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THE SYMMETRIC MINIMAL RANK SOLUTION OF THE MATRIX EQUATION $AX = B$ AND THE OPTIMAL APPROXIMATION*

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Abstract. By applying the matrix rank method, the set of symmetric matrix solutions with prescribed rank to the matrix equation $AX = B$ is found. An expression is provided for the optimal approximation to the set of the minimal rank solutions.

Key words. Symmetric matrix, Matrix equation, Maximal rank, Minimal rank, Fixed rank solutions, Optimal approximate solution.

AMS subject classifications. 65F15, 65F20.

1. Introduction. We first introduce some notation to be used. Let $C^{n\times m}$ denote the set of all $n \times m$ complex matrices; $R^{n\times m}$ denote the set of all $n \times m$ real matrices; $SR^{n\times n}$ and $ASR^{n\times n}$ be the sets of all $n \times n$ real symmetric and antisymmetric matrices respectively; $OR^{n\times n}$ be the sets of all $n\times n$ orthogonal matrices. The symbols A^T, A^+, A^-, $R(A)$, $N(A)$ and $r(A)$ stand, respectively, for the transpose, Moore-Penrose generalized inverse, any generalized inverse, range (column space), null space and rank of $A \in R^{n\times m}$. The symbols E_A and F_A stand for the two projectors $E_A = I - AA^-$ and $F_A = I - A^-A$ induced by A. The matrices I and 0 denote, respectively, the identity and zero matrices of sizes implied by the context. We use $< A, B >= \mathrm{trace}(B^TA)$ to define the inner product of matrices A and B in $R^{n\times m}$. Then $R^{n\times m}$ is an inner product Hilbert space. The norm of a matrix generated by the inner product is the Frobenius norm $\|\cdot\|$, that is, $\|A\| = \sqrt{< A, A >} = (\mathrm{trace}(A^TA))^{\frac{1}{2}}$.

Ranks of solutions of linear matrix equations have been considered previously by several authors. For example, Mitra [1] considered solutions with fixed ranks for the matrix equations $AX = B$ and $AXB = C$; Mitra [2] gave common solutions of minimal rank of the pair of matrix equations $AX = C, XB = D$; Uhlig [3] gave the maximal and minimal ranks of solutions of the equation $AX = B$; Mitra [4] examined common solutions of minimal rank of the pair of matrix equations $A_1X_1B_1 = C_1$ and $A_2X_2B_2 = C_2$. Recently, by applying the matrix rank method, Tian [5] obtained

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the minimal rank among solutions to the matrix equation $A = BX + YC$. Theoretically speaking, the general solution of a linear matrix equation can be written as linear matrix expressions involving variant matrices. Hence the maximal and minimal ranks among the solutions of a linear matrix equation can be determined through the corresponding linear matrix expressions.

Motivated by the work in [1,3], in this paper, we derive the minimal and maximal rank among symmetric solutions to the matrix equation $AX = B$ and obtain the symmetric matrix solution with prescribed rank. In addition, in corresponding minimal rank solution set of the equation, an explicit expression for the nearest matrix to a given matrix in the Frobenius norm is provided.

The problems studied in this paper are described below.

Problem I. Given $X \in R^{n\times m}, B \in R^{n\times m}$, and a positive integer s, find $A \in SR^{n\times n}$such that $AX = B$, and $r(A) = s$. Moreover, when the solution set $S_1 = \{A \in SR^{n\times n} | AX = B\}$ is nonempty, find

$$\tilde{m} = \min_{A\in S_1} r(A), \tilde{M} = \max_{A\in S_1} r(A),$$

and determine the symmetric minimal rank solution in S_1, that is $S_{\tilde{m}} = \{A \mid r(A) = \tilde{m}, A \in S_1\}$.

