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# A NEW SOLVABLE CONDITION FOR A PAIR OF GENERALIZED SYLVESTER EQUATIONS*

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**Abstract.** A necessary and sufficient condition is given for the quaternion matrix equations $A_iX + YB_i = C_i$ $(i = 1, 2)$ to have a pair of common solutions $X$ and $Y$. As a consequence, the results partially answer a question posed by Y.H. Liu (Y.H. Liu, Ranks of solutions of the linear matrix equation $AX + YB = C$, Comput. Math. Appl., 52 (2006), pp. 861-872).

**Key words.** Quaternion matrix equation, Generalized Sylvester equation, Generalized inverse, Minimal rank, Maximal rank.

**AMS subject classifications.** 15A03, 15A09, 15A24, 15A33.

**1. Introduction.** Throughout this paper, we denote the real number field by $\mathbb{R}$, the complex number field by $\mathbb{C}$, the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid \; i^2 = j^2 = k^2 = ijk = -1, \, a_0, a_1, a_2, a_3 \; \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, the identity matrix with the appropriate size by $I$, the transpose of a matrix $A$ by $A^T$, the column right space, the row left space of a matrix $A$ over $\mathbb{H}$ by $\mathcal{R}(A)$, $\mathcal{N}(A)$, respectively, a reflexive inverse of a matrix $A$ by $A^+$ which satisfies simultaneously $AA^+A = A$ and $A^+AA^+ = A^+$. Moreover, $R_A$ and $L_A$ stand for the two projectors $L_A = I - A^+A$, $R_A = I - AA^+$ induced by $A$. By [1], for a quaternion matrix $A$, $\dim \mathcal{R}(A) = \dim \mathcal{N}(A)$, which is called the rank of $A$ and denoted by $r(A)$.

Many problems in systems and control theory require the solution of the generalized Sylvester matrix equation $AX+YB = C$. Roth [2] gave a necessary and sufficient condition for the consistency of this matrix equation, which was called Roth's theorem on the equivalence of block diagonal matrices. Since Roth's paper appeared

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in 1952, Roth's theorem has been widely extended (see, e.g., [2]-[16]). Perturbation analysis of generalized Sylvester eigenspaces of matrix quadruples [17] leads to a pair of generalized Sylvester equations of the form

$$A_1X + YB_1 = C_1, A_2X + YB_2 = C_2. \tag{1.1}$$

In 1994, Wimmer [12] gave a necessary and sufficient condition for the consistency of (1.1) over $\mathbb{C}$ by matrix pencils. In 2002, Wang, Sun and Li [14] established a necessary and sufficient condition for the existence of constant solutions with bi(skew)symmetric constrains to (1.1) over a finite central algebra. Liu [16] in 2006 presented a necessary and sufficient condition for the pair of equations in (1.1) to have a common solution $X$ or $Y$ over $\mathbb{C}$, respectively, and proposed an open problem: find a necessary and sufficient condition for system (1.1) to have a pair of solutions $X$ and $Y$ by ranks.

Motivated by the work mentioned above and keeping applications and interests of quaternion matrices in view (e.g., [18]-[34]), in this paper we investigate the above open problem over $\mathbb{H}$. In Section 2, we establish a necessary and sufficient condition for (1.1) to have a pair of solutions $X$ and $Y$ over $\mathbb{H}$. In section 3, we present a counterexample to illustrate the errors in Liu's paper [16]. A conclusion and a further research topic related to (1.1) are also given.

**2. Main results.** The following lemma is due to Marsaglia and Styan [35], which can also be generalized to $\mathbb{H}$.

LEMMA 2.1. *Let* $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$ *and* $C \in \mathbb{H}^{l\times n}$. *Then they satisfy the following:*
(*a*) $r\begin{bmatrix} A & B \end{bmatrix} = r(A) + r(R_AB) = r(B) + r(R_BA)$.
(*b*) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CL_A) = r(C) + r(AL_C)$.
(*c*) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(R_BAL_C)$.

From Lemma 2.1 we can easily get the following.

LEMMA 2.2. *Let* $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{j\times k}$ *and* $E \in \mathbb{H}^{l\times i}$. *Then*
(*a*) $r(CL_A) = r\begin{bmatrix} A \\ C \end{bmatrix} - r(A)$.
(*b*) $r\begin{bmatrix} B & AL_C \end{bmatrix} = r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$.
(*c*) $r\begin{bmatrix} C \\ R_BA \end{bmatrix} = r\begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$.

