

## ON THE INVERSE EIGENVALUE PROBLEMS: THE CASE OF SUPERSTARS\*

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**Abstract.** Let  $T$  be a tree and let  $x_0$  be a vertex of  $T$ .  $T$  is called a superstar with central vertex  $x_0$  if  $T - x_0$  is a union of paths. The General Inverse Eigenvalue Problem for certain trees is partially answered. Using this description, some superstars are presented for which the problem of ordered multiplicity lists and the Inverse Eigenvalue Problem are not equivalent.

**Key words.** Eigenvalues, Tree, Graph, Symmetric matrices.

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**1. Introduction.** Let  $A = [a_{ij}]$  be an  $n \times n$  real symmetric matrix. We denote by  $G(A) = (X, U)$  the simple graph on  $n$  vertices,  $\{1, \dots, n\}$ , such that  $\{i, j\} \in U$ ,  $i \neq j$ , if and only if  $a_{ij} \neq 0$ . Let  $A(i)$  denote the principal submatrix of  $A$  obtained by deleting row and column  $i$ .

Let  $G = (X, U)$  a simple graph, where  $X = \{x_1, \dots, x_n\}$  is the vertex set of  $G$  and let  $\mathcal{S}(G)$  be the set of all  $n \times n$  real symmetric matrices  $A$  such that  $G(A) \cong G$ . One of the most important problems of Spectral Graph Theory is the General Inverse Eigenvalue Problem for  $\mathcal{S}(G)$  (GIEP for  $\mathcal{S}(G)$ ):

“What are all the real numbers  $\lambda_1 \leq \dots \leq \lambda_n$  and  $\mu_1 \leq \dots \leq \mu_{n-1}$  that may occur as the eigenvalues of  $A$  and  $A(i)$ , respectively, as  $A$  runs over  $\mathcal{S}(G)$ ?”

Another important problem is the Inverse Eigenvalue Problem for  $\mathcal{S}(G)$  (IEP for  $\mathcal{S}(G)$ ):

“What are all the real numbers  $\lambda_1 \leq \dots \leq \lambda_n$  that may occur as the eigenvalues of  $A$ , as  $A$  runs over  $\mathcal{S}(G)$ ?”

First, we remind the reader of some results concerning the GIEP.

Perhaps the most well known result on this subject is the Interlacing Theorem for Eigenvalues of Hermitian matrices:

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**THEOREM 1.1.** [11] *If  $A$  is an  $n \times n$  Hermitian matrix with eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$  and if  $A(i)$  has eigenvalues  $\mu_1 \leq \dots \leq \mu_{n-1}$  then*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \dots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

If  $\lambda$  is a real number and  $A$  is an  $n \times n$  real symmetric matrix, we denote by  $m_A(\lambda)$  the multiplicity of  $\lambda$  as an eigenvalue of  $A$ . As a Corollary of Theorem 1.1 we have the following result:

**PROPOSITION 1.2.** [11] *Let  $A$  be an  $n \times n$  Hermitian matrix and let  $\lambda$  be an eigenvalue of  $A$ . Then*

$$m_A(\lambda) - 1 \leq m_{A(i)}(\lambda) \leq m_A(\lambda) + 1, \quad i = 1, \dots, n.$$

When the graph  $G$  is a path, we have the well-known fact:

**PROPOSITION 1.3.** *Let  $T$  be a path. If  $A$  is a matrix in  $\mathcal{S}(T)$  and  $j$  is a pendant vertex of  $T(A)$ , all the eigenvalues of  $A$  have multiplicity 1 and the eigenvalues of  $A(j)$  strictly interlace those of  $A$ .*

Several authors proved the converse of Proposition 1.3.

The solution of the GIEP for  $\mathcal{S}(G)$  when  $G$  is a cycle is also well known, see [3, 4, 5, 6].

Leal Duarte generalized the converse of Proposition 1.3 to any tree, [13].

**PROPOSITION 1.4.** *Let  $T$  be a tree on  $n$  vertices and let  $i$  be a vertex of  $T$ . Let  $\lambda_1 < \dots < \lambda_n$  and  $\mu_1 < \dots < \mu_{n-1}$  be real numbers. If*

$$\lambda_1 < \mu_1 < \lambda_2 < \dots < \mu_{n-1} < \lambda_n,$$

*then there exists a matrix  $A$  in  $\mathcal{S}(T)$ , with eigenvalues  $\lambda_1 < \dots < \lambda_n$ , and such that,  $A(i)$  has eigenvalues  $\mu_1 < \dots < \mu_{n-1}$ .*

In [7], Johnson and Leal Duarte studied this problem for vertices, of a generic path  $T$ , of degree two and solved it for the particular case that occurs when  $A$  is a matrix in  $\mathcal{S}(T)$  and  $A(i)$  has eigenvalues of multiplicity two.

In 2003, Johnson, Leal Duarte and Saiago, [8], rewrote the GIEP for  $\mathcal{S}(G)$ :

“Let  $G$  be a simple connected graph  $G$  on  $n$  vertices,  $x_0$  be a vertex of  $G$  of degree  $k$  and  $G_1, \dots, G_k$  be the connected components of  $G - x_0$ . Let  $\lambda_1, \dots, \lambda_n$  be real numbers,  $g_1(t), \dots, g_k(t)$  be monic polynomials having only real roots and such that

deg  $g_i(t)$  is equal to the number of vertices of  $G_i$ . Is it possible to construct a matrix  $A$  in  $\mathcal{S}(G)$  such that  $A$  has eigenvalues  $\lambda_1, \dots, \lambda_n$  and the characteristic polynomial of  $A[G_i]$  is  $g_i(t)$  ( $A[G_i]$  is the principal submatrix of  $A$  obtained by deleting rows and columns that correspond to vertices of  $G - G_i$ )?"

With respect to GIEP for  $\mathcal{S}(G)$  as written above, the following results are proven in [8].

**THEOREM 1.5.** *Let  $T$  be a tree on  $n$  vertices and  $x_0$  be a vertex of  $T$  of degree  $k$  whose neighbors are  $x_1, \dots, x_k$ . Let  $T_i$  be the branch (connected component) of  $T$  at  $x_0$  containing  $x_i$  and  $s_i$  be the number of vertices in  $T_i$ ,  $i = 1, \dots, k$ .*

*Let  $g_1(t), \dots, g_k(t)$  be monic polynomials having only distinct real roots, with deg  $g_i(t) = s_i$ ,  $p_1, \dots, p_s$  be the distinct roots among polynomials  $g_i(t)$  and  $m_i$  be the multiplicity of root  $p_i$  in  $\prod_{i=1}^k g_i(t)$ .*

*Let  $g(t)$  be a monic polynomial of degree  $s + 1$ .*

*There exists a matrix  $A$  in  $\mathcal{S}(T)$  with characteristic polynomial*

$$f(t) = g(t) \prod_{i=1}^s (t - p_i)^{m_i - 1}$$

*and such that*

- 1)  $A[T_i]$  has characteristic polynomial  $g_i(t)$ ,  $i = 1, \dots, k$ ,
- 2) If  $1 \leq i \leq k$  and  $s_i > 1$ , the eigenvalues of  $A[T_i - x_i]$  strictly interlace those of  $A[T_i]$ ,

*if and only if the roots of  $g(t)$  strictly interlace those of  $\prod_{i=1}^s (t - p_i)$ .*

The statement of the previous theorem is shorter when  $T$  is a generalized tree, [8].

