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POLYNOMIAL NUMERICAL HULLS OF ORDER 3*

H.R. AFSHIN†, M.A. MEHRJOOFARD†, AND A. SALEMI‡

Dedicated to Professor Chandler Davis for his outstanding contributions to Mathematics

Abstract. In this note, analytic description of $V^3(A)$ is given for normal matrices of the form $A = A_1 \oplus iA_2$ or $A = A_1 \oplus e^{i\frac{2\pi}{3}} A_2 \oplus e^{i\frac{4\pi}{3}} A_3$, where A_1, A_2, A_3 are Hermitian matrices. The new concept "k^{th} roots of a convex set" is used to study the polynomial numerical hulls of order k for normal matrices.

Key words. Polynomial numerical hull, Numerical order, K^{th} roots of a convex set.

AMS subject classifications. 15A60, 15A18, 15A57.

1. Introduction. Let $A \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices. The numerical range of A is denoted by

$$W(A) := \{x^* Ax : \|x\| = 1\}.$$

Let $p(\lambda)$ be any complex polynomial. Define

$$V_p(A) := \{\lambda : |p(\lambda)| \leq \|p(A)\|\}.$$

If p is not constant, $V_p(A)$ is a compact convex set which contains $\sigma(A)$ (for more details see [5]). The polynomial numerical hull of A of order k, denoted by $V^k(A)$ is defined by

$$V^k(A) := \bigcap V_p(A),$$

where the intersection is taken over all polynomials p of degree at most k.

The intersection over all polynomials is called the polynomial numerical hull of A and is denoted by

$$V(A) := \bigcap_{k=1}^{\infty} V^k(A).$$

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The following proposition due to O. Nevanlinna states the relationship between polynomial numerical hull of order one and the numerical range of a bounded operator.

PROPOSITION 1.1. *Let A be a bounded linear operator on a Hilbert space H, then* $V^1(A) = \overline{W(A)}$ (see [5, 4]).

In the finite dimensional case $V^1(A) = W(A)$. If $A \in M_n(\mathbb{C})$ and the degree of the minimal polynomial of A is k, then $V^i(A) = \sigma(A)$ for all $i \geq k$. The integer m is called the numerical order of A and is denoted by $num(A)$ provided that $V^m(A) = V(A)$ and $V^{m-1}(A) \neq V(A)$. So the numerical order of A is less than or equal to the degree of the minimal polynomial of A. Nevanlinna in [6] proved the following result and Greenbaum later in [4] showed this proposition with a shorter proof.

PROPOSITION 1.2. *Let* $A \in M_n(\mathbb{C})$ *be Hermitian. Then* $num(A) \leq 2$ *and* $V^2(A) = \sigma(A)$.

The joint numerical range of $(A_1, \ldots, A_m) \in M_n \times \cdots \times M_n$ is denoted by

$$W(A_1, \ldots, A_m) = \{(x^*A_1x, x^*A_2x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

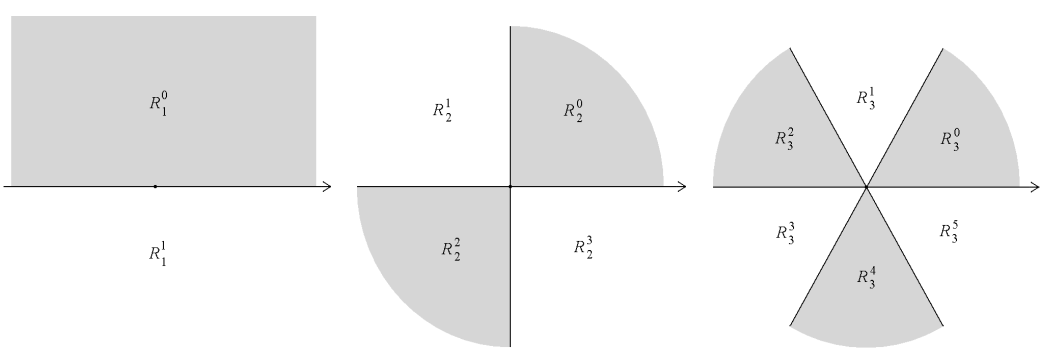
By the result in [3] (see also [1]),

$$V^k(A) = \left\{\xi \in \mathbb{C} : (0, \ldots, 0) \in \text{conv}\left(W\left((A - \xi I), (A - \xi I)^2, \ldots, (A - \xi I)^k\right)\right)\right\}$$

where $conv(X)$ denotes the convex hull of $X \subseteq \mathbb{C}^k$.

