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SOME RESULTS ON GROUP INVERSES OF BLOCK MATRICES OVER SKEW FIELDS*

CHANGJIANG BU†, JIEMEI ZHAO†, AND KUIZE ZHANG†

Abstract. In this paper, necessary and sufficient conditions are given for the existence of the group inverse of the block matrix $\left(\begin{smallmatrix} A & A \\ B & 0 \end{smallmatrix}\right)$ over any skew field, where A, B are both square and $rank(B) \geq rank(A)$. The representation of this group inverse and some relative additive results are also given.

Key words. Skew, Block matrix, Group inverse.

AMS subject classifications. 15A09.

1. Introduction. Let K be a skew field and $K^{n\times n}$ be the set of all matrices over K. For $A \in K^{n\times n}$, the matrix $X \in K^{n\times n}$ is said to be the group inverse of A, if

$$AXA = A, XAX = X, AX = XA.$$

We then write $X = A^\sharp$. It is well known that if $A^\sharp$ exists, it is unique; see [16].

Research on representations of the group inverse of block matrices is an important effort in generalized inverse theory of matrices; see [14] and [13]. Indeed, generalized inverses are useful tools in areas such as special matrix theory, singular differential and difference equations and graph theory; see [5], [9] [11], [12] and [15]. For example, in [9] it is shown that the adjacency matrix of a bipartite graph can be written in the form of $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, and necessary and sufficient conditions are given for the existence and representation of the group inverse of a block matrix $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$.

In 1979, Campbell and Meyer proposed the problem of finding an explicit representation for the Drazin (group) inverse of a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in terms of its sub-blocks, where A and D are required to be square matrices; see [5]. In [10] a condition for the existence of the group inverse of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is given under the as-

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†Department of Applied Mathematics, College of Science, Harbin Engineering University, Harbin 150001, P. R. China (buchangjiang@hrbeu.edu.cn, zhaojiemei500@163.com, zkz0017@163.com).
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sumption that A and $(I+CA^{-2}B)$ are both invertible over any field; however, the representation of the group inverse is not given. The representation of the group inverse of a block matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ over skew fields has been given in 2001; see [6]. The representation of the Drazin (group) inverse of a block matrix of the form $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ (A is square, 0 is square null matrix) has not been given since it was proposed as a problem by Campbell in 1983; see [4]. However, there are some references in the literature about representations of the Drazin (group) inverse of the block matrices $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ under certain conditions. Some results are on matrices over the field of complex numbers, e.g., in [8]; or when $A = B = I_n$ in [7]; or when $A, B, C \in \{P, P^*, PP^*\}$, $P^2 = P$ and P^* is the conjugate transpose of P. Some results are over skew fields, e.g., in [1], when $A = I_n$ and $rank(CB)^2 = rank(B) = rank(C)$; in [3] when $A = B, A^2 = A$. In addition, in [2] results are given on the group inverse of the product of two matrices over a skew field, as well as some related properties.

In this paper, we mainly give necessary and sufficient conditions for the existence and the representation of the group inverse of a block matrix $\begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ or $\begin{pmatrix} A & B \\ A & 0 \end{pmatrix}$, where $A, B \in K^{n\times n}, rank(B) \geq rank(A)$. We also give a sufficient condition for AB to be similar to BA.

Letting $A \in K^{m\times n}$, the order of the maximum invertible sub-block of A is said to be the rank of A, denoted by $rank(A)$; see [17]. Let $A, B \in K^{n\times n}$. If there is an invertible matrix $P \in K^{n\times n}$ such that $B = PAP^{-1}$, then A and B are similar; see [17].

2. Some Lemmas.

LEMMA 2.1. *Let $A, B \in K^{n\times n}$. If $rank(A) = r$, $rank(B) = rank(AB) = rank(BA)$, then there are invertible matrices $P, Q \in K^{n\times n}$ such that*

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q,\ B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1},$$

where $B_1 \in K^{r\times r}$, $X \in K^{r\times(n-r)}$, and $Y \in K^{(n-r)\times r}$.