**THEOREM 1.6.** *Let  $T$  be a generalized star on  $n$  vertices with central vertex  $x_0$ , let  $T_1, \dots, T_k$  be the branches of  $T$  at  $x_0$ , and let  $l_0, \dots, l_k$  be the number of vertices of  $T_1, \dots, T_k$ , respectively.*

*Let  $g_1(t), \dots, g_k(t)$  be monic polynomials having only real roots, with deg  $g_i(t) = l_i$ , let  $p_1, \dots, p_l$  be the distinct roots among polynomials  $g_i(t)$  and let  $m_i$  be the multiplicity of root  $p_i$  in  $\prod_{i=1}^k g_i(t)$ , ( $m_i \geq 1$ ).*

*Let  $g(t)$  be a monic polynomial of degree  $l + 1$ .*

*There exists a matrix  $A$  in  $\mathcal{S}(T)$  with characteristic polynomial*

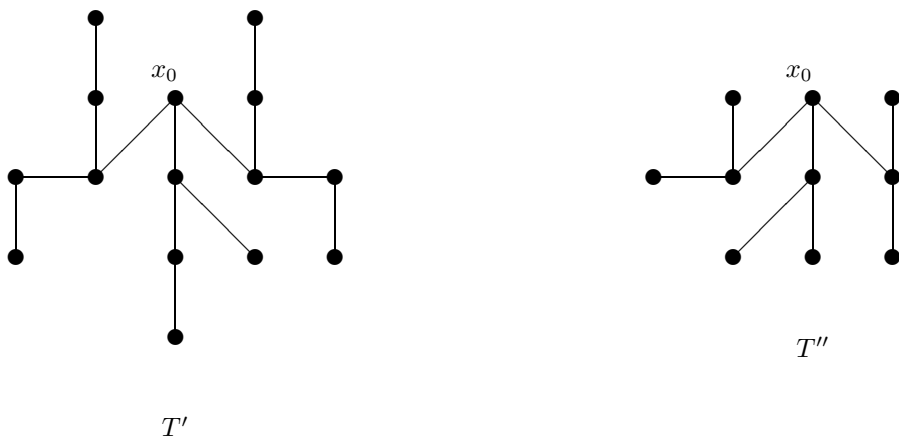
$$f(t) = g(t) \prod_{i=1}^l (t - p_i)^{m_i - 1}$$

and such that  $A[T_i]$  has characteristic polynomial  $g_i(t)$ ,  $i = 1, \dots, k$ , if and only if each  $g_i(t)$  has only simple roots and the roots of  $g(t)$  strictly interlace those of  $\prod_{i=1}^l (t - p_i)$ .

In [8], the GIEP was solved for  $\mathcal{S}(T)$  when  $T$  is a generalized star. Moreover, the authors of [8] proved that the IEP for  $\mathcal{S}(G)$  when  $G$  is a generalized star,  $T$ , is equivalent to the determination of all possible ordered multiplicity lists of  $T$ ; that is, if  $A \in \mathcal{S}(T)$  has eigenvalues  $\lambda_1 < \dots < \lambda_t$  of multiplicities  $m_1, \dots, m_t$ , respectively, then for any set of real numbers  $\lambda'_1 < \dots < \lambda'_t$ , there exists a matrix  $A' \in \mathcal{S}(T)$  having eigenvalues  $\lambda'_1 < \dots < \lambda'_t$  of multiplicities  $m_1, \dots, m_t$ , respectively.

The case of double generalized stars has also been studied by Barioli and Fallat in [1].

In [2], Barioli and Fallat gave the first example of a graph (the graph  $T'$ ) for which the equivalence between the ordered multiplicity lists and the IEP does not occur. Another example appears in [10], the graph  $T''$ .



Bearing in mind these two graphs, we give now the following definitions.

DEFINITION 1.7. Let  $T$  be a tree and  $x_0$  be a vertex of  $T$ . A superstar  $T$  with central vertex  $x_0$  is a tree such that  $T - x_0$  is a union of paths.

DEFINITION 1.8. Let  $T$  be a tree and  $x_0$  be a vertex of  $T$ . Let  $T_i$  be a connected component of  $T - x_0$  and  $x_i$  be the vertex of  $T_i$  adjacent to  $x_0$  in  $T$ . We say that  $T_i$  is a cut branch at  $x_i$  if  $T_i - x_i$  has at most two connected components.

The aforementioned trees  $T'$  and  $T''$  are superstars. All the paths, stars and generalized stars (defined in [8]) are also superstars. Sometimes, when  $T$  is a superstar

with central vertex  $x_0$ , we say that  $T$  is a superstar at  $x_0$ .

More recently, ([12] Lemma 16) Kim and Shader proved a necessary condition for the GIEP to have a solution for some superstars. Notice that in [12] the  $k$ -whirls are the superstars of the present paper.

LEMMA 1.9. ([12] Lemma 16) *Let  $T$  be a superstar on  $n$  vertices with central vertex  $x_0$  whose neighbors are  $x_1, \dots, x_k$ . Let  $T_i$  be the branch of  $T$  at  $x_0$  containing  $x_i$ ,  $i = 1, \dots, k$ . Suppose that  $T_1, \dots, T_k$  are cut branches at  $x_1, \dots, x_r$ , respectively, and let  $\alpha_1^i, \alpha_2^i$  be the paths of  $T_i - x_i$ ,  $i = 1, \dots, k$ .*

*Let  $A$  in  $\mathcal{S}(T)$  and let  $A'$  be the direct sum of  $A[\alpha_j^i]$ , for all  $i \in \{1, \dots, k\}$  and  $j \in \{1, 2\}$ . If  $n_r$  denotes the number of eigenvalues of  $A$  with multiplicity  $r$ , then the following holds:*

- (a)  $n_{k+1} \leq 1$  and  $n_j = 0$  for  $j \geq k + 2$ ;
- (b) if  $\lambda$  is an eigenvalue of  $A$  and  $m_A(\lambda) = k + 1$ , then  $\lambda$  is a simple eigenvalue of  $A[\alpha_j^i]$ , for all  $i \in \{1, \dots, k\}$  and  $j \in \{1, 2\}$ , and  $m_{A'}(\lambda) = 2k$ ;
- (c) if  $\mu$  is an eigenvalue of  $A$  and  $m_A(\mu) = k$ , then for all  $i \neq s$ ,  $j$  and  $t$ ,  $\mu$  is a simple eigenvalue of at least one of  $A[\alpha_j^i]$ ,  $A[\alpha_t^s]$ , and  $m_{A'}(\mu) \geq 2k - 2$ ; and
- (d)  $(2k - 2)n_k + (2k)n_{k+1} \leq n - (k + 1)$ .

In section 3, the methods used to prove Theorem 1.5 allows us to generalize it. As in this generalization we suppose that some branches of the tree  $T$  are cut branches we obtain a much more general result than Lemma 1.9. Using this generalization we prove in section 4:

- 1) there is no matrix  $A' \in \mathcal{S}(T')$  (where  $T'$  is the above mentioned tree) having eigenvalues

$$(0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7),$$

but there exists a matrix  $A \in \mathcal{S}(T')$  having eigenvalues

$$(1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7).$$

- 2) there is no matrix  $A' \in \mathcal{S}(T'')$  (where  $T''$  is the above mentioned tree) having eigenvalues

$$(-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2}, 2),$$

but there exists a matrix  $A \in \mathcal{S}(T'')$  having eigenvalues

$$(-\sqrt{5}, -\sqrt{2}, -\sqrt{2}, 0, 0, 0, 0, \sqrt{2}, \sqrt{2}, \sqrt{5}).$$

In section 4, we also prove that the converse of Lemma 1.9 is not true.

**2. Prior Results.** The key tool used to prove Proposition 1.4 and Theorem 1.5 was the decomposition of a real rational function into partial fractions. We recall here the following well known results, which will be useful for the present work.