Throughout this paper all direct sums are assumed to be orthogonal and we fix the following notations. Define $i[a, b] = \{it : a \leq t \leq b\}$ and $i(a, b) = \{it : a < t < b\}$, where a and b are real numbers. Also $|AB|$ means the length of the line segment AB, and $S^{\frac{1}{n}} = \{z \in \mathbb{C} : z^n \in S\}$. Let $k \in \mathbb{N}$. Define

$$R_k^j := \left\{re^{i\theta} : r \geq 0, \frac{j\pi}{k} \leq \theta \leq \frac{(j+1)\pi}{k}\right\}, \quad 0 \leq j \leq 2k - 1. \tag{1.1}$$



$k = 1$ $k = 2$ $k = 3$

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In Section 2, we give an analytic description of $V^3(A)$ for any matrix $A \in M_n$ of the form $A = A_1 \oplus iA_2$, where $A_1^* = A_1, A_2^* = A_2$. Section 3 concerns matrices of the form $A = A_1 \oplus e^{i\frac{2\pi}{3}} A_2 \oplus e^{i\frac{4\pi}{3}} A_3$, where $A_1^* = A_1, A_2^* = A_2, A_3^* = A_3$. Additional results and remarks about the polynomial numerical hulls of order k of normal matrices are given by a new concept "k^{th} roots of a convex set" in section 4.

2. Matrices of the form $A = A_1 \oplus iA_2$. In this section we shall characterize $V^3(A)$, where

$$A = A_1 \oplus iA_2, \qquad A_1^* = A_1, \quad A_2^* = A_2. \tag{2.1}$$

Lemma 2.1. *Let H be a semi-definite Hermitian matrix and $k \geq 2$ be an integer such that $X^*H^kX = (X^*HX)^k$ for some unit vector $X = (x_1, ..., x_n)^t$. Then $X^*HX \in \sigma(H)$.*

Proof. Without loss of generality, we assume that $H = diag(h_1, h_2, \ldots, h_n)$, where $h_i \geq 0, i = 1, \ldots, n$. Define $P_i = (h_i, h_i^k) \in \mathbb{R}^2, i = 1, \ldots, n$. Let $\mu = X^*HX$. By assumption $\mu^k = (X^*HX)^k = X^*H^kX$. Hence $\|x_1\|^2 \left(h_1, h_1^k\right) + \cdots + \|x_n\|^2 \left(h_n, h_n^k\right) = \left(\mu, \mu^k\right) \in \mathbb{R}^2$. Since the graph of the function $y = x^k, x \geq 0$ is convex, we have $\mu = h_i$ for some $i = 1, \ldots, n$. Consequently, $\mu \in \sigma(A)$. □

Theorem 2.2. *Let A be of the form* (2.1) *and A_2 be a semi-definite matrix. Then $V^3(A) = \sigma(A)$.*

Proof. Without loss of generality, we assume that A_2 is a positive definite matrix. By [2, Theorem 2.2], we know that

$$V^3(A) \subseteq V^2(A) \subseteq \sigma(A_1) \cup \{i\gamma : 0 \leq \gamma \leq r(A_2)\},$$

where $r(A_2)$ is the spectral radius of A_2. Then, $V^3(A) \cap \mathbb{R} \subseteq \sigma(A)$. Now, let $i\eta \in V^3(A) \cap i\mathbb{R}$. Thus there exists a unit vector $x = x_1 \oplus x_2$ such that

$$\begin{aligned} \|x_1\|^2 + \|x_2\|^2 &= 1, \\ x_1^*A_1x_1 + ix_2^*A_2x_2 &= i\mu, \\ x_1^*A_1^2x_1 - x_2^*A_2^2x_2 &= -\mu^2, \\ x_1^*A_1^3x_1 - ix_2^*A_2^3x_2 &= -i\mu^3. \end{aligned}$$

The above relations imply that $(\mu, \mu^3) = (x_2^*A_2x_2, x_2^*A_2^3x_2)$. Define $H = 0 \oplus A_2$, where 0 is the zero matrix of the same size as A_1. Hence $H \geq 0$ and $X^*H^3X = (X^*HX)^3$. By Lemma 2.1, $\mu \in \sigma(H)$. Hence $\mu = 0$ or $\mu \in \sigma(A_2) \subseteq \sigma(A)$. It is enough to show that if $\mu = 0$, then $\mu \in \sigma(A)$. By[2, Lemma 2.3] we know that $0 \in \sigma(A)$ if and only if $0 \in V^2(A)$. Since $0 \in V^3(A) \subseteq V^2(A)$, we obtain $\mu = 0 \in \sigma(A)$. □

Corollary 2.3. *Let $A = diag(\alpha, -\beta, 0, i\gamma)$, where α, β and γ are positive numbers. Then $V^3(A) = \sigma(A)$ and therefore $num(A) = 3$.*

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Corollary 2.4. *Let* $A = diag(\alpha, -\beta, i\gamma, i\theta)$ *such that* $\alpha > 0, \beta > 0$ *and* $0 \le \gamma < \theta$. *Then* $V^3(A) = \sigma(A)$.