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$(d)$ $r\begin{bmatrix} A & BL_D \\ R_EC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E).$

The following three lemmas are due to Baksalary and Kala [6], Tian [36],[37], respectively, which can be generalized to $\mathbb{H}$.

Lemma 2.3. *Let $A \in \mathbb{H}^{m\times p}, B \in \mathbb{H}^{q\times n}$ and $C \in \mathbb{H}^{m\times n}$ be known and $X \in \mathbb{H}^{p\times q}$ unknown. Then the matrix equation $AX + YB = C$ is solvable if and only if*

$$r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} = r(A) + r(B).$$

*In this case, the general solution to the matrix equation is given by*

$$X = A^{+}C + UB + L_AV,$$
$$Y = R_AC - AU + L_AWR_B,$$

*where $U \in \mathbb{H}^{p\times q}, V \in \mathbb{H}^{p\times n}$ and $W \in \mathbb{H}^{m\times q}$ are arbitrary.*

Lemma 2.4. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times p}$, $C \in \mathbb{H}^{q\times n}$ be given, $Y \in \mathbb{H}^{p\times n}$, $Z \in \mathbb{H}^{m\times q}$ be two variant matrices. Then*

$$\max_{Y,Z} r(A - BY - ZC) = \min\left\{m,\ n,\ r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right\}; \tag{2.1}$$

$$\min_{Y,Z} r(A - BY - ZC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C). \tag{2.2}$$

Lemma 2.5. *The matrix equation $A_1X_1B_1 + A_2X_2B_2 + A_3Y + ZB_3 = C$ is solvable if and only if the following four rank equalities are all satisfied:*

$$r\begin{bmatrix} C & A_1 & A_2 & A_3 \\ B_3 & 0 & 0 & 0 \end{bmatrix} = r\left[A_1, A_2, A_3\right] + r(B_3),$$

$$r\begin{bmatrix} C & A_3 \\ B_1 & 0 \\ B_2 & 0 \\ B_3 & 0 \end{bmatrix} = r(A_3) + r\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix},$$

$$r\begin{bmatrix} C & A_1 & A_3 \\ B_2 & 0 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} + r\left[A_1, A_3\right],$$

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$$r\begin{bmatrix} C & A_2 & A_3 \\ B_1 & 0 & 0 \\ B_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} + r\left[A_2, A_3\right].$$

LEMMA 2.6. *(Lemma 2.3 in [38]) Let $A, B$ be matrices over $\mathbb{H}$ and*

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, B = \left[B_1, B_2\right], S = A_2 L_{A_1}, T = R_{B_1} B_2.$$

*Then*

$$A^+ = \left[A_1^+ - L_{A_1} S^+ A_2 A_1^+, L_{A_1} S^+\right], B^+ = \begin{bmatrix} B_1^+ - B_1^+ B_2 T^+ R_{B_1} \\ T^+ R_{B_1} \end{bmatrix}$$

*are reflexive inverses of $A$ and $B$, respectively.*

LEMMA 2.7. *Suppose $A_1, A_2 \in \mathbb{H}^{m\times p}, B_1, B_2 \in \mathbb{H}^{q\times n}$ and $\widehat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ are given, $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ and $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ are any matrices with compatible dimensions. Then*

*(a) $[I_p, 0]\, L_{[A_1,A_2]} V$ and $[0, I_p]\, L_{[A_1,A_2]} V$ are independent, that is, for any $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$, $[I_p, 0]\, L_{[A_1,A_2]} V$ only relates to $V_2$ and the change of $[0, I_p]\, L_{[A_1,A_2]} V$ only relates to $V_1$, if and only if*

$$r\left[A_1, A_2\right] = r\left(A_1\right) + r\left(A_2\right).$$

*(b) $W R_{\widehat{B}} \begin{bmatrix} I_q \\ 0 \end{bmatrix}$ and $W R_{\widehat{B}} \begin{bmatrix} 0 \\ I_q \end{bmatrix}$ are independent, that is, for any $W = \left[W_1, W_2\right]$, $W R_{\widehat{B}} \begin{bmatrix} I_q \\ 0 \end{bmatrix}$ only relates to $W_1$ and $W R_{\widehat{B}} \begin{bmatrix} 0 \\ I_q \end{bmatrix}$ only relates to $W_2$, if and only if*

$$r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = r\left(B_1\right) + r\left(B_2\right).$$