LEMMA 2.1. [13] *Let  $g(t)$  be a monic polynomial of degree  $n$ ,  $n > 1$ , having only distinct real roots and  $h(t)$  be a monic polynomial with  $\deg h(t) < \deg g(t)$ . Then  $h(t)$  has  $n - 1$  distinct real roots strictly interlacing the roots of  $g(t)$  if and only if the coefficients of the partial fraction decomposition (pfd) of  $\frac{h(t)}{g(t)}$  are positive real numbers.*

REMARK 2.2. If  $\lambda_1, \dots, \lambda_n$  and  $\mu_1, \dots, \mu_{n-1}$  are real numbers such that

$$\lambda_1 < \mu_1 < \lambda_2 < \dots < \mu_{n-1} < \lambda_n$$

and,  $g(t)$  and  $h(t)$  are the monic polynomials

$$\begin{aligned} g(t) &= (t - \lambda_1)(t - \lambda_2) \dots (t - \lambda_n), \\ h(t) &= (t - \mu_1)(t - \mu_2) \dots (t - \mu_{n-1}), \end{aligned}$$

it is easy to show that  $\frac{g(t)}{h(t)}$  can be represented in a unique way as

$$\frac{g(t)}{h(t)} = (t - a) - \sum_{i=1}^{n-1} \frac{x_i}{t - \mu_i}$$

in which  $a = \sum_{i=1}^n \lambda_i - \sum_{i=1}^{n-1} \mu_i$  and  $x_i, i = 1, \dots, n - 1$ , are positive real numbers such that

$$x_i = -\frac{g(\mu_i)}{\prod_{j=1, j \neq i}^{n-1} (\mu_i - \mu_j)} = -\frac{\prod_{j=1}^n (\mu_i - \lambda_j)}{\prod_{j=1, j \neq i}^{n-1} (\mu_i - \mu_j)}.$$

If  $T$  is a tree on  $n$  vertices,  $A \in \mathcal{S}(T)$  and  $T_i$  is a subgraph of  $T$ , we denote by  $A[T_i]$  (respectively,  $A(T_i)$ ) the principal submatrix of  $A$  obtained by deleting rows and columns that correspond to vertices of  $T \setminus T_i$  (respectively,  $T_i$ ). We will also need the expansion of the characteristic polynomial at a particular vertex of  $T$  with neighbors  $x_1, \dots, x_k$ .

LEMMA 2.3. [9] *Let  $T$  be a tree on  $n$  vertices and let  $A = [a_{ij}]$  be a matrix in  $\mathcal{S}(T)$ . If  $x_0$  is a vertex of  $T$  of degree  $k$ , whose neighbors in  $T$  are  $x_1, \dots, x_k$ , then*

$$(2.1) \quad p_A(t) = (t - a_{x_0 x_0})p_{A[T-x_0]}(t) - \sum_{i=1}^k |a_{x_0 x_i}|^2 p_{A[T_i-x_i]}(t) \prod_{j=1, j \neq i}^k p_{A[T_j]}(t),$$

with the convention that  $p_{A[T_i-x_i]}(t) = 1$  whenever the vertex set of  $T_i$  is  $\{x_i\}$ .

Since  $T$  is a tree, if  $A$  is a matrix in  $\mathcal{S}(T)$  and  $x_0$  is a vertex of degree  $k$ , we have  $A(x_0) = A[T_1] \oplus \dots \oplus A[T_k]$ , where  $T_i$  is the branch of  $T - x_0$  containing the neighbor  $x_i$  of  $x_0$  in  $T$ . The following result was shown in [8].

LEMMA 2.4. [8] *Let  $T$  be a tree and  $A$  be a matrix in  $\mathcal{S}(T)$ . Let  $x_0$  be a vertex of  $T$  and  $\lambda$  be an eigenvalue of  $A(x_0)$ . Let  $x_i$  be a neighbor of  $x_0$  in  $T$  and  $T_i$  be the branch of  $T$  at  $x_0$  containing  $x_i$ . If  $\lambda$  is an eigenvalue of  $A[T_i]$  and*

$$m_{A[T_i-x_i]}(\lambda) = m_{A[T_i]}(\lambda) - 1,$$

then  $m_{A(x_0)}(\lambda) = m_A(\lambda) + 1$ .

When  $T$  is a generalized star with central vertex  $x_0$ , each branch  $T_i$  of  $T$  at  $x_0$  is a path. Thus, if  $B$  is a matrix in  $\mathcal{S}(T_i)$  then all the eigenvalues of  $B$  have multiplicity 1 and the eigenvalues of  $B(x_i)$  strictly interlace those of  $B$ .

LEMMA 2.5. [8] *Let  $T$  be a generalized star with central vertex  $x_0$ . If  $A$  is a matrix in  $\mathcal{S}(T)$  and  $\lambda$  is an eigenvalue of  $A(x_0)$  then  $m_{A(x_0)}(\lambda) = m_A(\lambda) + 1$ .*

Let  $T$  be a tree and  $A$  be a matrix in  $\mathcal{S}(T)$ . A result concerning the multiplicity of the largest and smallest eigenvalues of  $A$  was proved in [9].

LEMMA 2.6. *If  $T$  is a tree, the largest and smallest eigenvalues of each matrix  $A$  in  $\mathcal{S}(T)$ , have multiplicity 1. Moreover, the largest or smallest eigenvalue of a matrix  $A$  in  $\mathcal{S}(T)$  cannot occur as an eigenvalue of a submatrix  $A(x_0)$ , for any vertex  $x_0$  in  $T$ .*

Using this result we can prove the following lemma.

LEMMA 2.7. *Let  $T$  be a tree on  $n > 1$  vertices,  $x_0$  be a vertex in  $T$  and  $A$  be a matrix in  $\mathcal{S}(T)$ . Then there exists an eigenvalue  $\lambda$  of  $A(x_0)$  such that  $m_A(\lambda) = m_{A(x_0)}(\lambda) - 1$ .*

*Proof.* Suppose that for each eigenvalue  $\lambda$  of  $A(x_0)$  we have  $m_A(\lambda) \neq m_{A(x_0)}(\lambda) - 1$ . Consequently, if  $\lambda$  is an eigenvalue of  $A(x_0)$  then  $m_A(\lambda) \geq m_{A(x_0)}(\lambda)$ . Since  $A(x_0)$  has  $n - 1$  eigenvalues and  $A$  has  $n$  eigenvalues, there exists at most one eigenvalue of  $A$  which is not an eigenvalue of  $A(x_0)$ . Using Lemma 2.6 we obtain a contradiction, and the result follows.  $\square$

Let  $T$  be a tree and  $A$  be a matrix in  $\mathcal{S}(T)$ . We have the following result, which can be obtained collectively from [14, 15]. Note that if  $A$  is a matrix in  $\mathcal{S}(T)$ , then  $A(i)$  will be a direct sum of matrices, and we refer to the direct summands as blocks of  $A(i)$ .

THEOREM 2.8. *Let  $T$  be a tree,  $A$  be a matrix in  $\mathcal{S}(T)$  and  $\lambda$  be an eigenvalue of  $A$  such that  $m_A(\lambda) \geq 2$ . Then there exists a vertex  $i$  in  $G(A)$  such that*

- 1)  $m_{A(i)}(\lambda) = m_A(\lambda) + 1$ ;
- 2)  $\lambda$  is an eigenvalue of at least three blocks of  $A(i)$ .

**3. General Inverse Eigenvalue Problem.** The following theorem generalizes Theorem 1.5 and gives a partial answer of the GIEP for  $\mathcal{S}(T)$  when some of the branches of  $T$ , at a fixed vertex  $x_0$ , are cut branches.