Theorem 2.5. *Let* $A = diag(\alpha, -\beta, i\gamma, -i\theta)$ *and* α, β, γ *and* θ *be positive numbers. Then*

(a) $\alpha = \beta$ *and* $\gamma = \theta$ *if and only if* $V^3(A) = \sigma(A) \cup \{0\}$.

(b) *If* $\alpha = \beta$ *and* $\gamma \neq \theta$, *then* $V^3(A) = \sigma(A) \cup \left(\left\{ \frac{\alpha^2(\theta-\gamma)}{\alpha^2+\theta\gamma} \right\} i \cap W(A) \right)$.

(c) *If* $\alpha \neq \beta$ *and* $\gamma = \theta$, *then* $V^3(A) = \sigma(A) \cup \left(\left\{ \frac{\gamma^2(\beta-\alpha)}{\gamma^2+\beta\alpha} \right\} \cap W(A) \right)$.

(d) *If* $\alpha \neq \beta$ *and* $\gamma \neq \theta$, *then* $V^3(A) = \sigma(A)$.

Proof. (a) Let $\alpha = \beta$ and $\gamma = \theta$. Define $X = (x, y, z, t)^t$, where

$$x = \left(\frac{\gamma^2 + \theta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}, \ y = \left(\frac{\gamma^2 + \theta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}},$$
$$z = \left(\frac{\alpha^2 + \beta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}, \ t = \left(\frac{\alpha^2 + \beta^2}{2(\alpha^2 + \beta^2 + \gamma^2 + \theta^2)} \right)^{\frac{1}{2}}.$$

It is easy to show that X is a unit vector and $X^*AX = X^*A^2X = X^*A^3X = 0$ and hence $0 \in V^3(A)$.

Now, let $\eta \in V^3(A)$. Then there exists a unit vector $X = (x, y, z, t)^t$ such that

$$|x|^2 + |y|^2 + |z|^2 + |t|^2 = 1, \tag{2.2}$$

$$X^*AX = \alpha|x|^2 - \beta|y|^2 + i\gamma|z|^2 - i\theta|t|^2 = \eta, \tag{2.3}$$

$$X^*A^2X = \alpha^2|x|^2 + \beta^2|y|^2 - \gamma^2|z|^2 - \theta^2|t|^2 = \eta^2, \tag{2.4}$$

$$X^*A^3X = \alpha^3|x|^2 - \beta^3|y|^2 - i\gamma^3|z|^2 + i\theta^3|t|^2 = \eta^3. \tag{2.5}$$

Conversely, let $\eta = 0$. The relations (2.3) and (2.5) imply that ($\beta = \alpha$ or $|x|^2 = |y|^2 = 0$) and ($\theta = \gamma$ or $|z|^2 = |t|^2 = 0$). Since $\alpha, \beta, \gamma, \theta$ are positive numbers and $X \neq 0$, by (2.4), we obtain $\alpha = \beta$ and $\gamma = \theta$.

(b) By [3, Theorem 2.6], we know that $V^2(A) \subseteq [-\alpha, \alpha] \cup i[-\theta, \gamma]$. Let $\eta \in V^3(A)$, then $\eta \in [-\alpha, \alpha]$ or $\eta \in i[-\theta, \gamma]$. If $\eta \in \mathbb{R}$, then the relations (2.3) and (2.5) imply that $|z|^2 = |t|^2 = 0$. Therefore, $|x|^2 + |y|^2 = 1$ and hence $\eta = \pm\alpha$. Thus, $V^3(A) \cap \mathbb{R} = \{-\alpha, \alpha\} \subseteq \sigma(A)$.

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Let $i\eta \in V^3(A) \cap i\mathbb{R}$. Then $\eta \in [-\theta, \gamma]$. By (2.3) and (2.5), we obtain

$$|x|^2 = |y|^2 = \frac{-\eta^2 + \gamma^2 |z|^2 + \theta^2 |t|^2}{2\alpha^2}, \ |z|^2 = \frac{\eta\left(\eta^2 - \theta^2\right)}{\gamma\left(\gamma^2 - \theta^2\right)}, \ |t|^2 = \frac{\eta\left(\eta^2 - \gamma^2\right)}{\theta\left(\gamma^2 - \theta^2\right)}.$$

Now, replacing the above equations in (2.2), we can write

$$1 = \frac{[\gamma\theta + \alpha^2]\eta^3 - [\gamma\theta(\gamma - \theta)]\eta^2 - [\gamma^2\theta^2 + \theta^2 - \alpha^2\gamma\theta - \alpha^2\gamma^2]\eta}{\alpha^2\gamma\theta(\gamma - \theta)}.$$