*Proof.* From Lemma 2.6, we have

$$\begin{aligned}
&[I_p, 0]\, L_{[A_1,A_2]} V \\
&= [I_p, 0]\left(I - \begin{bmatrix} A_1^+ - A_1^+ A_2[(I - A_1A_1^+)A_2]^+(I - A_1A_1^+) \\ [(I - A_1A_1^+)A_2]^+(I - A_1A_1^+) \end{bmatrix}\left[A_1, A_2\right]\right) V \\
&= [I_p, 0]\left(I - \begin{bmatrix} A_1A_1^+ & A_1^+A_2 - A_1^+A_2[(I - A_1A_1^+)A_2]^+(I - A_1A_1^+)A_2 \\ 0 & [(I - A_1A_1^+)A_2]^+(I - A_1A_1^+)A_2 \end{bmatrix}\right)\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \\
&= V_1 - \left[A_1A_1^+, A_1^+A_2 - A_1^+A_2[(I - A_1A_1^+)A_2]^+(I - A_1A_1^+)A_2\right]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.
\end{aligned}$$

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Similarly, we have

$$
\begin{aligned}
&[0, I_p]\, L_{[A_1,A_2]} V \\
&= V_2 - \left[0, [(I - A_1A_1^+)A_2]^+(I - A_1A_1^+)A_2\right] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.
\end{aligned}
$$

Thus, $[I_p, 0]\, L_{[A_1,A_2]} V$ and $[0, I_p]\, L_{[A_1,A_2]} V$ are independent if and only if

$$A_1^+A_2 - A_1^+A_2[(I - A_1A_1^+)A_2]^+(I - A_1A_1^+)A_2 = 0.$$

According to Lemma 2.2, we have

$$
\begin{aligned}
&r\left(A_1^+A_2 - A_1^+A_2[(I - A_1A_1^+)A_2]^+(I - A_1A_1^+)A_2\right) \\
&= r\begin{bmatrix} (I - A_1A_1^+)A_2 \\ A_1^+A_2 \end{bmatrix} - r\left((I - A_1A_1^+)A_2\right) \\
&= r\begin{bmatrix} A_2 & A_1 \\ A_1^+A_2 & 0 \end{bmatrix} - r\,[A_2, A_1] \\
&= r\begin{bmatrix} A_2 & 0 \\ 0 & A_1 \end{bmatrix} - r\,[A_2, A_1]\,.
\end{aligned}
$$

That is $r\,[A_1, A_2] = r\,(A_1) + r\,(A_2)\,.$

Similarly, we can prove $(b)\,.$ ☐

Now we give the main result of this article.

THEOREM 2.8. *Suppose that every matrix equation in system (1.1) is consistent and*

$$r\,[A_1, A_2] = r\,(A_1) + r\,(A_2)\,, r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = r\,(B_1) + r\,(B_2)\,. \tag{2.3}$$

*Then system (1.1) has a pair of solutions $X$ and $Y$ if and only if*

$$r\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ -C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{2.4}$$

$$r\begin{bmatrix} A_1 & A_2 & -C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{bmatrix} = r\,[A_1, A_2] + r\,[B_1, B_2]\,, \tag{2.5}$$

$$r\begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & F \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\,[B_1, B_2]\,, \tag{2.6}$$

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$$r\begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & \widehat{F} \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r\,[B_1, B_2]\,, \tag{2.7}$$

*where*

$$F = A_1\left(A_2^{+}C_2 - A_1^{+}C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{+}\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \Omega B_1 \tag{2.8}$$

*and*

$$\widehat{F} = A_2\left(A_2^{+}C_2 - A_1^{+}C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{+}\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \Omega B_2 \tag{2.9}$$

*with* $\Omega = [-A_1, A_2]\,[-A_1, A_2]^{+}\left(R_{A_2}C_2B_2^{+} - R_{A_1}C_1B_1^{+}\right).$

