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MINIMUM RANK OF EDGE SUBDIVISIONS OF GRAPHS[*]

WAYNE BARRETT[†], RYAN BOWCUTT[‡], MARK CUTLER[‡], SETH GIBELYOU[‡], AND
KAYLA OWENS[§]

Abstract. Let F be a field, let G be an undirected graph on n vertices, and let $S(F,G)$ be the set of all F-valued symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. The minimum rank of G over F is defined to be $\mathrm{mr}(F,G) = \min\{\mathrm{rank}\,A \mid A \in S(F,G)\}$. The problem of finding the minimum rank (maximum nullity) of edge subdivisions of a given graph G is investigated. Is is shown that if an edge is adjacent to a vertex of degree 1 or 2, its maximum nullity is unchanged upon subdividing the edge. This enables us to reduce the problem of finding the minimum rank of any graph obtained from G by subdividing edges to finding the minimum rank of those graphs obtained from G by subdividing each edge at most once. The graph obtained by subdividing each edge of G once is called its subdivision graph and is denoted by $\overset{\circ}{G}$. It is shown that its maximum nullity is an upper bound for the maximum nullity of any graph obtained from G by subdividing edges. It is also shown that the minimum rank of $\overset{\circ}{G}$ often depends only upon the number of vertices of G. In conclusion, some illustrative examples and open questions are presented.

Key words. Combinatorial matrix theory, Edge subdivision, Graph, Maximum nullity, Minimum rank, Symmetric.

AMS subject classifications. 05C50, 15A03, 15B57.

1. Introduction. Given any field F and any (simple, undirected) graph $G = (V, E)$ on n vertices, let $S(F,G)$ be the set of all symmetric $n \times n$ matrices $A = [a_{ij}]$ with entries in F such that $a_{ij} \neq 0$, $i \neq j$ if and only if $ij \in E$; there is no restriction on the diagonal entries of A. Let

$$\mathrm{mr}(F,G) = \min\{\mathrm{rank}\,A \mid A \in S(F,G)\}.$$

The problem of determining $\mathrm{mr}(F,G)$ has been intensively studied and many results can be found in the survey paper [FH]. We also define

$$M(F,G) = \max\{\mathrm{nullity}\,A \mid A \in S(F,G)\}.$$

Since $\mathrm{mr}(F,G) + M(F,G) = n$, the problems of determining $\mathrm{mr}(F,G)$ and $\mathrm{M}(F,G)$ are equivalent.

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In this paper we investigate the problem of finding the minimum rank of edge subdivisions of a given graph.

DEFINITION 1.1. Let $G = (V, E)$ be a graph and let $e = vw$ be an edge of G. Let G_e be the graph obtained from G by inserting a new vertex u into V, inserting edges uv and uw into E, and deleting vw from E. We say that the edge e has been *subdivided once* and call G_e an *edge subdivision* of G. A graph obtained from a finite number of edge subdivisions of G is called an sG. The graph G itself is considered an sG. We denote the class of all sG's by $\mathcal{SC}(G)$, and call it the *subdivision class of G*.

The question we will consider is:

Fix a field F and a graph G. If H is in $\mathcal{SC}(G)$, what is $M(F, H)$?

In principle, this question can be completely answered by repeated application of Theorem 17 in [vdH] (also see section 2) which enables one to calculate $M(F, G_e)$ in terms of the maximum nullity of two multigraphs on fewer vertices. However, this can be a laborious process. We will see that this question can be answered by investigating the maximum nullity of just those graphs obtained from G by subdividing each of its edges at most once. The graph obtained from G by subdividing each edge exactly once (the subdivision graph) plays a fundamental role and its maximum nullity can be found for most graphs whether or not $M(F, G)$ is known. Consequently, for most graphs we can either find $M(F, H)$ for every graph $H \in \mathcal{SC}(G)$ or at least infinitely many sG's.

Before proceeding we pause to recall and introduce some terms from graph theory.