Define $P(\eta) := [\gamma\theta + \alpha^2]\eta^3 - [\gamma\theta(\gamma - \theta)]\eta^2 - [\gamma^2\theta^2 + \theta^2 - \alpha^2\gamma\theta - \alpha^2\gamma^2]\eta - \alpha^2\gamma\theta(\gamma - \theta)$ Since $\{i\gamma, -i\theta\} \subseteq V^3(A)$, the polynomial $P(\eta)$ is divided by $(\eta - \gamma)(\eta + \theta)$. Hence

$$P(\eta) = (\eta - \gamma)(\eta + \theta)[(\gamma\theta + \alpha^2)\eta - (\theta - \gamma)\alpha^2]. \tag{2.6}$$

Therefore, $V^3(A) \cap i\mathbb{R} \subseteq \left\{i\gamma, -i\theta, i\frac{(\theta-\gamma)\alpha^2}{\alpha^2+\theta\gamma}\right\}$. We are looking to find $\eta \in \mathbb{R}$ such that $P(\eta) = 0$ and

$$\frac{-\eta^2 + \gamma^2 |z|^2 + \theta |t|^2}{2\alpha^2} \geq 0, \quad \frac{\eta\left(\eta^2 - \theta^2\right)}{\gamma\left(\gamma^2 - \theta^2\right)} \geq 0, \quad \frac{\eta\left(\eta^2 - \gamma^2\right)}{\theta\left(\gamma^2 - \theta^2\right)} \geq 0. \tag{2.7}$$

Let $\eta = \frac{(\theta-\gamma)\alpha^2}{\alpha^2+\theta\gamma} \in [-\theta, \gamma]$. It is readily seen that the relations in (2.7) hold and by (2.6), $P(\eta) = 0$. Therefore, $V^3(A) \cap i\mathbb{R} = \{i\gamma, -i\theta\} \cup \left\{i\frac{\alpha^2(\theta-\gamma)}{\alpha^2+\gamma\theta} \cap i[-\theta, \gamma]\right\}$.

(c) It is enough to consider iA instead of A.

(d) Let $\eta \in V^3(A) \cap \mathbb{R}$. Then, there exists a unit vector X such that $X^*AX = \eta, X^*A^2X = \eta^2$ and $X^*A^3X = \eta^3$. These relations imply that $|x|^2 = \frac{\eta+\beta}{\alpha+\beta}$, $|y|^2 = \frac{\alpha-\eta}{\alpha+\beta}$, and $|z|^2 = |t|^2 = 0$. Also, we have $\eta^2 + (\beta - \alpha)\eta - \alpha\beta = 0$. Therefore, $\eta = -\beta$ or $\eta = \alpha$ which are in $\sigma(A)$. Similarly, if $\eta \in V^3(A) \cap i\mathbb{R}$ is pure imaginary, then $\eta = -i\theta$ or $i\gamma$ which are in $\sigma(A)$. □

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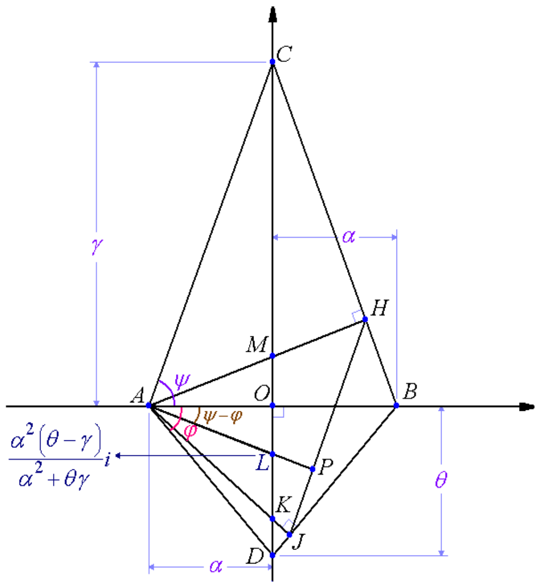

Remark 2.6. In the above Figure, we find a geometric interpretations for the 5^{th} point in $V^3(A)$, where A is a 4×4 normal matrix as in Theorem 2.5(b), see [1, Theorem 5.1]. The points M and K are the orthocenters of the triangles ABC and ABD, respectively. Let L be the intersection of the line CD and the line passing through A and perpendicular to HJ. It is readily seen that the slope of the lines HJ and AP are $\cot(\psi-\varphi)$ and $-\tan(\psi-\varphi)$, respectively. Also, $-\tan(\psi-\varphi)=\frac{\tan(\varphi)-\tan(\psi)}{1+\tan(\psi)\tan(\varphi)}=\frac{\theta/\alpha-\gamma/\alpha}{1+(\gamma/\alpha)(\theta/\alpha)}$. Hence $L=\left(0,\frac{\alpha^2(\theta-\gamma)}{\alpha^2+\gamma\theta}\right)$.