*Proof.* Clearly, system (1.1) has a pair of solutions $X$ and $Y$ if and only if

$$A_1X_1 + Y_1B_1 = C_1 \tag{2.10}$$

$$A_2X_2 + Y_2B_2 = C_2 \tag{2.11}$$

are consistent and $X_1 = X_2$ and $Y_1 = Y_2$. It follows from Lemma 2.3 that $A_iX_i + Y_iB_i = C_i, i = 1, 2$, are consistent if and only if

$$C_i - A_iA_i^{+}C_i - C_iB_i^{+}B_i + A_iA_i^{+}C_iB_i^{+}B_i = 0, i = 1, 2.$$

In that case, the general solutions can be written as

$$X_i = A_i^{+}C_i + U_iB_i + L_{A_i}V_i, \tag{2.12}$$

$$Y_i = R_{A_i}C_i - A_iU_i + L_{A_i}W_iR_{B_i}, \tag{2.13}$$

where $U_i \in \mathbb{H}^{p\times q}, V_i \in \mathbb{H}^{p\times n}, W_i \in \mathbb{H}^{m\times q}$, $i = 1, 2$, are arbitrary. Hence,

$$\begin{aligned} &X_1 - X_2 \\ &= A_1^{+}C_1 - A_2^{+}C_2 + [U_1, U_2]\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + [L_{A_1}, -L_{A_2}]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &Y_1 - Y_2 \\ &= R_{A_1}C_1B_1^{+} - R_{A_2}C_2B_2^{+} + [-A_1, A_2]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + [W_1, W_2]\begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix}. \end{aligned} \tag{2.15}$$

Obviously, the equations (2.10) and (2.11) have common solutions, $X_1 = X_2, Y_1 = Y_2$, if and only if there exist $U_1$ and $U_2$ in (2.14) and (2.15) such that

$$\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r(X_1 - X_2) = 0, \tag{2.16}$$

$$\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r(Y_1 - Y_2) = 0, \tag{2.17}$$

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which is equivalent to the existence of $U_1$ and $U_2$ such that

$$A_1^+C_1 - A_2^+C_2 + [U_1, U_2]\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + [L_{A_1}, -L_{A_2}]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0, \tag{2.18}$$

and

$$R_{A_1}C_1B_1^+ - R_{A_2}C_2B_2^+ + [-A_1, A_2]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + [W_1, W_2]\begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} = 0. \tag{2.19}$$

It follows from (2.16-2.17) and Lemma 2.3 that

$$\begin{aligned} &\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r\left(X_1 - X_2\right) \\ &= r\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ -C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0 \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\min_{A_1X_1+Y_1B_1=C_1, A_2X_2+Y_2B_2=C_2} r\left(Y_1 - Y_2\right) \\ &= r\begin{bmatrix} A_1 & A_2 & -C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{bmatrix} - r\left[A_1, A_2\right] - r\left[B_1, B_2\right] = 0 \end{aligned} \tag{2.21}$$

implying, from Lemma 2.3, that (2.18) and (2.19) are solvable for $[U_1, U_2]$ and $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$, respectively, and

$$\begin{aligned} &[U_1, U_2] \\ &= R_{[L_{A_1}, -L_{A_2}]}\left(A_2^+C_2 - A_1^+C_1\right)\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ - [L_{A_1}, -L_{A_2}]\widetilde{U} + WR_{\widehat{B}}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} &\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= [-A_1, A_2]^+\left(R_{A_2}C_2B_2^+ - R_{A_1}C_1B_1^+\right) + \widehat{U}\begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} + L_{[-A_1, A_2]}V, \end{aligned} \tag{2.23}$$

where $\widehat{U}, \widetilde{U}, W$ and $V$ are any matrices over $\mathbb{H}$ with appropriate dimensions. Clearly,

$$[U_1, U_2]\begin{bmatrix} I_q \\ 0 \end{bmatrix} = [I_p, 0]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \tag{2.24}$$

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and

$$[U_1, U_2] \begin{bmatrix} 0 \\ I_q \end{bmatrix} = [0, I_p] \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}. \tag{2.25}$$

Substituting (2.22) and (2.23) into (2.24) and (2.25) yields

$$\begin{aligned} &R_{[L_{A_1}, -L_{A_2}]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [I_p, 0]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U} \begin{bmatrix} I_q \\ 0 \end{bmatrix} + [I_p, 0]\, \widehat{U} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W R_{\widehat{B}} \begin{bmatrix} I_q \\ 0 \end{bmatrix} + [I_p, 0]\, L_{[-A_1, A_2]} V, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &R_{[L_{A_1}, -L_{A_2}]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I_q \end{bmatrix} - [0, I_p]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U} \begin{bmatrix} 0 \\ I_q \end{bmatrix} + [0, I_p]\, \widehat{U} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W R_{\widehat{B}} \begin{bmatrix} 0 \\ I_q \end{bmatrix} + [0, I_p]\, L_{[-A_1, A_2]} V \end{aligned} \tag{2.27}$$

where

$$\alpha = [-A_1, A_2]^+ \left(R_{A_2} C_2 B_2^+ - R_{A_1} C_1 B_1^+\right), \widehat{B} = \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}.$$