Problem II. Given $A^* \in R^{n\times n}$, find $\tilde{A} \in S_{\tilde{m}}$ such that

$$||A^* - \tilde{A}|| = \min_{A\in S_{\tilde{m}}} ||A^* - A||.$$

The paper is organized as follows. First, in Section 2, we will introduce several lemmas which will be used in the later sections. Then, in Section 3, applying the matrix rank method, we will discuss the rank of the general symmetric solution to the matrix equation $AX = B$, where X, B are given matrices in $R^{n\times m}$. Based on this, the symmetric solution set with prescribed ranks to the matrix equation $AX = B$ will be presented. Lastly, in Section 4, an expression for the optimal approximation to the set of the minimal rank solution $S_{\tilde{m}}$ will be provided.

2. Some lemmas. The following lemmas are essential for deriving the solution to Problem 1.

LEMMA 2.1. [6] *Let A, B, C, and D be $m \times n$, $m \times k$, $l \times n$, $l \times k$ matrices, respectively. Then*

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(C(I - A^{-}A)), \tag{2.1}$$

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$$r\begin{pmatrix} A & B \\ C & D \end{pmatrix} = r\begin{pmatrix} A \\ C \end{pmatrix} + r\begin{pmatrix} A & B \end{pmatrix} - r(A) + r[E_G(D - CA^{-}B)F_H], \tag{2.2}$$

where $G = CF_A$ *and* $H = E_AB$.

LEMMA 2.2. [7] *Assume* $K \in R^{m\times n}$, $Y \in R^{p\times m}$ *are full-column rank matrices,* $Z \in R^{n\times q}$ *is full-row rank matrix. Then*

$$r(K) = r(YK) = r(KZ) = r(YKZ).$$

LEMMA 2.3. [7] *Suppose that* $L \in R^{n\times n}$ *satisfies* $L^2 = L$. *Then*

$$r(L) = trace(L).$$

LEMMA 2.4. [7]

$$r(MM^+) = r(M).$$

LEMMA 2.5. [7] *Given* $S \in R^{m\times n}$, *and* $J \in R^{k\times m}, W \in R^{l\times n}$ *satisfying* $J^TJ = I_m, W^TW = I_n$, *then*

$$(JSW^T)^+ = WS^+J^T.$$

LEMMA 2.6 (8). *Given* $X, B \in R^{n\times m}$, *let the singular value decompositions of* X *be*

$$X = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^T = U_1\Sigma V_1^T, \tag{2.3}$$

where $U = (U_1, U_2) \in OR^{n\times n}$, $U_1 \in R^{n\times k}$, $V = (V_1, V_2) \in OR^{m\times m}$, $V_1 \in R^{m\times k}$, $k = r(X)$, $\Sigma = diag(\sigma_1, \sigma_2, \cdots \sigma_k)$, $\sigma_1 \geq \cdots \geq \sigma_k > 0$. *Then* $AX = B$ *is solvable in* $SR^{n\times n}$ *if and only if*

$$BV_2 = 0, X^TB = B^TX, \tag{2.4}$$

and its general solution can be expressed as

$$A = U\begin{pmatrix} U_1^TBV_1\Sigma^{-1} & \Sigma^{-1}V_1^TB^TU_2 \\ U_2^TBV_1\Sigma^{-1} & A_{22} \end{pmatrix}U^T, \quad \forall A_{22} \in SR^{(n-k)\times(n-k)}.$$

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3. General expression of the solutions to Problem I. Now consider Problem 1 and suppose that (2.4) holds. Then the general symmetric solution of the equation $AX = B$ can be expressed as

$$A = U \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} U^T \tag{3.1}$$

where

$$A_{11} = U_1^T B V_1 \Sigma^{-1} \in SR^{k\times k}, \quad A_{12} = \Sigma^{-1} V_1^T B^T U_2 \in R^{k\times(n-k)}$$

and

$$A_{21} = U_2^T B V_1 \Sigma^{-1} \in R^{(n-k)\times k}$$

satisfy

$$A_{11} = A_{11}^T, \quad A_{21} = A_{12}^T, \quad A_{22} = A_{22}^T.$$

By Lemma 2.2 and the orthogonality of U, we obtain

$$r(A) = r \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{3.2}$$

Let

$$r \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} + r \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} - r(A_{11}) = t.$$

