{*^{-1}}(t) \\ &= M^{*^{-1}}(s)(M(a) + M(b))M^{*^{-1}}(t) \\ &= (M^{*^{-1}}(s)M(a)M^{*^{-1}}(t)) + (M^{*^{-1}}(s)M(b)M^{*^{-1}}(t)) \\ &= M^{*^{-1}}(sat) + M^{*^{-1}}(sbt). \ \square \end{aligned}$$

Lemma 4.6. *Let $a_{11} \in \mathcal{T}_{11}$ and $b_{12} \in \mathcal{T}_{12}$, then*

(i) $M(a_{11} + b_{12}) = M(a_{11}) + M(b_{12})$,

(ii) $M^{*^{-1}}(a_{11} + b_{12}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{12})$.

Proof. We only prove (i). We choose $c \in \mathcal{T}$ such that $M(c) = M(a_{11}) + M(b_{12})$. For arbitrary $s_{11} \in \mathcal{T}_{11}$ and $t_{22} \in \mathcal{T}_{22}$, by Lemma 4.5, we have

$$M^{*^{-1}}(s_{11}ct_{22}) = M^{*^{-1}}(s_{11}a_{11}t_{22}) + M^{*^{-1}}(s_{11}b_{12}t_{22}) = M^{*^{-1}}(s_{11}b_{12}t_{22}).$$

It follows that $s_{11}ct_{22} = s_{11}b_{12}t_{22}$, i.e., $s_{11}c_{12}t_{22} = s_{11}b_{12}t_{22}$, which yields that $c_{12} = b_{12}$.

Now for any s_{11} and t_{11} in $\mathcal{T}_{11}$, we have

$$M^{*^{-1}}(s_{11}ct_{11}) = M^{*^{-1}}(s_{11}a_{11}t_{11}) + M^{*^{-1}}(s_{11}b_{12}t_{11}) = M^{*^{-1}}(s_{11}a_{11}t_{11}).$$

This implies that $c_{11} = a_{11}$.

Similarly, for any $s_{22}, t_{22} \in \mathcal{T}_{22}$, we obtain

$$M^{*^{-1}}(s_{22}ct_{22}) = M^{*^{-1}}(s_{22}a_{11}t_{22}) + M^{*^{-1}}(s_{22}b_{12}t_{22}) = 0.$$

Hence $c_{22} = 0$ follows from the fact that $s_{22}ct_{22} = 0$. $\square$

Similarly, we can get the following result.

Lemma 4.7. *Let $a_{22} \in \mathcal{T}_{22}$ and $b_{12} \in \mathcal{T}_{12}$, then*

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(i) $M(a_{22}+b_{12}) = M(a_{22}) + M(b_{12})$,
(ii) $M^{*^{-1}}(a_{22}+b_{12}) = M^{*^{-1}}(a_{22}) + M^{*^{-1}}(b_{12})$.

LEMMA 4.8. *For any $t_{11}, a_{11} \in \mathcal{T}_{11}$, $b_{12}, c_{12} \in \mathcal{T}_{12}$, and $d_{22} \in \mathcal{T}_{22}$, we have*

(i) $M(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) = M(t_{11}a_{11}b_{12}) + M(t_{11}c_{12}d_{22})$,
(ii) $M^{*^{-1}}(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) = M^{*^{-1}}(t_{11}a_{11}b_{12}) + M^{*^{-1}}(t_{11}c_{12}d_{22})$.