**THEOREM 3.1.** *Let  $T$  be a tree on  $n$  vertices and  $x_0$  be a vertex of  $T$  of degree  $k$  whose neighbors are  $x_1, \dots, x_k$ . Let  $T_i$  be the branch of  $T$  at  $x_0$  containing  $x_i$  and  $s_i$  be the number of vertices in  $T_i$ ,  $i = 1, \dots, k$ .*

*Let  $T_1, \dots, T_r$ , with  $0 \leq r \leq k$ , be cut branches at  $x_1, \dots, x_r$ , respectively, and  $l_j = \min\{\text{number of vertices in each connected component of } T_j - x_j\}$ ,  $j = 1, \dots, r$ .*

*Let  $g_1(t), \dots, g_k(t)$  be monic polynomials having only distinct real roots, with  $\deg g_i(t) = s_i$ ,  $p_1, \dots, p_s$  be the distinct roots among polynomials  $g_i(t)$  and  $m_j$  be the multiplicity of root  $p_j$  in  $\prod_{i=1}^k g_i(t)$ .*

*Let  $g(t)$  be a monic polynomial of degree  $s + 1$ ,  $p_1, \dots, p_l$  be the common roots of  $g(t)$  and  $\prod_{i=1}^s (t - p_i)$  and*

$$\bar{g}(t) = \frac{g(t)}{\prod_{i=1}^l (t - p_i)}.$$

*For each  $1 \leq j \leq r$ , let  $q_{j1} < q_{j2} < \dots < q_{jv_j}$ , with  $1 \leq v_j \leq l_j$ , be roots of  $g_j(t)$ .*

*Let*

$$\bar{g}_j(t) = \begin{cases} \frac{g_j(t)}{(t - q_{j1}) \dots (t - q_{jv_j})} & \text{if } 1 \leq j \leq r \\ g_j(t) & \text{if } r < j \leq k. \end{cases}$$

*and*

$$m_{ij} = \begin{cases} 1 & \text{if } p_i \text{ is a root of } \bar{g}_j(t) \\ 0 & \text{otherwise.} \end{cases}$$

*There exists a matrix  $A$  in  $\mathcal{S}(T)$  with characteristic polynomial*

$$f(t) = g(t) \prod_{i=1}^s (t - p_i)^{m_i - 1}$$

*and such that*

- 1)  $A[T_i]$  has characteristic polynomial  $g_i(t)$ ,  $i = 1, \dots, k$ ,
- 2) for each  $1 \leq j \leq r$ ,  $q_{j1}, q_{j2}, \dots, q_{jv_j}$  are the common eigenvalues of  $A[T_j]$  and  $A[T_j - x_j]$ , and  $m_{[T_j - x_j]}(q_{j1}) = \dots = m_{[T_j - x_j]}(q_{jv_j}) = 2$ ,



3) if  $r < i \leq k$  and  $s_i > 1$ , the eigenvalues of  $A[T_i - x_i]$  strictly interlace those of  $A[T_i]$ ,

if and only if

- I)  $s - l \geq 1$ ,
- II) the roots of  $\bar{g}(t)$  strictly interlace those of  $\prod_{i=l+1}^s (t - p_i)$ ,
- III)  $p_1, \dots, p_l$  are not roots of  $\bar{g}_j(t)$  and there exist positive real numbers  $y_{ij}$ ,  $i \in \{l+1, \dots, s\}$ ,  $j \in \{1, \dots, k\}$  such that for each  $i \in \{l+1, \dots, s\}$

$$(3.1) \quad -\frac{\bar{g}(p_i)}{\prod_{j=l+1, j \neq i}^s (p_i - p_j)} = \sum_{j=1}^k m_{ij} y_{ij},$$

and for each  $j \in \{1, \dots, r\}$ ,  $u \in \{1, \dots, v_j\}$

$$(3.2) \quad \sum_{i=l+1, m_{ij}=1}^s \frac{m_{ij} y_{ij}}{q_{ju} - p_i} = 0.$$

*Proof.* We start by proving the necessity of the stated conditions for the existence of the matrix  $A$ . Firstly notice that the characteristic polynomial of  $A(x_0) = A[T_1] \oplus \dots \oplus A[T_k]$  is  $\prod_{i=1}^k g_i(t) = \prod_{i=1}^s (t - p_i)^{m_i}$ . By hypothesis,  $f(t) = g(t) \prod_{i=1}^s (t - p_i)^{m_i - 1}$  is the characteristic polynomial of  $A$ . So, using Lemma 2.7 there exists  $1 \leq i \leq s$  such that  $p_i$  is not a root of  $g(t)$ . Because  $p_1, \dots, p_l$  are roots of  $g(t)$  then  $s - l \geq 1$  and we have I).

As  $\sum_{i=1}^s m_i = n - 1$  then  $n - s - 1 + l = \sum_{i=1}^s (m_i - 1) + l$  is the degree of the polynomial  $\prod_{i=1}^s (t - p_i)^{m_i - 1} \prod_{j=1}^l (t - p_j)$ . Because  $p_A(t) = f(t) = \bar{g}(t) \prod_{j=1}^l (t - p_j) \prod_{i=1}^s (t - p_i)^{m_i - 1}$  is a polynomial of degree  $n$  then  $\bar{g}(t)$  is a polynomial of degree  $s - l + 1$ . By hypothesis,  $p_1, \dots, p_l$  are the common roots of  $g(t)$  and  $\prod_{i=1}^s (t - p_i)$  then  $\bar{g}(t)$  must have  $s - l + 1$  real roots (because  $A$  is symmetric), each one different from each  $p_{l+1}, \dots, p_s$ . By the interlacing theorem for eigenvalues of Hermitian matrices, the roots of  $p_A(t)$  must interlace the roots of  $p_{A(x_0)}(t) = \prod_{i=1}^s (t - p_i)^{m_i}$ . Then the roots of  $\bar{g}(t)$  must strictly interlace those of  $\prod_{i=l+1}^s (t - p_i)$ . So, we have II).

Now we are going to prove III). Let  $j \in \{1, \dots, k\}$ . If  $s_j > 1$ , denote by  $h_j(t)$  the characteristic polynomial of  $A[T_j - x_j]$ ; if  $s_j = 1$ , define  $h_j(t) = 1$ . Let

$$\bar{h}_j(t) = \begin{cases} \frac{h_j(t)}{(t - q_{j1}) \dots (t - q_{jv_j})} & \text{if } 1 \leq j \leq r \\ h_j(t) & \text{if } r < j \leq k. \end{cases}$$

By the interlacing theorem for eigenvalues of Hermitian matrices, and by hypothesis 2), if  $1 \leq j \leq r$  or by hypothesis 3), if  $r < j \leq k$  and  $s_j > 1$ , then the roots of  $\bar{g}_j(t)$  must strictly interlace those of  $\bar{h}_j(t)$ . By Lemma 2.1, if  $s_j > 1$ , or by construction, if

$s_j = 1$ , there exist positive real numbers  $b_{ij}$ , with  $1 \leq i \leq s$ ,  $1 \leq j \leq k$ , such that for each  $j \in \{1, \dots, k\}$ ,

$$\frac{h_j(t)}{g_j(t)} = \frac{\overline{h_j}(t)}{\overline{g_j}(t)} = \sum_{i=1}^s \frac{m_{ij} b_{ij}}{t - p_i}.$$

By hypothesis, if  $1 \leq i \leq l$  then  $p_i$  is a root of  $g(t)$ . So, if  $1 \leq i \leq l$  then  $m_i = m_{A(x_0)}(p_i) \neq m_A(p_i) + 1$ . Therefore, using Lemma 2.4, if  $1 \leq i \leq l$  and  $p_i$  is a root of  $A[T_j]$  then  $p_i$  is a root of  $A[T_j - x_j]$ . By hypothesis, it follows that  $m_{A[T_j]}(p_i)$  is equal to 1 or to 0. Consequently,  $p_1, \dots, p_l$  are not roots of  $\overline{g_j}(t)$  (if  $r < j \leq k$  and  $s_j > 1$  we use hypothesis 3); if  $1 \leq j \leq r$ , we use hypothesis 2) and the fact that  $g_j(t)$  has distinct real roots).