Applying (2.2) to A in (3.2), we have

$$r(A) = t + r[E_{G_1}(A_{22} - A_{21}A_{11}^+ A_{12})F_{H_1}],$$

where $G_1 = A_{21}(I - A_{11}^- A_{11})$, $H_1 = (I - A_{11}A_{11}^-)A_{12}$. Thus the minimal and the maximal ranks of A relative to A_{22} are, in fact, determined by the term $E_{G_1}(A_{22} - A_{21}A_{11}^+ A_{12})F_{H_1}$. It is quite easy to see that

$$\min_A r(A) = \min_{A_{22}} r[E_{G_1}(A_{22} - A_{21}A_{11}^+ A_{12})F_{H_1}] + t, \tag{3.3}$$

$$\max_A r(A) = \max_{A_{22}} r[E_{G_1}(A_{22} - A_{21}A_{11}^+ A_{12})F_{H_1}] + t. \tag{3.4}$$

Since $A_{11} = A_{11}^T$, $A_{12} = A_{21}^T$, we have $G_1 = H_1^T$. By $E_{G_1} = I - G_1G_1^+$, $F_{H_1} = I - H_1^+ H_1$, it is easy to verify that $E_{G_1}^T = F_{H_1} = E_{G_1}$.

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By Lemma 2.2 and $BV_2 = 0$, $A_{11} = A_{11}^T$, we have

$$r\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = r(U^T BV_1\Sigma^{-1}) = r(BV_1) = r(B(V_1, V_2)) = r(BV) = r(B), \tag{3.5}$$

$$\begin{aligned} r\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} &= r\begin{pmatrix} A_{11}^T & A_{12} \end{pmatrix} = r(\Sigma^{-1}V_1^T B^T U) = r(BV_1) \\ &= r(B(V_1, V_2)) = r(BV) = r(B), \end{aligned} \tag{3.6}$$

$$\begin{aligned} r(A_{11}) &= r(U_1^T BV_1\Sigma^{-1}) = r(\Sigma U_1^T BV_1) = r(V_1\Sigma U_1^T B(V_1, V_2)) \\ &= r(X^T BV) = r(X^T B). \end{aligned} \tag{3.7}$$

By (3.2), (3.3) and Lemma 1, when $r[E_{G_1}(A_{22} - A_{21}A_{11}^+A_{12})F_{H_1}] = 0$, $r(A)$ is minimal. By (3.5), (3.6), (3.7) we know that the minimal rank of symmetric solution for the matrix equation $AX = B$ is

$$\tilde{m} = 2r(B) - r(X^T B). \tag{3.8}$$

If the matrix A_{22} satisfies $r[E_{G_1}(A_{22} - A_{21}A_{11}^+A_{12})F_{H_1}] = 0$, we obtain the expression of the symmetric minimal rank solution. Let

$$A_{22} = A_{21}A_{11}^+A_{12} + N. \tag{3.9}$$

where $N \in SR^{(n-k)\times(n-k)}$ satisfies $E_{G_1}NF_{H_1} = 0$. Then the symmetric minimal rank solution of the matrix equation $AX = B$ can be expressed as

$$A = BX^+ + (BX^+)^T(I - XX^+) + U_2A_{22}U_2^T, \tag{3.10}$$

where A_{22} is as (3.9).

When $r[E_{G_1}(A_{22} - A_{21}A_{11}^+A_{12})F_{H_1}]$ is maximal, we can obtain the expression for the symmetric maximal rank solution of the matrix equation $AX = B$. Since $E_{G_1} = F_{H_1}$, and $r[E_{G_1}(A_{22} - A_{21}A_{11}^+A_{12})E_{G_1}]$ being maximal is equivalent to $r(A)$ being maximal, we have

$$\max_{A_{22}} r[E_{G_1}(A_{22} - A_{21}A_{11}^+A_{12})E_{G_1}] = r(E_{G_1}). \tag{3.11}$$