DEFINITION 1.2. The complement of the graph $G = (V, E)$ is the graph $G^c = (V, E^c)$. If $S \subset V$, $G[S]$ denotes the subgraph of G induced by S.

DEFINITION 1.3. Given two graphs G, and H, with $V(G)$ and $V(H)$ disjoint,

- the *union*, $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$, and edge set $E(G) \cup E(H)$.
- the *join*, $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set

$$E(G) \cup E(H) \cup \{uv | u \in V(G) \text{ and } v \in V(H)\}$$

DEFINITION 1.4. Let G and H be graphs on at least two vertices, each with a vertex labeled v. Then $G \underset{v}{\oplus} H$ is the graph on $|G| + |H| - 1$ vertices obtained by identifying the vertex v in G with the vertex v in H.

DEFINITION 1.5. Let $G = (V, E)$ be a graph and let $v, w \in V$.

a) If $vw \in E$, $G\backslash vw$ is the graph obtained from G by removing the edge vw from E.

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b) G/vw is the multigraph obtained from G by removing vw from E if it is an edge, and identifying the vertices v and w. (If there is a vertex in G adjacent to both v and w, there will be multiple edges in G/vw.)

c) $G + vw$ is the multigraph obtained from G by adding an edge between v and w. (So $G + vw$ has a multiple edge if vw is an edge of G.)

DEFINITION 1.6. We denote the path on n vertices by P_n, the cycle on n vertices by C_n, and the complete graph on n vertices by K_n. The complete bipartite graph $K_{m,n}$ is the complement of $K_m \cup K_n$. The n-wheel W_n is $C_{n-1} \vee K_1$.

We also need a few standard terms from matrix theory.

DEFINITION 1.7. The $k \times k$ matrix with all entries equal to 1 is denoted by J_k. If $A \in S(F, G)$, and v is a vertex in G, then $A(v)$ is the matrix obtained from A by deleting the row and column labeled by v.

The following results are well known; see Observations 1–5 in [BvdHL].

OBSERVATION 1.8. *Let F be any field*

a) *For $n \geq 2$, $\mathrm{mr}(F, K_n) = 1$ and $M(F, K_n) = n - 1$.*
b) *If m, $n \geq 1$, and $m + n \geq 3$, then $\mathrm{mr}(F, K_{m,n}) = 2$.*
c) *If H is an induced subgraph of G, then $\mathrm{mr}(F, G) \geq \mathrm{mr}(F, H)$.*

2. Edge Subdivisions and Maximum Nullity. We begin with a basic lemma due to Johnson, Loewy, and Smith [JLS]; they did not explicitly mention that their proof also holds for any field.

LEMMA 2.1. *Let F be any field, let G be any graph, and let e be an edge of G. Then*

(2.1) $$\mathrm{mr}(F, G) \leq \mathrm{mr}(F, G_e) \leq \mathrm{mr}(F, G) + 1$$

(2.2) $$M(F, G) \leq M(F, G_e) \leq M(F, G) + 1.$$

Proof. The inequalities (2.2) follow immediately from (2.1) but it is convenient to state both sets of inequalities. Let v, w be the vertices of e and let u be the new vertex in G_e that is adjacent to v and w.

We first prove that $\mathrm{mr}(F, G) \leq \mathrm{mr}(F, G_e)$. Let

$$A = \begin{bmatrix} d_1 & a & b & 0^T \\ a & d_2 & 0 & x^T \\ b & 0 & d_3 & y^T \\ 0 & x & y & C \end{bmatrix} \in S(F, G_e)$$

with rank $A = \mathrm{mr}(F, G_e)$ and with the first three rows and columns of A labeled by