Proof. Since $rank(A) = r$, there are nonsingular matrices $P, Q \in K^{n\times n}$ such that

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q,\ B = Q^{-1}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}P^{-1},$$

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where $B_1 \in K^{r\times r}$, $B_2 \in K^{r\times(n-r)}$, $B_3 \in K^{(n-r)\times r}$, and $B_4 \in K^{(n-r)\times(n-r)}$. From $rank(B) = rank(AB)$, we have

$$B_3 = YB_1,\ B_4 = YB_2,\ Y \in K^{(n-r)\times r}.$$

Since $rank(B) = rank(BA)$, we obtain

$$B_2 = B_1X,\ B_4 = B_3X,\ X \in K^{r\times(n-r)}.$$

So

$$B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1}. \square$$

LEMMA 2.2. [6] *Let* $A \in K^{r\times r}$, $B \in K^{(n-r)\times r}$, $M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \in K^{n\times n}$. *Then the group inverse of* M *exists if and only if the group inverse of* A *exists and* $rank(A) = rank\begin{pmatrix} A \\ B \end{pmatrix}$. *If the group inverse of* M *exists, then*

$$M^\sharp = \begin{pmatrix} A^\sharp & 0 \\ B(A^\sharp)^2 & 0 \end{pmatrix}.$$

LEMMA 2.3. [6] *Let* $A \in K^{r\times r}$, $B \in K^{r\times(n-r)}$, $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in K^{n\times n}$. *Then the group inverse of* M *exists if and only if the group inverse of* A *exists and* $rank(A) = rank\begin{pmatrix} A & B \end{pmatrix}$. *If the group inverse of* M *exists, then*

$$M^\sharp = \begin{pmatrix} A^\sharp & (A^\sharp)^2B \\ 0 & 0 \end{pmatrix}.$$

LEMMA 2.4. [2] *Let* $A \in K^{m\times n}$, $B \in K^{n\times m}$. *If* $rank(A) = rank(BA)$, $rank(B) = rank(AB)$, *then the group inverse of* AB *and* BA *exist.*

LEMMA 2.5. *Let* $A, B \in K^{n\times n}$. *If* $rank(A) = rank(B) = rank(AB) = rank(BA)$, *then the following conclusions hold:*

(i) $AB(AB)^\sharp A = A$,
(ii) $A(BA)^\sharp BA = A$,
(iii) $BA(BA)^\sharp B = B$,
(iv) $B(AB)^\sharp A = BA(BA)^\sharp$,
(v) $A(BA)^\sharp = (AB)^\sharp A$.

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Proof. Suppose $rank(A) = r$. By Lemma 2.1, we have

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \; B = Q^{-1} \begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1},$$

where $B_1 \in K^{r\times r}$, $X \in K^{r\times(n-r)}$, $Y \in K^{(n-r)\times r}$. Then

$$AB = P \begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix} P^{-1}, \; BA = Q^{-1} \begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix} Q.$$

Since $rank(A) = rank(B)$, we have that B_1 is invertible. By using Lemma 2.2 and Lemma 2.3, we get

$$(AB)^\sharp = P \begin{pmatrix} {B_1}^{-1} & {B_1}^{-1}X \\ 0 & 0 \end{pmatrix} P^{-1}, \; (BA)^\sharp = Q^{-1} \begin{pmatrix} {B_1}^{-1} & 0 \\ Y{B_1}^{-1} & 0 \end{pmatrix} Q.$$

Then

(i) $AB(AB)^\sharp A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q = A,$

(ii) $A(BA)^\sharp BA = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q = A,$

(iii) $BA(BA)^\sharp B = Q^{-1} \begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1} = B,$

(iv) $B(AB)^\sharp A = Q^{-1} \begin{pmatrix} I_r & 0 \\ Y & 0 \end{pmatrix} Q = BA(BA)^\sharp,$

(v) $A(BA)^\sharp = P \begin{pmatrix} {B_1}^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q = (AB)^\sharp A.$ □

3. Conclusions.

THEOREM 3.1. *Let* $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$, *where* $A, B \in K^{n\times n}$, $rank(B) \geq rank(A)$ $= r$. *Then*

(i) The group inverse of M *exists if and only if* $rank(A) = rank(B) = rank(AB) = rank(BA)$.

(ii) If the group inverse of M *exists, then* $M^\sharp = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, *where*

$M_{11} = (AB)^\sharp A - (AB)^\sharp A^2(BA)^\sharp B,$
$M_{12} = (AB)^\sharp A,$
$M_{21} = (BA)^\sharp B - B(AB)^\sharp A^2(BA)^\sharp + B(AB)^\sharp A(AB)^\sharp A^2(BA)^\sharp B,$
$M_{22} = -B(AB)^\sharp A^2(BA)^\sharp.$

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Proof. (i) It is obvious that the condition is sufficient. Now we show that the condition is necessary.

$$rank(M) = rank \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = rank \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = rank(A) + rank(B),$$

$$rank(M^2) = rank \begin{pmatrix} A^2 + AB & A^2 \\ BA & BA \end{pmatrix} = rank \begin{pmatrix} AB & A^2 \\ 0 & BA \end{pmatrix}.$$