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Since E_{G_1} is an idempotent matrix, by Lemma 2.3, 2.4, 2.5, we have

$$\begin{aligned} r(E_{G_1}) = trace(E_{G_1}) &= n - k - r(G_1 G_1^+) = n - k - r(G_1) \\ &= n - k - r(A_{21}(I - A_{11}^+ A_{11})) \\ &= n - k - r \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} + r(A_{11}) \\ &= n - k - r(B) + r(X^T B) = n - r(X) - r(B) + r(X^T B). \end{aligned}$$

By (3.2), (3.4), (3.5), (3.6), (3.7) and Lemma 2.1, we know the maximal rank of symmetric solution of the matrix equation $AX = B$ is

$$\tilde{M} = n + r(B) - r(X). \tag{3.12}$$

Similar to the discussion of the minimal rank solution, the symmetric maximal rank solution of the matrix equation $AX = B$ can be expressed as

$$A = BX^+ + (BX^+)^T (I - XX^+) + U_2 A_{22} U_2^T, \tag{3.13}$$

where $A_{22} = A_{21} A_{11}^+ A_{12} + N$, the arbitrary matrix $N \in SR^{(n-k)\times(n-k)}$ satisfies $r(E_{G_1} N E_{G_1}) = n + r(X^T B) - r(X) - r(B)$.

Combining the above, we can immediately obtain the following theorem about the general solution to Problem 1.

THEOREM 3.1. *Given $X, B \in R^{n\times m}$, and a positive integer s, consider the singular value decomposition of X in (2.3). Then $AX = B$ has symmetric solution with rank of s if and only if*

$$BX^+ X = B, X^T B = B^T X, \tag{3.14}$$

$$2r(B) - r(X^T B) \le s \le n + r(B) - r(X). \tag{3.15}$$

Moreover, the general solution can be written as

$$A = BX^+ + (BX^+)^T (I - XX^+) + U_2 A_{22} U_2^T, \tag{3.16}$$

where $U_2 \in R^{n\times(n-k)}$, $U_2^T U_2 = I_{n-k}$, $N(X^T) = R(U_2)$, and $A_{22} = A_{21} A_{11}^+ A_{12} + N$, $N \in SR^{(n-k)\times(n-k)}$ satisfies $r(E_{G_1} N E_{G_1}) = s + r(X^T B) - 2r(B)$.

Now we discuss further the expression of the symmetric minimal rank solution of $AX = B$ and the solution set $S_{\tilde{m}}$. From the foregoing analysis, we know that given

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$X, B \in R^{n\times m}$, if the singular value decomposition of X is as in (2.3), and $AX = B$ satisfies (2.4), then the minimal rank of symmetric solution is $2r(B) - r(X^TB)$. Also the symmetric minimal rank solution can expressed as

$$A = BX^+ + (BX^+)^T(I - XX^+) + U_2A_{21}A_{11}^+A_{12}U_2^T + U_2NU_2^T, \tag{3.17}$$

where $N \in SR^{(n-k)\times(n-k)}$ satisfies $E_{G_1}NE_{G_1} = 0$.

Next we will discuss (3.17) further.

Equation (3.1) says $A_{11} = U_1^TBV_1\Sigma^{-1}$, $A_{12} = \Sigma^{-1}V_1^TB^TU_2$, $A_{21} = U_2^TBV_1\Sigma^{-1}$. Combining (2.3) and Lemma 2.5, we obtain

$$X^+ = V_1\Sigma^{-1}U_1^T, \quad XX^+ = U_1U_1^T, \tag{3.18}$$

that is,

$$XX^+BX^+ = U_1U_1^TBV_1\Sigma^{-1}U_1^T = U_1A_{11}U_1^T \Rightarrow (XX^+BX^+)^+ = U_1A_{11}^+U_1^T.$$

Thus

$$A_{11}^+ = U_1^T(XX^+BX^+)^+U_1, \tag{3.19}$$

hence

$$\begin{aligned} U_2A_{21}A_{11}^+A_{12}U_2^T &= U_2U_2^TBV_1\Sigma^{-1}U_1^T(XX^+BX^+)^+U_1\Sigma^{-1}V_1^TB^TU_2U_2^T \\ &= (I - XX^+)BX^+(XX^+BX^+)^+(BX^+)^T(I - XX^+). \end{aligned}$$

Substituting the above formula into (3.17), we obtain that the symmetric minimal rank solution of $AX = B$ can be expressed as

$$A = A_0 + U_2NU_2^T, \tag{3.20}$$

where $A_0 = BX^+ + (BX^+)^T(I - XX^+) + (I - XX^+)BX^+(XX^+BX^+)^+(BX^+)^T(I - XX^+)$, and $N \in SR^{(n-k)\times(n-k)}$ satisfies $E_{G_1}NE_{G_1} = 0$.