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u, v and w. Then $a, b \neq 0$. Let B be the matrix obtained from A by adding row 1 to row 2 and column 1 to column 2. Then $\operatorname{rank} B = \operatorname{rank} A$ and $B(u) \in S(F, G)$. It follows that

$$\operatorname{mr}(F, G) \leq \operatorname{rank} B(u) \leq \operatorname{rank} B = \operatorname{rank} A = \operatorname{mr}(F, G_e).$$

To prove the upper bound on $\operatorname{mr}(F, G_e)$, let

$$A = \begin{bmatrix} d_1 & a & b^T \\ a & d_2 & c^T \\ b & c & B \end{bmatrix} \in S(F, G)$$

with $\operatorname{rank} A = \operatorname{mr}(F, G)$ and the first two rows and columns of A labeled by v and w. Then $a \neq 0$ and

$$A_e = \begin{bmatrix} 0 & 0 & 0 & 0^T \\ 0 & d_1 & a & b^T \\ 0 & a & d_2 & c^T \\ 0 & b & c & B \end{bmatrix} - \begin{bmatrix} a & a & a & 0^T \\ a & a & a & 0^T \\ a & a & a & 0^T \\ 0 & 0 & 0 & 0 \end{bmatrix} \in S(F, G_e).$$

It follows that

$$\operatorname{mr}(F, G_e) \leq \operatorname{rank} A_e \leq \operatorname{rank} A + 1 = \operatorname{mr}(F, G) + 1. \qquad \square$$

Our next aim is to give an important case of equality for the first inequality in (2.2), but first we need some additional results.

PROPOSITION 2.2. *Let F be a field and let G be a graph with a vertex u of degree 2. Assume the neighbors v, w of u are adjacent, and let $e = vw$. Then we have*

a) if $F \neq F_2$, $\operatorname{mr}(F, G) \leq \operatorname{mr}(F, G - u) + 1$.
b) if $F \neq F_2$, $M(F, G - u) \leq M(F, G)$.
c) $\operatorname{mr}(F, G) \leq \operatorname{mr}(F, (G - u)\backslash e) + 1$.
d) $M(F, (G - u)\backslash e) \leq M(F, G)$.

Proof.

a) It is given that $F \neq F_2$. Let

$$A = \begin{bmatrix} a & b & x^T \\ b & c & y^T \\ x & y & D \end{bmatrix} \in S(F, G - u)$$

with $\operatorname{rank} A = \operatorname{mr}(F, G - u)$ and with the first two rows and columns of A labeled by v and w so that $b \neq 0$. Since $F \neq F_2$, there exists $d \in F$ such that

$d \neq 0$ and $d \neq -b$. Then

$$B = \begin{bmatrix} 0 & 0^T \\ 0 & A \end{bmatrix} + d \begin{bmatrix} J_3 & O \\ O & O \end{bmatrix} \in S(F, G)$$

and $\mathrm{mr}(F, G) \leq \mathrm{rank}\, B \leq \mathrm{rank}\, A + 1 = \mathrm{mr}(F, G - u) + 1$.

b) This follows from a) and the equations $\mathrm{mr}(F, G) = n - M(F, G)$ and $\mathrm{mr}(F, G - u) = n - 1 - M(F, G - u)$.

c) Let

$$A = \begin{bmatrix} d_1 & 0 & b^T \\ 0 & d_2 & c^T \\ b & c & D \end{bmatrix} \in S(F, (G - u) \backslash e).$$

with $\mathrm{rank}\, A = \mathrm{mr}(F, (G - u) \backslash e)$. Then

$$B = \begin{bmatrix} 0 & 0^T \\ 0 & A \end{bmatrix} + \begin{bmatrix} J_3 & O \\ O & O \end{bmatrix} \in S(F, G)$$

and as before, $\mathrm{mr}(F, G) \leq \mathrm{rank}\, B \leq \mathrm{rank}\, A + 1 = \mathrm{mr}(F, (G - u) \backslash e) + 1$.

d) This follows from c) in the same way that b) follows from a). □

We need the following definition, extending $S(F, G)$ to graphs which may have multiple edges, and a result from [vdH].