Proof. We only prove (i). Using Lemma 4.6 and Lemma 4.7, we compute

$$\begin{aligned}
& M(t_{11}a_{11}b_{12} + t_{11}c_{12}d_{22}) \\
&= M(t_{11}(a_{11}+c_{12})(b_{12}+d_{22})) \\
&= M(t_{11}M^*M^{*^{-1}}(a_{11}+c_{12})(b_{12}+d_{22})) \\
&= M(t_{11})M^{*^{-1}}(a_{11}+c_{12})M(b_{12}+d_{22}) \\
&= M(t_{11})M^{*^{-1}}(a_{11})M(b_{12}) + M(t_{11})M^{*^{-1}}(a_{11})M(d_{22}) \\
&\quad + M(t_{11})M^{*^{-1}}(c_{12})M(b_{12}) + M(t_{11})M^{*^{-1}}(c_{12})M(d_{22}) \\
&= M(t_{11})(M^{*^{-1}}(a_{11}) + M^{*^{-1}}(c_{12}))M(b_{12}) \\
&\quad + M(t_{11})(M^{*^{-1}}(a_{11}) + M^{*^{-1}}(c_{12}))M(d_{22}) \\
&= M(t_{11})M^{*^{-1}}(a_{11}+c_{12})M(b_{12}) + M(t_{11})M^{*^{-1}}(a_{11}+c_{12})M(d_{22}) \\
&= M(t_{11}(a_{11}+c_{12})b_{12}) + M(t_{11}(a_{11}+c_{12})d_{22}) \\
&= M(t_{11}a_{11}b_{12}) + M(t_{11}c_{12}d_{22}). \quad \square
\end{aligned}$$

LEMMA 4.9. *Both M and $M^{*^{-1}}$ are additive on $\mathcal{T}_{12}$.*

Proof. Let a_{12} and b_{12} be in $\mathcal{T}_{12}$. We pick $c \in \mathcal{T}$ such that $M(c) = M(a_{12}) + M(b_{12})$.

For arbitrary $t_{11}, s_{11} \in \mathcal{T}_{11}$, by Lemma 4.5, we have

$$M^{*^{-1}}(t_{11}cs_{11}) = M^{*^{-1}}(t_{11}a_{12}s_{11}) + M^{*^{-1}}(t_{11}b_{12}s_{11}) = 0,$$

this implies that $t_{11}cs_{11} = 0$, and so $c_{11} = 0$.

Similarly, we can get $c_{22} = 0$.

We now show that $c_{12} = a_{12} + b_{12}$. For any $t_{11}, r_{11} \in \mathcal{T}_{11}$ and $s_{22} \in \mathcal{T}_{22}$, by Lemma 4.5 and Lemma 4.8, we obtain

$$\begin{aligned}
M^{*^{-1}}(r_{11}t_{11}cs_{22}) &= M^{*^{-1}}(r_{11}t_{11}a_{12}s_{22}) + M^{*^{-1}}(r_{11}t_{11}b_{12}s_{22}) \\
&= M^{*^{-1}}(r_{11}t_{11}a_{12}s_{22} + r_{11}t_{11}b_{12}s_{22}) = M^{*^{-1}}(r_{11}t_{11}(a_{12}+b_{12})s_{22}).
\end{aligned}$$

It follows that

$$r_{11}t_{11}cs_{22} = r_{11}t_{11}(a_{12}+b_{12})s_{22}.$$

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Equivalently,

$$r_{11}t_{11}c_{12}s_{22} = r_{11}t_{11}(a_{12} + b_{12})s_{22}.$$

Then we get $c_{12} = a_{12} + b_{12}$.

With the similar argument, one can see that ${M^*}^{-1}$ is also additive on $\mathcal{T}_{12}$. □

LEMMA 4.10. *Both M and ${M^*}^{-1}$ are additive on $\mathcal{T}_{11}$.*

Proof. We only show the additivity of M on $\mathcal{T}_{11}$. Suppose that a_{11} and b_{11} are two elements of $\mathcal{T}_{11}$. Let $c \in \mathcal{T}$ be chosen satisfying $M(c) = M(a_{11}) + M(b_{11})$. Now for any $t_{22}, s_{22} \in \mathcal{T}_{22}$, by Lemma 4.5, we have

$${M^*}^{-1}(t_{22}cs_{22}) = {M^*}^{-1}(t_{22}a_{11}s_{22}) + {M^*}^{-1}(t_{22}b_{11}s_{22}) = 0.$$

Consequently, $t_{22}cs_{22} = 0$, i.e., $t_{22}c_{22}s_{22} = 0$, and so $c_{22} = 0$.