Therefore, there exist positive real numbers  $b_{ij}$ , with  $l + 1 \leq i \leq s$ ,  $1 \leq j \leq k$ , such that for each  $j \in \{1, \dots, k\}$ ,

$$(3.3) \quad \frac{h_j(t)}{g_j(t)} = \frac{\overline{h_j}(t)}{\overline{g_j}(t)} = \sum_{i=l+1}^s \frac{m_{ij} b_{ij}}{t - p_i}$$

Since  $A = [a_{ij}]$  is a matrix in  $\mathcal{S}(T)$ , we define for each  $j \in \{1, \dots, k\}$ ,  $x_j = |a_{x_0 x_j}|^2$  and  $a = a_{x_0 x_0}$ . According to (2.1), the characteristic polynomial of  $A$  may be written as

$$\begin{aligned} p_A(t) &= (t - a_{x_0 x_0}) \prod_{j=1}^k p_{A[T_j]}(t) - \sum_{j=1}^k |a_{x_0 x_j}|^2 p_{A[T_j - x_j]}(t) \prod_{u=1, u \neq j}^k p_{A[T_u]}(t) \\ &= (t - a) \prod_{j=1}^k g_j(t) - \sum_{j=1}^k x_j \frac{p_{A[T_j - x_j]}(t)}{p_{A[T_j]}(t)} \prod_{u=1}^k g_u(t) \\ &= \left( (t - a) - \sum_{j=1}^k x_j \sum_{i=l+1}^s \frac{m_{ij} b_{ij}}{t - p_i} \right) \prod_{i=1}^s (t - p_i)^{m_i} \\ &= \left( (t - a) - \sum_{i=l+1}^s \frac{\sum_{j=1}^k x_j m_{ij} b_{ij}}{t - p_i} \right) \prod_{i=1}^s (t - p_i) \prod_{i=1}^s (t - p_i)^{m_i - 1} \end{aligned}$$

Consequently,

$$g(t) = \left( (t - a) - \sum_{i=l+1}^s \frac{\sum_{j=1}^k x_j m_{ij} b_{ij}}{t - p_i} \right) \prod_{i=1}^s (t - p_i)$$

and

$$\bar{g}(t) = \frac{g(t)}{\prod_{i=1}^l (t - p_i)} = (t - a) \prod_{i=l+1}^s (t - p_i) - \sum_{i=l+1}^s \sum_{j=1}^k (x_j m_{ij} b_{ij}) \prod_{u=l+1, u \neq i}^s (t - p_u)$$

So, for each  $i \in \{l + 1, \dots, s\}$ ,  $\bar{g}(p_i) = - \sum_{j=1}^k (x_j m_{ij} b_{ij}) \prod_{u=l+1, u \neq i}^s (p_i - p_u)$ . Then

$$- \frac{\bar{g}(p_i)}{\prod_{u=l+1, u \neq i}^s (p_i - p_u)} = \sum_{j=1}^k (x_j m_{ij} b_{ij}).$$

For each  $i \in \{l + 1, \dots, s\}$  and  $j \in \{1, \dots, k\}$ , we denote  $x_j b_{ij}$  by  $y_{ij}$ . Since  $b_{ij}$ ,  $x_j$  are positive real numbers, then  $y_{ij}$  is a positive real number and we have III)(3.1).

Let  $j \in \{1, \dots, r\}$ . Since  $q_{j1}, \dots, q_{jv_j}$  are the common eigenvalues of  $A[T_j]$  and  $A[T_j - x_j]$ , and  $m_{A[T_j - x_j]}(q_{j1}) = \dots = m_{A[T_j - x_j]}(q_{jv_j}) = 2$ , by hypothesis 2), we have that  $q_{j1}, \dots, q_{jv_j}$  are roots of  $\bar{h}_j(t)$  but not of  $\bar{g}_j(t)$ . Thus, if  $p_i$  is a root of  $\bar{g}_j(t)$  then  $p_i \notin \{q_{j1}, \dots, q_{jv_j}\}$ . Therefore, by (3.3), for each  $u \in \{1, \dots, v_j\}$ ,

$$0 = \frac{x_j \bar{h}_j(q_{ju})}{\bar{g}_j(q_{ju})} = \sum_{i=l+1, m_{ij}=1}^s \frac{m_{ij} (x_j b_{ij})}{q_{ju} - p_i} = \sum_{i=l+1, m_{ij}=1}^s \frac{m_{ij} y_{ij}}{q_{ju} - p_i}.$$

So, we have III)(3.2).

Next, we prove the sufficiency of the stated conditions. Because of the strict interlacing between the roots of  $\bar{g}(t)$  and those of  $\prod_{i=l+1}^s (t - p_i)$  (hypothesis II) and because  $\bar{g}(t)$  is a polynomial of degree  $s + 1 - l > 1$ , due to Remark 2.2, we conclude the existence of a real number  $a$  and positive real numbers  $x_{l+1}, \dots, x_s$  such that

$$\frac{\bar{g}(t)}{\prod_{i=l+1}^s (t - p_i)} = (t - a) - \sum_{i=l+1}^s \frac{x_i}{(t - p_i)}$$

i.e.,

$$\begin{aligned} \bar{g}(t) &= \left( (t - a) - \sum_{i=l+1}^s \frac{x_i}{(t - p_i)} \right) \prod_{i=l+1}^s (t - p_i) \\ &= (t - a) \prod_{i=l+1}^s (t - p_i) - \sum_{i=l+1}^s x_i \prod_{j=l+1, j \neq i}^s (t - p_j). \end{aligned}$$

Therefore, if  $l + 1 \leq i \leq s$ ,

$$\bar{g}(p_i) = -x_i \prod_{j=l+1, j \neq i}^s (p_i - p_j)$$

and

$$x_i = -\frac{\bar{g}(p_i)}{\prod_{j=l+1, j \neq i}^s (p_i - p_j)}.$$

Using hypothesis III)(3.1), there exist real positive numbers  $y_{ij}$ , with  $l + 1 \leq i \leq s$ ,  $1 \leq j \leq k$  such that  $x_i = \sum_{j=1}^k m_{ij}y_{ij}$ . So,

$$\begin{aligned} \bar{g}(t) &= \left( (t - a) - \sum_{i=l+1}^s \frac{\sum_{j=1}^k m_{ij}y_{ij}}{(t - p_i)} \right) \prod_{i=l+1}^s (t - p_i) \\ &= \left( (t - a) - \left( \sum_{i=l+1}^s \frac{m_{i1}y_{i1}}{(t - p_i)} + \dots + \sum_{i=l+1}^s \frac{m_{ik}y_{ik}}{(t - p_i)} \right) \right) \prod_{i=l+1}^s (t - p_i). \end{aligned}$$

By hypothesis III) then  $\bar{g}_j(t) = \prod_{i=l+1}^s (t - p_i)^{m_{ij}}$ . Notice that, when  $\deg \bar{g}_j(t) > 1$ ,  $\sum_{i=l+1}^s \frac{m_{ij}y_{ij}}{(t - p_i)}$  is a pfd of  $\frac{\bar{h}_j(t)}{\bar{g}_j(t)}$  for some polynomial  $\bar{h}_j(t)$ . By Lemma 2.1, the coefficients of this pfd are positive which means that  $\deg \bar{h}_j(t) = \deg \bar{g}_j(t) - 1$  and  $\bar{h}_j(t)$  has only real roots strict interlacing those of  $\bar{g}_j(t)$ . If  $\deg \bar{g}_j(t) = 1$ ,  $\sum_{i=l+1}^s \frac{m_{ij}y_{ij}}{(t - p_i)} = \frac{m_{uj}y_{uj}}{t - p_u}$ ,  $m_{uj}y_{uj} > 0$ , for some  $u \in \{l + 1, \dots, s\}$ . In this case, for convenience we denote  $m_{uj}y_{uj}$  by  $\bar{h}_j(t)$ .