Since the group inverse of M exists if and only if $rank(M) = rank(M^2)$, we have

$$\begin{aligned} rank(A) + rank(B) &= rank(M^2) \\ &\leq rank(AB) + rank \begin{pmatrix} A^2 \\ BA \end{pmatrix} \\ &\leq rank(AB) + rank(A), \\ rank(A) + rank(B) &= rank(M^2) \\ &\leq rank \begin{pmatrix} AB & A^2 \end{pmatrix} + rank(BA) \\ &\leq rank(BA) + rank(A). \end{aligned}$$

Then $rank(B) \leq rank(AB)$, and $rank(B) \leq rank(BA)$. Therefore

$$rank(B) = rank(AB) = rank(BA).$$

From $rank(B) = rank(AB) \leq rank(A)$, and $rank(A) \leq rank(B)$, we have

$$rank(A) = rank(B).$$

Since $rank(A)+rank(B) \leq rank \begin{pmatrix} AB & A^2 \end{pmatrix}+rank(BA)$, and $rank \begin{pmatrix} AB & A^2 \end{pmatrix} \leq rank(A)$, we get $rank \begin{pmatrix} AB & A^2 \end{pmatrix} = rank(A)$. Thus

$$rank \begin{pmatrix} AB & A^2 \end{pmatrix} = rank(AB).$$

Then there exists a matrix $U \in K^{n\times n}$ such that $ABU = A^2$. Then

$$rank(M^2) = rank \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = rank(AB) + rank(BA).$$

So we get

$$rank(A) = rank(B) = rank(AB) = rank(BA).$$

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(ii) Let $X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$. We will prove that the matrix X satisfies the conditions of the group inverse. Firstly, we compute

$$MX = \begin{pmatrix} AM_{11} + AM_{21} & AM_{12} + AM_{22} \\ BM_{11} & BM_{12} \end{pmatrix},$$

$$XM = \begin{pmatrix} M_{11}A + M_{12}B & M_{11}A \\ M_{21}A + M_{22}B & M_{21}A \end{pmatrix}.$$

Applying Lemma 2.5 (i), (ii), and (v), we have

$$\begin{aligned} AM_{11} + AM_{21} &= A(AB)^\sharp A - A(AB)^\sharp A^2(BA)^\sharp B + A(BA)^\sharp B - AB(AB)^\sharp A^2(BA)^\sharp \\ &\quad + AB(AB)^\sharp A(AB)^\sharp A^2(BA)^\sharp B \\ &= A(BA)^\sharp B, \\ M_{11}A + M_{12}B &= (AB)^\sharp A^2 - (AB)^\sharp A^2(BA)^\sharp BA + (AB)^\sharp AB \\ &= (AB)^\sharp A^2 - (AB)^\sharp A^2 + (AB)^\sharp AB \\ &= A(BA)^\sharp B. \end{aligned}$$

Using Lemma 2.5 (i), (ii), and (v), we get

$$\begin{aligned} AM_{12} + AM_{22} &= A(AB)^\sharp A - AB(AB)^\sharp A^2(BA)^\sharp \\ &= A(AB)^\sharp A - A^2(BA)^\sharp \\ &= 0, \\ M_{11}A &= (AB)^\sharp A^2 - (AB)^\sharp A^2(BA)^\sharp BA \\ &= (AB)^\sharp A^2 - (AB)^\sharp A^2 \\ &= 0. \end{aligned}$$

From Lemma 2.5 (ii), we obtain

$$\begin{aligned} BM_{11} &= B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B, \\ M_{21}A + M_{22}B &= (BA)^\sharp BA - B(AB)^\sharp A^2(BA)^\sharp A + B(AB)^\sharp A^2(BA)^\sharp A(BA)^\sharp BA \\ &\quad - B(AB)^\sharp A^2(BA)^\sharp B \\ &= (BA)^\sharp BA - [B(AB)^\sharp A^2(BA)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp A] \\ &\quad - B(AB)^\sharp A^2(BA)^\sharp B \\ &= B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B. \end{aligned}$$

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Using Lemma 2.5 (ii), we have

$$\begin{aligned} BM_{12} &= B(AB)^\sharp A, \\ M_{21}A &= (BA)^\sharp BA - B(AB)^\sharp A^2(BA)^\sharp A + B(AB)^\sharp A^2(BA)^\sharp A(BA)^\sharp BA \\ &= B(AB)^\sharp A. \end{aligned}$$

So

$$MX = XM = \begin{pmatrix} A(BA)^\sharp B & 0 \\ B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B & B(AB)^\sharp A \end{pmatrix}.$$