Similarly, we can infer that $c_{12} = 0$.

To complete the proof, we need to show that $c_{11} = a_{11} + b_{11}$. For each $t_{11} \in \mathcal{T}_{11}$ and $s_{12} \in \mathcal{T}_{12}$, by Lemma 4.5, we obtain

$${M^*}^{-1}(t_{11}cs_{12}) = {M^*}^{-1}(t_{11}a_{11}s_{12}) + {M^*}^{-1}(t_{11}b_{11}s_{12}) = {M^*}^{-1}(t_{11}a_{11}s_{12} + t_{11}b_{11}s_{12}).$$

Note that in the last equality we apply Lemma 4.9. It follows that

$$t_{11}cs_{12} = t_{11}a_{11}s_{12} + t_{11}b_{11}s_{12}.$$

This leads to $c_{11} = a_{11} + b_{11}$, as desired. □

LEMMA 4.11. *M and ${M^*}^{-1}$ are additive on $\mathcal{T}_{22}$.*

Proof. Suppose that a_{22} and b_{22} are in $\mathcal{T}_{22}$. We choose $c \in \mathcal{T}$ such that $M(c) = M(a_{22}) + M(b_{22})$. For any $t_{12} \in \mathcal{T}_{12}$ and $s_{22} \in \mathcal{T}_{22}$, using Lemma 4.5 and Lemma 4.9, we have

$${M^*}^{-1}(t_{12}cs_{22}) = {M^*}^{-1}(t_{12}a_{22}s_{22}) + {M^*}^{-1}(t_{12}b_{22}s_{22}) = {M^*}^{-1}(t_{12}(a_{22} + b_{22})s_{22}).$$

Accordingly, $t_{12}cs_{22} = t_{12}(a_{22} + b_{22})s_{22}$, which yields that $c_{22} = a_{22} + b_{22}$.

With the similar argument, we can verify that $c_{11} = c_{12} = 0$.

The additivity of ${M^*}^{-1}$ on $\mathcal{T}_{22}$ follows similarly. □

LEMMA 4.12. *For any $a_{11} \in \mathcal{T}_{11}$, $b_{12} \in \mathcal{T}_{12}$, and $c_{22} \in \mathcal{T}_{22}$, the following is true.*

$$M(a_{11} + b_{12} + c_{22}) = M(a_{11}) + M(b_{12}) + M(c_{22}).$$

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Proof. Let $d \in \mathcal{T}$ be an element satisfying $M(d) = M(a_{11} + b_{12}) + M(c_{22})$. For any $s, t \in \mathcal{T}$, using Lemma 4.5 , we arrive at

$$M^{*^{-1}}(sdt) = M^{*^{-1}}(s(a_{11} + b_{12})t) + M^{*^{-1}}(sc_{22}t). \tag{4.2}$$

Letting $s = s_{11}$ and $t = t_{11}$ in the above equality, we get $d_{11} = a_{11}$.

In the same fashion for $s = s_{22}$ and $t = t_{22}$ in equality (4.2), we can infer that $d_{22} = c_{22}$.

Finally, considering $s = s_{11}$ and $t = t_{22}$ in equality (4.2), we see that $d_{12} = b_{12}$. Thus, $d = a_{11} + b_{12} + c_{22}$. Then, by Lemma 4.6, we have

$$\begin{aligned} &M(a_{11} + b_{12} + c_{22}) = M(d) \\ &= M(a_{11} + b_{12}) + M(c_{22}) = M(a_{11}) + M(b_{12}) + M(c_{22}). \ \square \end{aligned}$$

We now prove our main result of this section.