For a 3×3 normal matrix A, the 4^{th} point in $V^2(A)$ (if any) is the orthocenter of the triangle generated by $\sigma(A)$. It is interesting that if $\gamma\to\infty$, then $i\frac{\alpha^2(\theta-\gamma)}{\alpha^2+\gamma\theta}\to i\frac{-\alpha^2}{\theta}$, where $i\frac{-\alpha^2}{\theta}$ is the orthocenter of the triangle generated by $\{\alpha,-\alpha,-i\theta\}$ [2, Theorem 2.4].

3. Matrices of the form $A=A_1\oplus e^{i\frac{2\pi}{3}}A_2\oplus e^{i\frac{4\pi}{3}}A_3$. In this section, we study the polynomial numerical hull of order 3 of matrices of the form

$$A=A_1\oplus e^{i\frac{2\pi}{3}}A_2\oplus e^{i\frac{4\pi}{3}}A_3,\qquad A_1^*=A_1,\ A_2^*=A_2 \text{ and } A_3^*=A_3. \tag{3.1}$$

Theorem 3.1. *Let A be a normal matrix such that $\sigma(A)\subseteq R_3^1\cup R_3^3\cup R_3^5$. Then $V^3(A)\subseteq R_3^1\cup R_3^3\cup R_3^5$.*

Proof. we know that $z\in R_3^1\cup R_3^3\cup R_3^5$ if and only if $z^3\in R_1^1$ (lower half plane), whereas $\sigma(A^3)=\{z^3: z\in\sigma(A)\}$ and $\sigma(A)\subseteq R_3^1\cup R_3^3\cup R_3^5$. Then $\sigma(A^3)\subseteq R_1^1$ and hence $W\left(A^3\right)=\operatorname{conv}\left(\sigma\left(A^3\right)\right)\subseteq R_1^1$. Thus, $V^3\left(A\right)\subseteq R_3^1\cup R_3^3\cup R_3^5$. $\square$

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COROLLARY 3.2. *Let A be a normal matrix such that $\sigma(A) \subset S = \mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R} \cup e^{i\frac{4\pi}{3}}\mathbb{R}$. Then $V^3(A) \subset S$.*

Proof. Since $\sigma(A) \subseteq S$ and $S = \left(R_3^0 \cup R_3^2 \cup R_3^4\right) \cap \left(R_3^1 \cup R_3^3 \cup R_3^5\right)$, by Theorem 3.1, we obtain $V^3(A) \subset S$. □

REMARK 3.3. Let A be as in (3.1). Then $V^3(A) \subseteq \mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R} \cup e^{i\frac{4\pi}{3}}\mathbb{R}$. Since $V^3(e^{i\frac{2\pi}{3}}A) \cap \mathbb{R} = V^3(A) \cap e^{i\frac{4\pi}{3}}\mathbb{R}$, it is enough to find $V^3(A) \cap \mathbb{R}$.

LEMMA 3.4. *Let A be as in* (3.1)*. Then*

$$V^3(A) \cap \mathbb{R} = \left\{ \eta = x_1^* A_1 x_1 - x_2^* A_2 x_2 : \left[\begin{array}{l} x_1^* x_1 + x_2^* x_2 + x_3^* x_3 = 1, \\ x_2^* A_2 x_2 = x_3^* A_3 x_3, \\ x_2^* A_2^2 x_2 = x_3^* A_3^2 x_3, \\ \eta^2 = x_1^* A_1^2 x_1 - x_2^* A_2^2 x_2, \\ \eta^3 = x_1^* A_1^3 x_1 + x_2^* A_2^3 x_2 + x_3^* A_3^3 x_3 \end{array} \right. \right\}.$$

Proof. Suppose that $x = x_1 \oplus x_2 \oplus x_3$ and $\eta = x^* A x \in V^3(A) \cap \mathbb{R}$. So

$$\left\{ \begin{array}{l} x_1^* x_1 + x_2^* x_2 + x_3^* x_3 = x^* x = 1, \\ \eta = x^* A x = x_1^* A_1 x_1 + e^{i\frac{2\pi}{3}} x_2^* A_2 x_2 + e^{i\frac{4\pi}{3}} x_3^* A_3 x_3, \\ \eta^2 = x^* A^2 x = x_1^* A_1^2 x_1 + e^{i\frac{4\pi}{3}} x_2^* A_2^2 x_2 + e^{i\frac{2\pi}{3}} x_3^* A_3^2 x_3, \\ \eta^3 = x^* A^3 x = x_1^* A_1^3 x_1 + x_2^* A_2^3 x_2 + x_3^* A_3^3 x_3. \end{array} \right.$$