Secondly,

$$\begin{aligned} MXM &= \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} \begin{pmatrix} A(BA)^\sharp B & 0 \\ B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B & B(AB)^\sharp A \end{pmatrix} \\ &= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & 0 \end{pmatrix}. \end{aligned}$$

Applying Lemma 2.5 (i) and (iii), we compute

$$\begin{aligned} X_{11} &= A^2(BA)^\sharp B + AB(AB)^\sharp A - AB(AB)^\sharp A^2(BA)^\sharp B \\ &= AB(AB)^\sharp A \\ &= A, \\ X_{12} &= AB(AB)^\sharp A = A, \\ X_{21} &= BA(BA)^\sharp B = B. \end{aligned}$$

Hence

$$MXM = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}.$$

Finally,

$$\begin{aligned} XMX &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A(BA)^\sharp B & 0 \\ B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B & B(AB)^\sharp A \end{pmatrix} \\ &= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} Y_{11} &= (AB)^\sharp A^2(BA)^\sharp B - (AB)^\sharp A^2(BA)^\sharp BA(BA)^\sharp B + (AB)^\sharp AB(AB)^\sharp A \\ &\quad - (AB)^\sharp AB(AB)^\sharp A^2(BA)^\sharp B \\ &= (AB)^\sharp A - (AB)^\sharp A^2(BA)^\sharp B \\ &= M_{11}, \end{aligned}$$

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and

$$\begin{aligned} Y_{12} &= M_{12}B(AB)^{\sharp}A \\ &= (AB)^{\sharp}AB(AB)^{\sharp}A \\ &= (AB)^{\sharp}A \\ &= M_{12}. \end{aligned}$$

We can easily get

$$\begin{aligned} Y_{21} &= M_{21}A(BA)^{\sharp}B + M_{22}B(AB)^{\sharp}A - M_{22}B(AB)^{\sharp}A^2(BA)^{\sharp}B \\ &= M_{21}; \\ Y_{22} &= M_{22}B(AB)^{\sharp}A = M_{22}. \end{aligned}$$

So we have $X = M^{\sharp}$. ∎

THEOREM 3.2. *Let* $M = \begin{pmatrix} A & B \\ A & 0 \end{pmatrix}$, *where* $A, B \in K^{n\times n}$, $rank(B) \geq rank(A)$ $= r$. *Then*

(i) The group inverse of M exists if and only if $rank(A) = rank(B) = rank(AB) = rank(BA)$.

(ii) If the group inverse of M exists, then $M^{\sharp} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$, where

$$\begin{aligned} Z_{11} &= (AB)^{\sharp}A - B(AB)^{\sharp}A^2(BA)^{\sharp}, \\ Z_{12} &= B(AB)^{\sharp} - (AB)^{\sharp}A^2(BA)^{\sharp}B + B(AB)^{\sharp}A^2(BA)^{\sharp}A(BA)^{\sharp}B, \\ Z_{21} &= (AB)^{\sharp}A, \\ Z_{22} &= -(AB)^{\sharp}A^2(BA)^{\sharp}B. \end{aligned}$$

Proof. Let $X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$. By Lemma 2.5, we have

$$MX = XM = \begin{pmatrix} B(AB)^{\sharp}A & A(BA)^{\sharp}B - B(AB)^{\sharp}A^2(BA)^{\sharp}B \\ 0 & A(BA)^{\sharp}B \end{pmatrix}.$$

Furthermore, we can prove $MXM = M, XMX = X$ easily. Thus, $X = M^{\sharp}$. ∎

THEOREM 3.3. *Let $A, B \in K^{n\times n}$, if $rank(B) = rank(AB) = rank(BA)$. Then AB and BA are similar.*

Proof. Suppose $rank(A) = r$, using Lemma 2.1, there are invertible matrices $P, Q \in K^{n\times n}$ such that

$$A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}Q, \ B = Q^{-1}\begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix}P^{-1},$$

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where $B_1 \in K^{r\times r}, X \in K^{r\times(n-r)}, Y \in K^{(n-r)\times r}$. Hence

$$\begin{aligned} AB &= P\begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix}P^{-1} \\ &= P\begin{pmatrix} I_r & -X \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & X \\ 0 & I_{n-r} \end{pmatrix}P^{-1}, \end{aligned}$$

$$\begin{aligned} BA &= Q^{-1}\begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix}Q \\ &= Q^{-1}\begin{pmatrix} I_r & 0 \\ Y & I_{n-r} \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ -Y & I_{n-r} \end{pmatrix}Q. \end{aligned}$$

So AB and BA are similar. ∎

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