If  $1 \leq j \leq r$ , since  $q_{j1}, \dots, q_{jv_j}$  are not roots of  $\bar{g}_j(t)$ , using III)(3.2) we have that  $q_{j1}, \dots, q_{jv_j}$  are roots of  $\bar{h}_j(t)$  but not of  $\bar{g}_j(t)$ . Remark that the leading coefficient of  $\bar{h}_j(t)$  is the positive real number  $\sum_{i=l+1}^s m_{ij}y_{ij}$ . Let  $h_j(t)$  be the monic polynomial such that  $\bar{h}_j(t)(t - q_{j1}) \dots (t - q_{jv_j}) = (\sum_{i=l+1}^s m_{ij}y_{ij})h_j(t)$ . So,

$$\bar{g}(t) = \left( (t - a) - \sum_{j=1}^k \left( \sum_{i=l+1}^s m_{ij}y_{ij} \right) \frac{h_j(t)}{g_j(t)} \right) \prod_{i=l+1}^s (t - p_i).$$

By hypothesis, if  $1 \leq j \leq r$  then  $T_j$  is a cut branch at  $x_i$ . Since  $g_j(t) = \bar{g}_j(t) \prod_{i=1}^{v_j} (t - q_{ji})$ , the roots of  $\bar{h}_j(t)$  strictly interlace those of  $\bar{g}_j(t)$  and  $q_{j1}, \dots, q_{jv_j}$  are roots of  $\bar{h}_j(t)$ , then by Theorem 1.5, there exists a matrix  $A_j$  in  $\mathcal{S}(T_j)$  such that  $p_{A_j}(t) = g_j(t)$  and  $p_{A_j[T_j - x_j]}(t) = h_j(t)$ . So, we have 2).

If  $r + 1 \leq j \leq k$ , by Proposition 1.4, there exists a matrix  $A_j$  in  $\mathcal{S}(T_j)$  such that  $p_{A_j}(t) = g_j(t)$  and  $p_{A_j[T_j - x_j]}(t) = h_j(t)$  (recall the convention that  $p_{A_j[T_j - x_j]}(t) = 1$  whenever the vertex set of  $T_j$  is  $\{x_j\}$ ). Therefore, we have 3) and 1).

Now define a matrix  $A = [a_{ij}]$  in  $\mathcal{S}(T)$  in the following way:

- $a_{x_0 x_0} = a$
- $a_{x_0 x_j} = a_{x_j x_0} = \sqrt{\sum_{i=l+1}^s m_{ij}y_{ij}}$ , for  $j = 1, \dots, k$

- $A[T_j] = A_j$ , for  $j = 1, \dots, k$
- the remaining entries of  $A$  are 0.

According to (2.1), the characteristic polynomial of  $A$  may be written as

$$(t - a_{x_0x_0})p_{A[T-x_0]}(t) - \sum_{j=1}^k |a_{x_0x_j}|^2 p_{A[T_j-x_j]}(t) \prod_{i=1, i \neq j}^k p_{A[T_i]}(t).$$

Note that  $A[T-x_0] = A[T_1] \oplus \dots \oplus A[T_k]$  which implies  $p_{A[T-x_0]}(t) = \prod_{j=1}^k p_{A[T_j]}(t)$ . Moreover the characteristic polynomial of  $A[T_j]$  is  $g_j(t)$  and the characteristic polynomial of  $A[T_j-x_j]$  is  $h_j(t)$ . Consequently, by hypothesis,  $\prod_{j=1}^k g_j(t) = \prod_{i=1}^s (t-p_i)^{m_i}$  and

$$\begin{aligned} p_A(t) &= (t-a) \prod_{i=1}^s (t-p_i)^{m_i} - \sum_{j=1}^k \left( \sum_{i=l+1}^s m_{ij} y_{ij} \right) \frac{h_j(t)}{g_j(t)} \prod_{i=1}^s (t-p_i)^{m_i} \\ &= \bar{g}(t) \prod_{i=1}^l (t-p_i) \prod_{i=1}^s (t-p_i)^{m_i-1} = f(t) \end{aligned}$$

i.e.,  $p_A(t) = f(t)$ .  $\square$

Under the conditions and following the notation of Theorem 3.1, if  $T$  is a superstar and  $T_j$  is a cut branch at  $x_j$  of  $T$ , then  $T_j$  is a path. So, if  $q_{ji}$  is a common eigenvalue of  $A[T_j]$  and  $A[T_j-x_j]$  then  $m_{A[T_j-x_j]}(q_{ji}) = 2$ . Therefore, if we suppose that  $T$  is a superstar then condition 2) of Theorem 3.1 is shorter and we have the following result.

**THEOREM 3.2.** *Let  $T$  be a superstar on  $n$  vertices and  $x_0$  be a central vertex of  $T$  of degree  $k$  whose neighbors are  $x_1, \dots, x_k$ . Let  $T_i$  be the branch of  $T$  at  $x_0$  containing  $x_i$  and  $s_i$  be the number of vertices in  $T_i$ ,  $i = 1, \dots, k$ .*

*Let  $T_1, \dots, T_r$ , with  $0 \leq r \leq k$ , be cut branches at  $x_1, \dots, x_r$ , respectively, and  $l_j = \min\{\text{number of vertices in each connected component of } T_j - x_j\}$ ,  $j = 1, \dots, r$ .*

*Let  $g_1(t), \dots, g_k(t)$  be monic polynomials having only distinct real roots, with  $\deg g_i(t) = s_i$ ,  $p_1, \dots, p_s$  be the distinct roots among polynomials  $g_i(t)$  and  $m_j$  be the multiplicity of root  $p_j$  in  $\prod_{i=1}^k g_i(t)$ .*

*Let  $g(t)$  be a monic polynomial of degree  $s+1$ ,  $p_1, \dots, p_l$  be the common roots of  $g(t)$  and  $\prod_{i=1}^s (t-p_i)$  and*

$$\bar{g}(t) = \frac{g(t)}{\prod_{i=1}^l (t-p_i)}.$$

For each  $1 \leq j \leq r$ , let  $q_{j1} < q_{j2} < \dots < q_{jv_j}$ , with  $1 \leq v_j \leq l_j$ , be roots of  $g_j(t)$ .