Since $\eta \in \mathbb{R}$,

$$\left\{ \begin{array}{l} \eta = x_1^* A_1 x_1 + \cos\frac{2\pi}{3} x_2^* A_2 x_2 + \cos\frac{4\pi}{3} x_3^* A_3 x_3, \\ \sin\frac{2\pi}{3} x_2^* A_2 x_2 + \sin\frac{4\pi}{3} x_3^* A_3 x_3 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \eta = x_1^* A_1 x_1 - x_2^* A_2 x_2, \\ x_2^* A_2 x_2 = x_3^* A_3 x_3 \end{array} \right.$$

$$\left\{ \begin{array}{l} \eta^2 = x_1^* A_1^2 x_1 + \cos\frac{4\pi}{3} x_2^* A_2^2 x_2 + \cos\frac{2\pi}{3} x_3^* A_3^2 x_3, \\ \sin\frac{4\pi}{3} x_2^* A_2^2 x_2 + \sin\frac{2\pi}{3} x_3^* A_3^2 x_3 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \eta^2 = x_1^* A_1^2 x_1 - x_2^* A_2^2 x_2, \\ x_2^* A_2^2 x_2 = x_3^* A_3^2 x_3 \end{array} \right.$$

and

$$\eta^3 = x^* A^3 x = x_1^* A_1^3 x_1 + x_2^* A_2^3 x_2 + x_3^* A_3^3 x_3. \ \square$$

THEOREM 3.5. *Let $A = A_1 \oplus e^{i\frac{2\pi}{3}} A_2$ and $A_1^* = A_1$, $A_2^* = A_2$. Then $V^3(A) = \sigma(A)$.*

Proof. By using [2, Lemma 2.3], $V^2(A) \subseteq R_3^2 \cup R_3^5$ and by Corollary 3.2, $V^3(A) \subseteq \mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R} \cup e^{i\frac{4\pi}{3}}\mathbb{R}$. Hence $V^3(A) \subseteq V^2(A) \cap \left(\mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R}\right)$. Now, we will show that

$$V^2(A) \cap \left(\mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R}\right) \subseteq \sigma(A).$$

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First, we show that $V^2(A) \cap \mathbb{R} \subseteq \sigma(A_1)$. Suppose that $x = x_1 \oplus x_2$ and $\eta = x^*Ax \in V^2(A) \cap \mathbb{R}$. By the same method as in the proof of Lemma 3.4, we have

$$V^2(A) \cap \mathbb{R} = \left\{ \eta = x_1^* A_1 x_1 : \left[\begin{array}{l} x_1^* x_1 + x_2^* x_2 = 1, \\ \eta^2 = x_1^* A_1^2 x_1 \end{array} \right. \right\}.$$

Then, $(x_1^* A_1 x_1)^2 = x_1^* A_1^2 x_1 = \|A_1 x_1\|^2$.

By the Cauchy-Schwarz Inequality, we have $(x_1^* A_1 x_1)^2 \leq \|x_1\|^2 \|A_1 x_1\|^2$. Hence $A_1 x_1 = 0$ or $\|x_1\| = 1$. In both cases $\eta = x_1^* A_1 x_1 \in \sigma(A_1) \subseteq \sigma(A)$. Since $V^2(e^{i\alpha}A) = e^{i\alpha}V^2(A)$, similarly, $V^2(A) \cap e^{i\frac{2\pi}{3}}\mathbb{R} \subseteq \sigma(e^{i\frac{2\pi}{3}}A_2) \subseteq \sigma(A)$. Therefore, $V^3(A) = \sigma(A)$.□

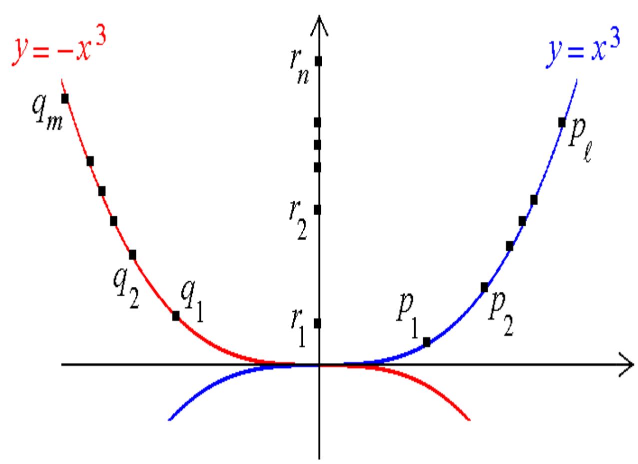
In the following Theorem, we show that if A_1, A_2 and A_3 are positive semi-definite matrices as in (3.1), then $V^3(A) = \sigma(A)$.