Assume the singular value decomposition of $G_1 = A_{21}(I - A_{11}^+A_{11})$ is

$$G_1 = P\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}Q^T = P_1\Gamma Q_1^T, \tag{3.21}$$

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where

$$P = (P_1, P_2) \in OR^{(n-k)\times(n-k)}, \quad P_1 \in R^{(n-k)\times t}, \quad Q = (Q_1, Q_2) \in OR^{k\times k},$$

$$Q_1 \in R^{k\times t}, \quad t = r(G_1), \quad \Gamma = diag(\alpha_1, \alpha_2, \cdots \alpha_t), \quad \alpha_1 \geq \cdots \geq \alpha_t > 0.$$

Then

$$G_1 G_1^+ = P_1 P_1^T, \quad E_{G_1} = I - P_1 P_1^T = P_2 P_2^T. \tag{3.22}$$

Thus from $E_{G_1} N E_{G_1} = 0$, i.e., $P_2 P_2^T N P_2 P_2^T = 0$, we have

$$N = P_1 P_1^T M P_1 P_1^T, \tag{3.23}$$

where $M \in SR^{(n-k)\times(n-k)}$ is arbitrary.

Substituting (3.23) into (3.20), we can obtain the following theorem.

Theorem 3.2. *Given $X, B \in R^{n\times m}$, assume the singular value decomposition of X is as in (2.3). If (2.4) is satisfied and the singular value decomposition of G_1 is as in (3.21), then the minimal rank of the symmetric solution of $AX = B$ is $2r(B) - r(X^T B)$, and the expression of the symmetric minimal rank solution is*

$$A = A_0 + U_2 P_1 P_1^T M P_1 P_1^T U_2^T, \tag{3.24}$$

where $A_0 = BX^+ + (BX^+)^T(I - XX^+) + (I - XX^+)BX^+(XX^+BX^+)^+(BX^+)^T(I - XX^+)$, and $M \in SR^{(n-k)\times(n-k)}$ is arbitrary.

4. The expression of the solution to Problem II. From (3.24), it is easy to verify that $S_{\tilde{m}}$ is a closed convex set. Therefore there exists a unique solution $\tilde{A}$ to Problem II. Now we give an expression for $\tilde{A}$.

The symmetric matrix set $SR^{n\times n}$ is a subspace of $R^{n\times n}$. Let $(SR^{n\times n})^\perp$ be the orthogonal complement space of $SR^{n\times n}$. Then for any $A^* \in R^{n\times n}$, we have

$$A^* = A_1^* + A_2^* \tag{4.1}$$

where $A_1^* \in SR^{n\times n}$, $A_2^* \in (SR^{n\times n})^\perp$. Partition the symmetric matrices

$$U^T A_0 U, \quad U^T A_1^* U$$

as

$$U^T A_0 U = \begin{pmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{pmatrix}, \quad U^T A_1^* U = \begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix}, \tag{4.2}$$

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where $A_{01} \in SR^{k\times k}$ and $A_{11}^* \in SR^{k\times k}$. Then we have the following theorem.

THEOREM 4.1. *Given $X, B \in R^{n\times m}$, assume the singular value decomposition of X is as in (2.3). Assume (2.4) is satisfied, the singular value decomposition of G_1 is as in (3.21) and let $A^* \in R^{n\times n}$ be given. Then Problem II has a unique solution $\tilde{A}$, which can be written as*

$$\tilde{A} = A_0 + U_2P_1P_1^T(A_{22}^* - A_{04})P_1P_1^TU_2^T, \tag{4.3}$$

where $A_0 = BX^+ + (BX^+)^T(I-XX^+) + (I-XX^+)BX^+(XX^+BX^+)^+(BX^+)^T(I-XX^+)$, and A_{22}^*, and A_{04} are given by (4.2).