Let

$$\widetilde{U} = \left[\widetilde{U}_1, \widetilde{U}_2\right], \widehat{U} = \begin{bmatrix} \widehat{U}_1 \\ \widehat{U}_2 \end{bmatrix}$$

in (2.26) and (2.27) where $\widetilde{U}_1, \widetilde{U}_2, \widehat{U}_1$ and $\widehat{U}_2$ are matrices over $\mathbb{H}$ with appropriate dimensions. Then it follows from (2.3) and Lemma 2.7 that (2.26) and (2.27) can be written as

$$\begin{aligned} &R_{[L_{A_1}, -L_{A_2}]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} I_q \\ 0 \end{bmatrix} - [I_p, 0]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U}_1 + \widehat{U}_1 \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W_1 R_{B_2 L_{B_1}} + V_1 R_{A_1}, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &R_{[L_{A_1}, -L_{A_2}]} \left(A_2^+ C_2 - A_1^+ C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^+ \begin{bmatrix} 0 \\ I_q \end{bmatrix} - [0, I_p]\, \alpha \\ &= [L_{A_1}, -L_{A_2}]\, \widetilde{U}_2 + \widehat{U}_2 \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} - W_2 L_{B_1} + V_2 L_{R_{A_1} A_2}. \end{aligned} \tag{2.29}$$

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Therefore, the equations (2.10) and (2.11) have common solutions, $X_1 = X_2, Y_1 = Y_2$, if and only if there exist $W_1, V_1, \widetilde{U_1}, \widehat{U_1}; W_2, V_2, \widetilde{U_2}, \widehat{U_2}$ such that (2.28) and (2.29) hold, respectively. By Lemma 2.5, the equation (2.28) is solvable if and only if

$$
(2.30) \qquad r\begin{bmatrix} C & [L_{A_1}, -L_{A_2}] & [I_p, 0]\, L_{[-A_1, A_2]} \\ \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} & 0 & 0 \\ R_{\widehat{B}}\begin{bmatrix} -I_q \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}
= r\begin{bmatrix} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} \\ R_{\widehat{B}}\begin{bmatrix} -I_q \\ 0 \end{bmatrix} \end{bmatrix} + r([L_{A_1}, -L_{A_2}]\, , [I_p, 0]\, L_{[-A_1, A_2]}),
$$

where

$$
C = \left(I - [L_{A_1}, -L_{A_2}]\, [L_{A_1}, -L_{A_2}]^{+}\right) \left(A_2^{+} C_2 - A_1^{+} C_1\right) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{+} \begin{bmatrix} I_q \\ 0 \end{bmatrix}
- [I_p, 0]\, [-A_1, A_2]^{+} \left(R_{A_2} C_2 B_2^{+} - R_{A_1} C_1 B_1^{+}\right).
$$

It follows from Lemma 2.2, (2.8) and block Gaussian elimination that

$$
r\begin{bmatrix} C & [L_{A_1}, -L_{A_2}] & [I_p, 0]\, L_{[-A_1, A_2]} \\ \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} & 0 & 0 \\ R_{\widehat{B}}\begin{bmatrix} -I_q \\ 0 \end{bmatrix} & 0 & 0 \end{bmatrix}
= r\begin{bmatrix} C & L_{A_1} & -L_{A_2} & I_p & 0 & 0 \\ R_{B_1} & 0 & 0 & 0 & 0 & 0 \\ -R_{B_2} & 0 & 0 & 0 & 0 & 0 \\ -I_q & 0 & 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 & 0 & -B_2 \\ 0 & 0 & 0 & -A_1 & A_2 & 0 \end{bmatrix} - r\,[-A_1, A_2] - r\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}
= r\begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & F \end{bmatrix} + p + q - r\,[-A_1, A_2] - r\begin{bmatrix} B_1 \\ -B_2 \end{bmatrix},
$$

$$
r\begin{bmatrix} \begin{bmatrix} R_{B_1} \\ -R_{B_2} \end{bmatrix} \\ R_{\widehat{B}}\begin{bmatrix} -I_q \\ 0 \end{bmatrix} \end{bmatrix} = r\,[B_1, B_2] + q - r\,(B_1) - r\,(B_2)\,,
$$

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$$r\left[[L_{A_1},-L_{A_2}],[I_p,0]\,L_{[-A_1,A_2]}\right]=r\begin{bmatrix}A_1\\A_2\end{bmatrix}+p-r(A_1)-r(A_2)$$

implying that (2.6) follows from (2.3) and (2.30).