Let

$$\overline{g}_j(t) = \begin{cases} \frac{g_j(t)}{(t-q_{j1}) \dots (t-q_{jv_j})} & \text{if } 1 \leq j \leq r \\ g_j(t) & \text{if } r < j \leq k. \end{cases}$$

and

$$m_{ij} = \begin{cases} 1 & \text{if } p_i \text{ is a root of } \overline{g}_j(t) \\ 0 & \text{otherwise.} \end{cases}$$

There exists a matrix  $A$  in  $\mathcal{S}(T)$  with characteristic polynomial  $f(t) = g(t) \prod_{i=1}^s (t - p_i)^{m_i - 1}$  and such that

- 1)  $A[T_i]$  has characteristic polynomial  $g_i(t)$ ,  $i = 1, \dots, k$ ,
- 2) for each  $1 \leq j \leq r$ ,  $q_{j1}, q_{j2}, \dots, q_{jv_j}$  are the common eigenvalues of  $A[T_j]$  and  $A[T_j - x_j]$ ,
- 3) if  $r < i \leq k$  and  $s_i > 1$ , the eigenvalues of  $A[T_i - x_i]$  strictly interlace those of  $A[T_i]$ ,

if and only if

- I)  $s - l \geq 1$ ,
- II) the roots of  $\overline{g}(t)$  strictly interlace those of  $\prod_{i=l+1}^s (t - p_i)$ ,
- III)  $p_1, \dots, p_l$  are not roots of  $\overline{g}_j(t)$  and there exist positive real numbers  $y_{ij}$ ,  $i \in \{l+1, \dots, s\}$ ,  $j \in \{1, \dots, k\}$  such that for each  $i \in \{l+1, \dots, s\}$

$$-\frac{\overline{g}(p_i)}{\prod_{j=l+1, j \neq i}^s (p_i - p_j)} = \sum_{j=1}^k m_{ij} y_{ij},$$

and for each  $j \in \{1, \dots, r\}$ ,  $u \in \{1, \dots, v_j\}$

$$\sum_{i=l+1, m_{ij}=1}^s \frac{m_{ij} y_{ij}}{q_{ju} - p_i} = 0.$$

REMARK 3.3. Under the conditions and following the notation of Theorem 3.1,

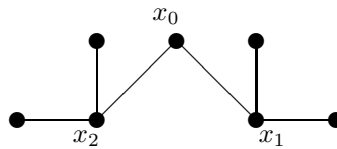
- 1) if we suppose that there are not common roots of the polynomials  $g(t)$  and  $\prod_{i=1}^k g_i(t)$ , conditions 2), I) and III) of Theorem 3.1 may be omitted and we obtain Theorem 1.5
- 2) if  $T$  is a generalized star and  $T_j$  is a branch of  $T$  then  $T_j - x_j$  is a path or an empty set. Then, conditions 2), 3), I) and III) of Theorem 3.1 may be omitted and we obtain Theorem 1.6.

REMARK 3.4. Under the mentioned hypothesis and conditions 1), 2), 3) of Theorem 3.1, we have condition III) of Theorem 3.1 which implies that  $p_1, \dots, p_l$  are not roots of  $\prod_{j=1}^k \overline{g}_j(t)$ . Because  $p_1, \dots, p_l$  are roots of  $\prod_{j=1}^k g_j(t)$  we must have

$$\{p_1, \dots, p_l\} \subseteq \bigcup_{j=1}^r \{q_{j1}, \dots, q_{jv_j}\}.$$

EXAMPLE 3.5.

1) Let  $T$  be the superstar



Let  $g_1(t) = t(t-1)(t-3)$ ,  $g_2(t) = (t+2)t(t-3)$ . Then  $p_1 = 1$ ,  $p_2 = -2$ ,  $p_3 = 0$ ,  $p_4 = 3$  are the distinct roots among polynomials  $g_1(t)$  and  $g_2(t)$ .

Consider the root  $p_1 = 1$  of  $g_1(t)$  and the root  $p_3 = 0$  of  $g_2(t)$ .

Let  $g(t) = (t+3)(t+1)(t-1)(t-\frac{6}{13})(t-4)$ .

Since  $p_1 = 1$  is a common root of  $g(t)$  and  $\prod_{i=1}^4 (t-p_i)$ , let

$$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4),$$

$$\overline{g}_1(t) = t(t-3) \text{ and}$$

$$\overline{g}_2(t) = (t+2)(t-3).$$

Because

I)  $s - l = 4 - 1 \geq 1$ ,

II) the roots of

$$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4)$$

strictly interlace those of

$$\prod_{i=2}^4 (t-p_i) = (t+2)t(t-3),$$

III) the positive real numbers  $y_{22} = \frac{96}{65}$ ,  $y_{31} = \frac{12}{13}$ ,  $y_{41} = \frac{24}{13}$  and  $y_{42} = \frac{144}{65}$  verify

$$-\frac{\overline{g}(-2)}{(-2) \times (-5)} = y_{22},$$

$$-\frac{\overline{g}(0)}{(2) \times (-3)} = y_{31},$$

$$\begin{aligned}
 -\frac{\overline{g}(3)}{(5) \times (3)} &= y_{41} + y_{42}, \\
 \frac{y_{31}}{1-0} + \frac{y_{41}}{1-3} &= 0, \\
 \frac{y_{22}}{0+2} + \frac{y_{42}}{0-3} &= 0,
 \end{aligned}$$

using Theorem 3.1, there exists a matrix  $A$  in  $\mathcal{S}(T)$  such that the characteristic polynomial of  $A$  is  $f(t) = g(t)t(t-3)$  and,  $A[T_1]$  has characteristic polynomial  $g_1(t)$ ,  $A[T_2]$  has characteristic polynomial  $g_2(t)$ ,  $p_1 = 1$  is the unique common eigenvalue of  $A[T_1]$  and  $A[T_1 - x_1]$ ,  $p_3 = 0$  is the unique common eigenvalue of  $A[T_2]$  and  $A[T_2 - x_2]$ . Moreover, following the proof of Theorem 3.1,

$$A = \begin{bmatrix}
 -\frac{7}{13} & \sqrt{\frac{36}{13}} & 0 & 0 & \sqrt{\frac{48}{13}} & 0 & 0 \\
 \sqrt{\frac{36}{13}} & 2 & 1 & 1 & 0 & 0 & 0 \\
 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 \sqrt{\frac{48}{13}} & 0 & 0 & 0 & 1 & \sqrt{3} & \sqrt{3} \\
 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\
 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0
 \end{bmatrix}$$

2) Let  $T$  be the superstar of the Example 3.5, 1).

Let  $g_1(t) = t(t-1)(t-3)$ ,  $g_2(t) = (t+2)t(t-3)$ .

Consider the root  $p_1 = 1$  of  $g_1(t)$  and let  $g(t) = (t+3)(t+1)(t-1)(t-\frac{6}{13})(t-4)$ .

Since  $p_1 = 1$  is a common root of  $g(t)$  and  $\prod_{i=1}^4 (t - p_i)$ , let

$$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4),$$

$$\overline{g}_1(t) = t(t-3) \text{ and}$$

$$\overline{g}_2(t) = (t+2)t(t-3).$$

In this case, because

I)  $s - l = 4 - 1 \geq 1$ ,

II) the roots of

$$\overline{g}(t) = (t+3)(t+1)(t-\frac{6}{13})(t-4)$$

strictly interlace those of

$$\prod_{i=2}^4 (t - p_i) = (t+2)t(t-3),$$

III) the positive real numbers  $y_{22} = \frac{96}{65}$ ,  $y_{31} = \frac{6}{13}$ ,  $y_{32} = \frac{6}{13}$ ,  $y_{41} = \frac{12}{13}$  and

$$y_{42} = \frac{204}{65} \text{ verify}$$

$$-\frac{\overline{g}(-2)}{(-2) \times (-5)} = y_{22},$$

$$-\frac{\overline{g}(0)}{(2) \times (-3)} = y_{31} + y_{32},$$

$$-\frac{\overline{g}(3)}{(5) \times (3)} = y_{41} + y_{42},$$



$$\frac{y_{31}}{1-0} + \frac{y_{41}}{1-3} = 0,$$

using Theorem 3.1, there exists a matrix  $A$  in  $\mathcal{S}(T)$  such that the characteristic polynomial of  $A$  is  $f(t) = g(t)t(t - 3)$  and,  $A[T_1]$  has characteristic polynomial  $g_1(t)$ ,  $A[T_2]$  has characteristic polynomial  $g_2(t)$ ,  $p_1 = 1$  is the unique common eigenvalue of  $A[T_1]$  and  $A[T_1 - x_1]$ . Moreover, following the proof of Theorem 3.1,

$$A = \begin{bmatrix} -\frac{7}{13} & \sqrt{\frac{18}{13}} & 0 & 0 & \sqrt{\frac{66}{13}} & 0 & 0 \\ \sqrt{\frac{18}{13}} & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \sqrt{\frac{66}{13}} & 0 & 0 & 0 & \frac{14}{11} & \sqrt{\frac{643}{22} + \frac{\sqrt{273}}{1001}} & \sqrt{\frac{643}{22} - \frac{\sqrt{273}}{1001}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{643}{22} + \frac{\sqrt{273}}{1001}} & \frac{-3-\sqrt{273}}{22} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{643}{22} - \frac{\sqrt{273}}{1001}} & 0 & \frac{-3+\sqrt{273}}{22} \end{bmatrix}$$

REMARK 3.6.