DEFINITION 2.3. Let $G = (V, E)$ be a multigraph on n vertices.

If $F \neq F_2$, define $S(F, G)$ as the set of all F-valued symmetric $n \times n$ matrices $A = [a_{ij}]$ with

1. $a_{ij} = 0$ if $i \neq j$ and i and j are not adjacent,
2. $a_{ij} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge,
3. $a_{ij} \in F$ if $i \neq j$ and i and j are connected by multiple edges, and
4. $a_{ii} \in F$ for all $i \in V$.

If $F = F_2$, we define $S(F_2, G)$ as the set of all F_2-valued symmetric $n \times n$ matrices $A = [a_{ij}]$ with

1. $a_{ij} \neq 0$ if $i \neq j$ and i and j are connected by an odd number of edges,
2. $a_{ij} = 0$ and i and j are connected by an even number of edges, and
3. $a_{ii} \in F_2$ for all $i \in V$.

We use the formulas found at the beginning of the paper to define $\mathrm{mr}(F, G)$ and $M(F, G)$ in this multigraph setting.

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THEOREM 2.4 (van der Holst). *Let F be a field, let G be a graph, and let u be a vertex of degree two in G with neighbors v and w. Then*

$$M(F, G) = \max\{M(F, (G - u) + vw), M(F, (G - u)/vw)\}.$$

We now prove that $M(F, G)$ is unchanged if an edge adjacent to a degree two vertex or a degree one vertex is subdivided.

THEOREM 2.5. *Let F be a field, let $G = (V, E)$ be a graph, let e be an edge adjacent to a vertex of degree at most 2, and let G_e be the graph obtained by subdividing e once. Then $M(F, G_e) = M(F, G)$.*

Proof. Let u, w be the vertices of e.

I. degree$(u) = 2$:

Let v be the other vertex adjacent to u. Let u' be the new vertex in G_e, By Theorem 2.4 applied to the degree two vertex u',

$$M(F, G_e) = \max\{M(F, (G_e - u') + uw), M(F, (G_e - u')/uw)\}.$$

But $(G_e - u') + uw = G$, so

(2.3) $$M(F, G_e) = \max\{M(F, G), M(F, (G_e - u')/uw)\}.$$

Case 1. $vw \notin E$. Then G is an edge subdivison of $(G_e - u')/uw$. By Lemma 2.1, $M(F, G) \geq M(F, (G_e - u')/uw)$ and we conclude that $M(F, G_e) = M(F, G)$.

Case 2. $vw \in E$. Then there are two edges from v to w in $(G_e - u')/uw$.

Subcase 1. $F \neq F_2$. It follows from Definition 2.3 that

$$M(F, (G_e - u')/uw) = \max\{M(F, G - u), M(F, (G - u)\backslash vw)\}.$$

Applying Proposition 2.2 b), d) yields $M(F, (G_e - u')/uw) \leq M(F, G)$ so by (2.3), $M(F, G_e) \leq M(F, G)$, and by Lemma 2.1, equality holds.

Subcase 2. $F = F_2$. By Definition 2.3, $M(F, (G_e - u')/uw) = M(F, (G - u)\backslash uw)$. By Proposition 2.2 d), this is less than or equal to $M(F, G)$, and again by (2.3) and Proposition 2.1, $M(F, G_e) = M(F, G)$.

II. degree$(u) = 1$:

Let v be the vertex of degree 2 in G_e that results from subdividing e. Then $G_e = (G_e - u) \underset{v}{\oplus} K_2$. Since $G_e - u$ is isomorphic to G, G_e is isomorphic to $G \underset{u}{\oplus} K_2$. Because

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the degree of u in $G \underset{u}{\oplus} K_2$ is 2,

$$\mathrm{mr}(F, G_e) = \mathrm{mr}(F, G \underset{u}{\oplus} K_2) = \mathrm{mr}(F, G) + \mathrm{mr}(F, K_2) = \mathrm{mr}(F, G) + 1$$

by Lemma 38 in [S]. (This lemma is also a corollary of a theorem in [H] and [BFH] which holds for any field - Theorem 57 in [BGL].) Therefore $M(F, G_e) = M(F, G)$. $\square$

Hein van der Holst has found an alternate proof of Theorem 2.5 using the formula in Theorem 14 of his paper [vdH].