Proof of Theorem 4.1 We first show the additivity of M. Let $a = a_{11} + a_{12} + a_{22}$ and $b = b_{11} + b_{12} + b_{22}$ be two arbitrary elements of $\mathcal{T}$. We have

$$\begin{aligned} &M(a + b) \\ &= M((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{22} + b_{22})) \\ &= M(a_{11} + b_{11}) + M(a_{12} + b_{12}) + M(a_{22} + b_{22}) \\ &= M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{22}) + M(b_{22}) \\ &= (M(a_{11}) + M(a_{12}) + M(a_{22})) + (M(b_{11}) + M(b_{12}) + M(b_{22})) \\ &= M(a_{11} + a_{12} + a_{22}) + M(b_{11} + b_{12} + b_{22}) = M(a) + M(b). \end{aligned}$$

That is, M is additive.

We now turn to prove that M^* is additive. For any $x, y \in \mathcal{R}'$, there exist $c = c_{11} + c_{12} + c_{22}$ and $d = d_{11} + d_{12} + d_{22}$ in $\mathcal{R}$ such that $c = M^*(x+y)$ and $d = M^*(x) + M^*(y)$.

For arbitrary $s, t \in \mathcal{T}$, by the additivity of M, we compute

$$\begin{aligned} M(sct) &= M(sM^*(x + y)t) = M(s)(x + y)M(t) \\ &= M(s)xM(t) + M(s)yM(t) = M(sM^*(x)t) + M(sM^*(y)t) \\ &= M(sM^*(x)t + sM^*(y)t) = M(s(M^*(x) + M^*(y))t) = M(sdt), \end{aligned}$$

which implies that $sct = sdt$. Consequently, we get $c = d$, i.e., $M^*(x + y) = M^*(x) + M^*(y)$. $\square$

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In particular, if both $\mathcal{A}$ and $\mathcal{B}$ are unital algebras, we have:

COROLLARY 4.13. *Let $\mathcal{R}'$ be an arbitrary ring. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras over a commutative ring $\mathcal{R}$, $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that (M, M^*) is an elementary map on $\mathcal{T} \times \mathcal{R}'$, and both M and M^* are surjective. Then both M and M^* are additive.*

We complete this note by considering elementary maps on triangular algebras provided $\mathcal{A}$ and $\mathcal{B}$ are standard operator algebras.

COROLLARY 4.14. *Let $\mathcal{R}'$ be an arbitrary ring. Let $\mathcal{A}$ and $\mathcal{B}$ be two standard operator algebras over a Banach space X, $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, and $\mathcal{T}$ be the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$. Suppose that (M, M^*) is an elementary map on $\mathcal{T} \times \mathcal{R}'$, and both M and M^* are surjective. Then both M and M^* are additive.*

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REFERENCES

[1] M. Brešar, L. Molnár, and P. Šerml. Elementary operators. *Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum*, 66:769–791, 2000.
[2] M. Brešar and P. Šerml. Elementary operators. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 129:1115–1135, 1999.
[3] W. S. Cheung. Commuting maps on triangular algebras. *Journal of the London Mathematical Society, Second Series*, 63:117–127, 2001.
[4] P. Ji. Jordan maps on triangular algebras. *Linear Algebra and its Applications*, 426:190–198, 2007.
[5] P. Li and W. Jing. Jordan elementary maps on rings. *Linear Algebra and its Applications*, 382:237–245, 2004.
[6] P. Li and F. Lu. Additivity of elementary maps on rings. *Communications in Algebra*, 32:3725–3737, 2004.
[7] F. Lu. Additivity of Jordan maps on standard operator algebras *Linear Algebra and its Applications*, 357:123–131, 2002.
[8] F. Lu. Jordan triple maps. *Linear Algebra and its Applications*, 375:311-317, 2003.
[9] F. Lu. Jordan maps on associative algebras. *Communications in Algebra*, 31:2273-2286, 2003.
[10] W. S. Martindale III. When are multiplicative mappings additive? *Proceedings of the American Mathematical Society*, 21:695–698, 1969.
[11] L. Molnár and P. Šerml. Elementary operators on standard operators. *Linear and Multilinear Algebra*, 50:315–319, 2002.