- 1) In Example 3.5, 1), we consider that the set of common roots of  $g(t)$  and  $g_1(t)g_2(t)$  is a subset of the set of fixed roots of  $g_1(t)$  and  $g_2(t)$ . This is,  $\{p_1, \dots, p_l\} \subset \bigcup_{j=1}^r \{q_{j1}, \dots, q_{jv_j}\}$ .  
 In 2), we consider that the set of common roots of  $g(t)$  and  $g_1(t)g_2(t)$  is the set of fixed roots of  $g_1(t)$  and  $g_2(t)$ . Thus,  $\{p_1, \dots, p_l\} = \bigcup_{j=1}^r \{q_{j1}, \dots, q_{jv_j}\}$ .  
 Following the proof of Theorem 3.1, we obtain different matrices or solution for the GIEP.
- 2) In Example 3.5, 1), the positive real numbers  $y_{22} = \frac{96}{65}$ ,  $y_{31} = \frac{12}{13}$ ,  $y_{41} = \frac{24}{13}$  and  $y_{42} = \frac{144}{65}$  verify conditions III)(3.1) and (3.2) of Theorem 3.1. If we consider the positive real numbers  $y_{22} = \frac{96}{65}$ ,  $y_{31} = \frac{12}{13}$ ,  $y_{41} = \frac{132}{65}$  and  $y_{42} = \frac{132}{65}$ , then these integers verify condition III)(3.1) but do not verify condition III)(3.2), this is,  
 $\frac{y_{31}}{1-0} + \frac{y_{41}}{1-3} \neq 0$ , and  $\frac{y_{22}}{0+2} + \frac{y_{42}}{0-3} \neq 0$ .  
 Consequently, it is possible to find positive real numbers that verify condition III)(3.1) but do not satisfy condition III)(3.2).

**4. Equivalence of ordered multiplicity lists and IEP.** As we have said in section 1 (Introduction), using the mentioned tree  $T'$ , Barioli and Fallat gave the first example for which the equivalence between the problem of ordered multiplicity lists and the IEP does not occur, [1]. Using the mentioned tree  $T''$ , a simpler example based on the same technique was given in [10].

Using Theorem 2.8 we can see that if

$$(1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$$

is the sequence of eigenvalues of a matrix  $A' \in \mathcal{S}(T')$  then

$$2, 3, 4, 5, 6$$

are the eigenvalues of  $A'[T_1]$  and of  $A'[T_2]$ , where  $T_1$  and  $T_2$  are the branches of  $T' - x_0$  with 5 vertices and

$$2, 3, 4, 6$$

are the eigenvalues of  $A'[T_3]$ , where  $T_3$  is the branch of  $T' - x_0$  with 4 vertices.

Thus,

- i)  $g_1(t) = (t-2)(t-3)(t-4)(t-5)(t-6) = g_2(t)$  and  $g_3(t) = (t-2)(t-3)(t-4)(t-6)$ ,
- ii) 3, 5 are roots of  $g_1(t)$  and of  $g_2(t)$ ,
- iii) 3 is a root of  $g_3(t)$ ,
- iv)  $g(t) = (t-1)(t-3)^2(t-5)^2(t-7)$ .

Consequently,  $\bar{g}(t) = (t-1)(t-3)(t-5)(t-7)$ . Using Theorem 3.1, it is possible to find positive real numbers  $y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}, y_{31}, y_{32}, y_{33}$  such that

$$\begin{aligned} -\frac{\bar{g}(2)}{(2-4) \times (2-6)} &= \frac{15}{8} = y_{11} + y_{12} + y_{13}, \\ -\frac{\bar{g}(4)}{(4-2) \times (4-6)} &= \frac{9}{4} = y_{21} + y_{22} + y_{23}, \\ -\frac{\bar{g}(6)}{(6-2) \times (6-4)} &= \frac{15}{8} = y_{31} + y_{32} + y_{33}, \\ \frac{y_{11}}{3-2} + \frac{y_{21}}{3-4} + \frac{y_{31}}{3-6} &= y_{11} - y_{21} - \frac{y_{31}}{3} = 0, \\ \frac{y_{12}}{3-2} + \frac{y_{22}}{3-4} + \frac{y_{32}}{3-6} &= y_{12} - y_{22} - \frac{y_{32}}{3} = 0, \\ \frac{y_{13}}{3-2} + \frac{y_{23}}{3-4} + \frac{y_{33}}{3-6} &= y_{13} - y_{23} - \frac{y_{33}}{3} = 0, \\ \frac{y_{11}}{5-2} + \frac{y_{21}}{5-4} + \frac{y_{31}}{5-6} &= \frac{y_{11}}{3} + y_{21} - y_{31} = 0, \\ \frac{y_{12}}{5-2} + \frac{y_{22}}{5-4} + \frac{y_{32}}{5-6} &= \frac{y_{12}}{3} + y_{22} - y_{32} = 0. \end{aligned}$$

Using the first six equalities we obtain a contradiction. Notice that there are positive real numbers that satisfy the first three equalities. Therefore, there is no matrix  $A' \in \mathcal{S}(T')$  having eigenvalues  $(1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$ .

REMARK 4.1. Using the sequence  $(1, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$ , consider the matrices

$$A_1 = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 3 \end{bmatrix}.$$

Considering the mentioned tree  $T'$ , let  $x_i$  be the vertex of  $T_i$  adjacent to  $x_0$ ,  $i \in \{1, 2, 3\}$ . Let  $\alpha_1^i$ ,  $\alpha_2^i$  be the paths of  $T_i - x_i$ ,  $i \in \{1, 2, 3\}$ , where  $\alpha_2^3$  is the shortest path of  $T_3 - x_3$ .

It is easy to prove that this sequence and the matrices

$$A[\alpha_1^1] = A[\alpha_2^1] = A[\alpha_1^2] = A[\alpha_2^2] = A[\alpha_1^3] = A_1$$

and

$$A[\alpha_2^3] = A_2$$

verify conditions (a), (b), (c), (d) of Lemma 1.9.

Thus, the converse of Lemma 1.9 is not true.

Using Theorem 3.1 it is possible to prove that

$$(0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7)$$

is the sequence of eigenvalues of a matrix  $A \in \mathcal{S}(T')$ .

Consider

- i)  $g_1(t) = (t-2)(t-3)(t-4)(t-5)(t-6) = g_2(t)$  and  $g_3(t) = (t-2)(t-3)(t-4)(t-6)$ ,
- ii) 3, 5 roots of  $g_1(t)$  and of  $g_2(t)$ ,
- iii) 3 a root of  $g_3(t)$ ,
- iv)  $g(t) = t(t-3)^2(t-5)^2(t-7)$ .

It is easy to find positive real numbers that satisfy condition III) of Theorem 3.1. By this theorem, there exists a matrix  $A \in \mathcal{S}(T')$  having eigenvalues

$$(0, 2, 2, 3, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7).$$

In the same way, we can prove the mentioned result for  $T''$  (see Introduction).

Using this technique, it is possible to find many superstars where the problem of ordered multiplicity lists and the IEP are not equivalent.

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