THEOREM 3.6. *Let A be as in (3.1). If A_1, A_2, A_3 are positive semi-definite matrices, then $V^3(A) = \sigma(A)$.*

Proof. By Lemma 3.4,

$$\begin{array}{l} V^3(A) \cap \mathbb{R} \subset \left\{ \eta : \left[\begin{array}{l} x_1^* x_1 + x_2^* x_2 + x_3^* x_3 = 1, \\ \eta = x_1^* A_1 x_1 - x_2^* A_2 x_2, \\ \eta^3 = x_1^* A_1^3 x_1 + x_2^* A_2^3 x_2 + x_3^* A_3^3 x_3 \end{array} \right. \right\} \\ = \left\{ \eta : (\eta, \eta^3) \in \operatorname{conv}\left(\left\{ (a, a^3) \right\}_{a \in \sigma(A_1)} \cup \left\{ (-b, b^3) \right\}_{b \in \sigma(A_2)} \cup \left\{ (0, c^3) \right\}_{c \in \sigma(A_3)} \right) \right\}. \end{array}$$

Assume $A_1 = \operatorname{diag}(a_1, \cdots, a_\ell)$, $A_2 = \operatorname{diag}(b_1, \ldots, b_m)$, and $A_3 = \operatorname{diag}(c_1, \ldots, c_n)$, where $0 \leq a_1 \leq \cdots \leq a_\ell$, $0 \leq b_1 \leq \cdots \leq b_m$, and $0 \leq c_1 \leq \cdots \leq c_n$ Let $p_i = (a_i, a_i^3)$, $q_j = (-b_j, b_j^3)$, $r_k = (0, c_k^3)$. By the following Figure, $V^3(A) \cap \mathbb{R} = \sigma(A_1)$. Similarly, $V^3(A) \cap e^{i\frac{2\pi}{3}}\mathbb{R} \subseteq \sigma(A_2)$ and $V^3(A) \cap e^{i\frac{4\pi}{3}}\mathbb{R} \subseteq \sigma(A_3)$. Hence, $V^3(A) = \sigma(A)$ and the proof is complete. □



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PROPOSITION 3.7. *Let A be as in* (3.1). *Assume A_1, A_2 are positive semidefinite matrices and A_3 is a negative semi definite matrix. Then $V^3(A) \subseteq \sigma(A) \cup e^{i\frac{\pi}{3}}(0,\infty)$.*

Proof. Without loss of generality, we assume that A_3 is a negative definite matrix. By [2, Theorem 1.4.], $V^2(A) \cap \left(\mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R}\right) \subseteq \sigma(A)$. Hence $V^3(A) \cap (\mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R}) \subseteq \sigma(A)$. By Corollary 3.2, $V^3(A) \subseteq \left(\mathbb{R} \cup e^{i\frac{2\pi}{3}}\mathbb{R}\right) \cup e^{i\frac{4\pi}{3}}\mathbb{R}$. Also, $V^3(A) \subseteq W(A)$, therefore, $V^3(A) \subseteq \sigma(A) \cup e^{i\frac{\pi}{3}}(0,\infty)$. ◻

In the following example, we show that Theorem 3.6 may not be true if A_1, A_2 are positive semi definite matrices and A_3 is a negative definite matrix.

EXAMPLE 3.8. Let $A = \mathrm{diag}(0, 2\sqrt{3}, \sqrt{1}2e^{i\frac{2\pi}{3}}, -\sqrt{1}2e^{i\frac{4\pi}{3}})$. After a rotation and a translation, by using Theorem 2.5 (a), it is readily seen that $V^3(A) = \sigma(A) \cup \{\sqrt{3}e^{i\frac{\pi}{3}}\}$.

4. K^{th} roots of a convex set. In this section we introduce the concept of k^{th} roots of a convex set and we show that the concepts "inner cross" and "outer cross" in [2, Section 3] are special cases of this concept.

DEFINITION 4.1. Let S be a convex set and $R := S^{\frac{1}{k}} = \left\{z \in \mathbb{C} : z^k \in S\right\}$. Then R is called k^{th} **root** of the convex set S.

In the following Lemma, we list some properties of the k^{th} roots of a convex set.

LEMMA 4.2. *Let P and Q be two convex sets. Then*

a) $(P \cap Q)^{\frac{1}{k}} = P^{\frac{1}{k}} \cap Q^{\frac{1}{k}}$.

b) $(P^c)^{\frac{1}{k}} = \left(P^{\frac{1}{k}}\right)^c$.

c) $\left(e^{ik\theta}P\right)^{\frac{1}{k}} = e^{i\theta}P^{\frac{1}{k}}$.