Similarly, the equation (2.29) is solvable if and only if

$$r\begin{bmatrix}\widehat{C} & J & K\\ \begin{bmatrix}R_{B_1}\\-R_{B_2}\end{bmatrix} & 0 & 0\\ R_{\widehat{B}}\begin{bmatrix}0\\I_q\end{bmatrix} & 0 & 0\end{bmatrix}=r\begin{bmatrix}\begin{bmatrix}R_{B_1}\\-R_{B_2}\end{bmatrix}\\ R_{\widehat{B}}\begin{bmatrix}0\\I_q\end{bmatrix}\end{bmatrix}+r(J,K),\tag{2.31}$$

where

$$\begin{aligned}J=&[L_{A_1},-L_{A_2}],K=[0,I_p]\,L_{[-A_1,A_2]},\\ \widehat{C}=&\left(I-[L_{A_1},-L_{A_2}][L_{A_1},-L_{A_2}]^{+}\right)\left(A_2^{+}C_2-A_1^{+}C_1\right)\begin{bmatrix}B_1\\-B_2\end{bmatrix}^{+}\begin{bmatrix}0\\I_q\end{bmatrix}\\ &-[0,I_p][-A_1,A_2]^{+}\left(R_{A_2}C_2B_2^{+}-R_{A_1}C_1B_1^{+}\right).\end{aligned}$$

Simplifying (2.31) yields (2.7) from (2.3) and (2.9). Moreover, (2.4) and (2.5) follow from (2.20) and (2.21), respectively. This proof is completed. □

Under an assumption, we have derived a necessary and sufficient condition for system (1.1) to have a pair of solutions $X$ and $Y$ over $\mathbb{H}$ by ranks. The open problem in [16] is, therefore, partially solved. By the way, we find that Corollary 2.3 in [16] is wrong.

Now we present a counterexample to illustrate the error. We first state the wrong corollary mentioned above: Suppose that the complex matrix equation $(A_0+A_1i)X+Y(B_0+B_1i)=(C_0+C_1i)$ is consistent. Then
$(a)$ Equation $(A_0+A_1i)X+Y(B_0+B_1i)=(C_0+C_1i)$ has a pair of real solutions $X=X_0$ and $Y=Y_0$ if and only if

$$r\begin{bmatrix}B_0 & 0\\B_1 & 0\\C_0 & A_0\\C_1 & A_1\end{bmatrix}=r\begin{bmatrix}A_0\\A_1\end{bmatrix}+r\begin{bmatrix}B_0\\B_1\end{bmatrix},\tag{2.32}$$

$$r\begin{bmatrix}A_0 & A_1 & C_0 & C_1\\0 & 0 & B_0 & B_1\end{bmatrix}=r[A_0,A_1]+r[B_0,B_1].\tag{2.33}$$

A counterexample is as follows. Let

$$A_0=B_0=\begin{bmatrix}1&0\\0&1\end{bmatrix},A_1=B_1=C_0=0,C_1=\begin{bmatrix}i&0\\0&i\end{bmatrix}.$$

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Then we have

$$r\begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_0 & A_0 \\ C_1 & A_1 \end{bmatrix} = r\begin{bmatrix} A_0 & A_1 & C_0 & C_1 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} = 4,$$

$$r\begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = r\begin{bmatrix} B_0 \\ B_1 \end{bmatrix} = r\,[A_0, A_1] = r\,[B_0, B_1] = 2,$$

i.e. (2.32) and (2.33) hold. However, the following matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X + Y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

has no real solution obviously.

Similarly, we can give a counterexample to illustrate that the part $(c)$ of Corollary 2.3 in [16] is also wrong.

Using the methods in this paper, we can correct the mistakes mentioned above. We are planning to present these corrections in a separate article.

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