COROLLARY 2.6. *Let G be a graph in which every edge is adjacent to a vertex of degree at most 2. Then $M(F, H) = M(F, G)$ for every graph $H \in \mathcal{SC}(G)$.*

We illustrate Corollary 2.6 with a few examples of graphs on 4 and 5 vertices. (The simple results we obtain here follow from other known results.)

EXAMPLE 2.7. *Consider the three graphs*

S_4 paw   folding stool
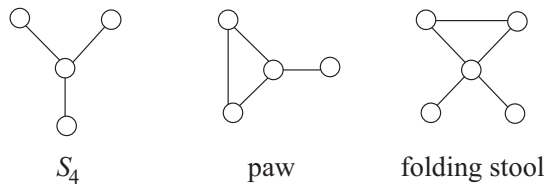
For each of these, $M(F, G) = 2$ for any field F. By Corollary 2.6, $M(F, H) = 2$ for every graph $H \in \mathcal{SC}(G)$.
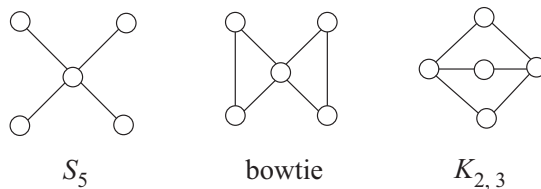
We summarize this example as:

PROPOSITION 2.8. *Let F be a field and let G be one of the following.*

  a) a tree with one vertex of degree 3 and all other vertices of degree at most 2.
  b) a unicyclic graph with one vertex whose degree is 3 or 4 and with all other vertices of degree at most 2.

Then $M(F, G) = 2$.

EXAMPLE 2.9. *Consider the three graphs*

S_5 bowtie $K_{2,3}$

For each, $M(F, G) = 3$ for any field F. By Corollary 2.6, $M(F, H) = 3$ for every

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graph $H \in \mathcal{SC}(G)$.

We summarize as:

PROPOSITION 2.10. *Let F be a field and let G be one of the following.*

  a) *a tree with one vertex of degree 4 and all other vertices of degree at most 2.*
  b) *$C_m \underset{v}{\oplus} C_n$ for any $m, n \geq 3$.*
  c) *an $sK_{2,3}$; i.e., a graph consisting of two vertices u, v and 3 disjoint paths, each of length at least 2, between them.*

Then $M(F, G) = 3$.

Finally we consider the first graph in the table on page 8 of [RW] that has an edge that is not adjacent to a vertex of degree one or two.

PROPOSITION 2.11. *Let F be a field, let the diamond $\equiv K_4 \backslash e$ be labeled*
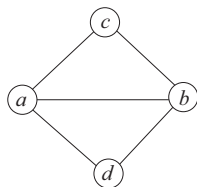
*and let $H \in \mathcal{SC}(\text{diamond})$. Then*

$$(2.4) \qquad M(F, G) = \begin{cases} 2 & \text{if a and b are adjacent in H} \\ 3 & \text{if a and b are not adjacent in H.} \end{cases}$$

Proof. We have $M(F, \text{diamond}) = 2$.

If a and b are adjacent in H, then H can be obtained by successively subdividing edges adjacent to a degree 2 vertex. By Theorem 2.5, $M(F, H) = M(F, \text{diamond}) = 2$.

If a and b are not adjacent in H, then H can be obtained from the diamond by first subdividing the edge ab and then subdividing the remaining edges. Then H is an $sK_{2,3}$ and by Proposition 2.10 c), $M(F, H) = 3$. □

We note that the remark on page 5 of [JLS] follows from this result.