The following is a key Theorem in this section:

THEOREM 4.3. *Let A be a normal matrix and S be an arbitrary convex set. If $\sigma(A) \subset S^{\frac{1}{k}}$, then $V^k(A) \subset S^{\frac{1}{k}}$.*

Proof. If $\sigma(A) \subset S^{\frac{1}{k}}$, then $\sigma\left(A^k\right) \subset S$. Since $W\left(A^k\right) = \mathrm{conv}(\sigma\left(A^k\right)) \subset S$. Thus, $\{z^k : z \in V^k(A)\} \subset S$, and hence $V^k(A) \subset S^{\frac{1}{k}}$. ◻

LEMMA 4.4. *The 2-roots of a line is a rectangular hyperbola with center at the origin.*

Proof. Suppose that $(a,b) \neq (0,0)$ and let $S = \{(x,y) : ax + by + c = 0\}$. Therefore

$$R = S^{\frac{1}{2}} = \left\{(x,y) : a\left(x^2 - y^2\right) + b\,(2xy) + c = 0\right\}.$$

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It is clear that R is an arbitrary rectangular hyperbola with center at the origin. $\square$

COROLLARY 4.5. [2, Theorem 3.1] *Let $A \in M_n$ be a normal matrix and $\sigma(A) \subseteq R$, where R is a rectangular hyperbola. Then $V^2(A) \subset R$.*

Proof. Since $V^2(\alpha A + \beta I) = \alpha V^2(A) + \beta$, we assume that the center of R is origin. Now, by Theorem 4.3 and Lemma 4.4 the result holds. $\square$

COROLLARY 4.6. [2, Lemma 3.3] *Let $A \in M_n(\mathbb{C})$ be a normal matrix and Δ be an inner or outer cross. If $\sigma(A) \subseteq \Delta$, then $V^2(A) \subseteq \Delta$.*

Proof. Without loss of generality we assume that $\Delta = \left\{x + iy : x^2 - y^2 \leq 1\right\}$. Then, $\Delta = \{z \in \mathbb{C}| : \Re(z^2) \leq 1\}$. Define $S := \{z \in \mathbb{C}| : \Re(z) \leq 1\}$. Thus, $\Delta = S^{1/2}$. This means that Δ is the 2^{nd} root of the convex set S. By Theorem 4.3, the result holds. $\square$

Let

$$R_k^e = \bigcup_{t=0}^{k-1} R_k^{2t} \qquad \text{and} \qquad \mathrm{R_k^o} = \bigcup_{\mathrm{t}=0}^{\mathrm{k}-1} \mathrm{R_k^{2t+1}}, \tag{4.1}$$

where R_k^t be as in (1.1). It is clear that $\mathbb{C} = R_k^e \cup R_k^o$ and $R_k^o = e^{\frac{i\pi}{k}} R_k^e$. The following is a generalization of Theorem 3.1.

THEOREM 4.7. *Let A be a normal matrix and let $z_0 \in \mathbb{C}$ and $\eta \in \mathbb{R}$. If $\sigma(A) \subseteq z_0 + e^{i\eta} R_k^e$, then $V^k(A) \subseteq z_0 + e^{i\eta} R_k^e$.*

Proof. Let $\hat{A} := e^{-i\eta}(A - z_0 I)$, then $\sigma\left(\hat{A}\right) \subseteq R_k^e$. Define $S = R_1^0$ (upper half plane), it is easy to show that $S^{1/k} = R_k^e$. Since $\sigma\left(\hat{A}\right) \subseteq S^{1/k} = R_k^e$, by Theorem 4.3 $V^k\left(\hat{A}\right) \subseteq S^{1/k} = R_k^e$. Also, $V^k(\hat{A}) = e^{-i\eta}(V^k(A) - z_0)$, hence

$$V^k(A) \subseteq z_0 + e^{i\eta} R_k^e. \quad \square$$

COROLLARY 4.8. *Let A be a normal matrix of the form*

$$A = A_1 \oplus e^{i\frac{2\pi}{k}} A_2 \oplus \cdots \oplus e^{i\frac{2(k-1)\pi}{k}} A_k, \quad A_i^* = A_i, \quad i = 1, \ldots, k.$$

Then, $V^k(A) \subseteq \mathbb{R} \cup e^{i\frac{2\pi}{k}} \mathbb{R} \cup \cdots \cup e^{i\frac{2(k-1)\pi}{k}} \mathbb{R}$.

Proof. It is clear that $\sigma(A) \subseteq \mathbb{R} \cup e^{i\frac{2\pi}{k}} \mathbb{R} \cup \cdots \cup e^{i\frac{2(k-1)\pi}{k}} \mathbb{R} = R_k^e \cap R_k^o$, where R_k^e and R_k^o be as in (4.1). By Theorem 4.7,

$$V^k(A) \subseteq R_k^e \cap R_k^o = \mathbb{R} \cup e^{i\frac{2\pi}{k}} \mathbb{R} \cup \cdots \cup e^{i\frac{2(k-1)\pi}{k}} \mathbb{R}. \quad \square$$

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