Proof. For any $A^* \in R^{n\times n}$, we have

$$A^* = A_1^* + A_2^*, \tag{4.4}$$

where $A_1^* \in SR^{n\times n}$, $A_2^* \in (SR^{n\times n})^\perp$.

Utilizing the invariance of Frobenius norm for orthogonal matrices, for $A \in S_{\tilde{m}}$, by (3.24), (4.2) and $P_1P_1^T + P_2P_2^T = I$, $P_1P_1^TP_2P_2^T = 0$, where $P_1P_1^T$, $P_2P_2^T$ are orthogonal projection matrices, we obtain

$$||A^* - A||^2 = ||A_1^* + A_2^* - A||^2 = ||A_1^* - A||^2 + ||A_2^*||^2. \tag{4.5}$$

Thus $\min\limits_{A\in S_{\tilde{m}}} \| A^* - A \|$ is equivalent to $\min\limits_{A\in S_{\tilde{m}}} \| A_1^* - A \|$. Also

$$\begin{aligned}
||A_1^* - A||^2 &= \left\|A_1^* - A_0 - U_2P_1P_1^TMP_1P_1^TU_2^T\right\|^2 \\
&= \left\|A_1^* - A_0 - U\begin{pmatrix} 0 & 0 \\ 0 & P_1P_1^TMP_1P_1^T \end{pmatrix}U^T\right\|^2 \\
&= \left\|U^TA_1^*U - U^TA_0U - \begin{pmatrix} 0 & 0 \\ 0 & P_1P_1^TMP_1P_1^T \end{pmatrix}\right\|^2 \\
&= \left\|\begin{pmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{pmatrix} - \begin{pmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & P_1P_1^TMP_1P_1^T \end{pmatrix}\right\|^2 \\
&= \|A_{11}^* - A_{01}\|^2 + \|A_{12}^* - A_{02}\|^2 + \|A_{21}^* - A_{03}\|^2 \\
&+ \left\|A_{22}^* - A_{04} - P_1P_1^TMP_1P_1^T\right\|^2 \\
&= \|A_{11}^* - A_{01}\|^2 + \|A_{12}^* - A_{02}\|^2 + \|A_{21}^* - A_{03}\|^2 + \left\|(A_{22}^* - A_{04})P_2P_2^T\right\|^2 \\
&+ \left\|(A_{22}^* - A_{04})P_1P_1^T - P_1P_1^TMP_1P_1^T\right\|^2 \\
&= \|A_{11}^* - A_{01}\|^2 + \|A_{12}^* - A_{02}\|^2 + \|A_{21}^* - A_{03}\|^2 + \left\|(A_{22}^* - A_{04})P_2P_2^T\right\|^2 \\
&+ \left\|P_2P_2^T(A_{22}^* - A_{04})P_1P_1^T\right\|^2 + \left\|P_1P_1^T(A_{22}^* - A_{04})P_1P_1^T - P_1P_1^TMP_1P_1^T\right\|^2.
\end{aligned}$$

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Hence $\min_{A \in S_{\tilde{m}}} \| A^* - A \|$ is equivalent to

$$\min_{M \in SR^{(n-k)\times(n-k)}} \left\| P_1 P_1^T (A_{22}^* - A_{04}) P_1 P_1^T - P_1 P_1^T M P_1 P_1^T \right\| . \tag{4.6}$$

Obviously, the solution of (4.6) can be written as

$$M = A_{22}^* - A_{04} + P_2 P_2^T \widetilde{M} P_2 P_2^T, \quad \forall \widetilde{M} \in SR^{(n-k)\times(n-k)} \tag{4.7}$$

Substituting (4.7) into (3.24), we get that the unique solution to Problem II can be expressed as in (4.3). $\square$

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