In contrast to Proposition 2.11 we state the following proposition about the bull graph, , without proof.

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PROPOSITION 2.12. *Let F be a field. Then $M(F, X) = 2$ for every s*bull*, X.*

REMARK 2.13. The last two propositions demonstrate, unlike Theorem 2.5, that if $e = vw$ is an edge in a graph G, and $\deg(v) = \deg(w) = 3$, then $M(F, G_e)$ may or may not be equal to $M(F, G)$.

REMARK 2.14. Similar results can be easily obtained for any of the following graphs on 5 vertices.
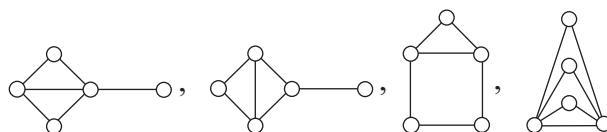
3. Zero Forcing Sets, the Subdivision Graph, and $\mathcal{SC}(K_4)$. In order to analyze edge subdivisions of graphs in which several or all edges are not adjacent to a degree 1 or degree 2 vertex, it is useful to employ the concept of zero forcing sets as defined in [AIM].

DEFINITION 3.1. [Zero forcing rule] Let G be a graph with each vertex colored white or black. If a black vertex has only one white neighbor that vertex may be colored black.

DEFINITION 3.2. Given a graph $G = (V, E)$, a subset Z of V is called a *zero forcing set* if it has the property that when the vertices of Z are colored black and the remaining vertices of V are colored white, then all vertices of V can be made black by successively applying the zero forcing rule to G.

DEFINITION 3.3. $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets Z of G. Any zero forcing set S with $|S| = Z(G)$ is called a *minimal zero forcing set*.

THEOREM 3.4. *Let F be a field and let G be a graph. Then $M(F, G) \leq Z(G)$.*

This is Proposition 2.4 of [AIM].

We next consider the class of sK_4's, the simplest class that illustrates a basic phenomenon. (This is more frequently called the class of hK_4's.)

DEFINITION 3.5. Given a graph G, let $\overset{\circ}{\widetilde{G}}$ be the graph obtained from G by applying one edge subdivision to each edge of G. We call $\overset{\circ}{\widetilde{G}}$ the *subdivision graph* of G.

EXAMPLE 3.6. $\overset{\circ}{\widetilde{K}}_4$ *is the graph in figure 3.1.*

Note that 1, 2, 3, 4 are the original vertices and each ij is a new vertex obtained
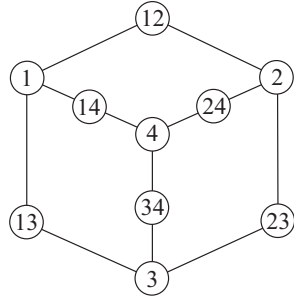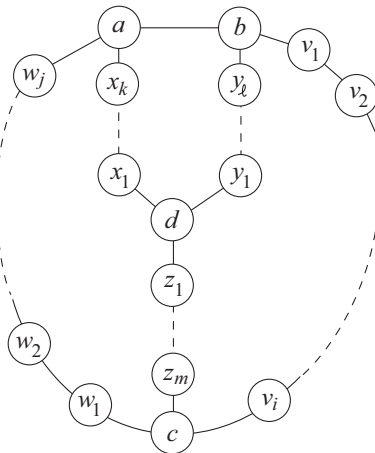
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Fig. 3.1.

from subdividing the edge ij in K_4.

PROPOSITION 3.7. *Let G be an sK_4. Then*

(3.1)
$$Z(G) \leq \begin{cases} 3 & \text{if } G \text{ is not an } s\overset{\circ\circ}{\tilde{K}}_4 \\ 4 & \text{if } G \text{ is an } s\overset{\circ\circ}{\tilde{K}}_4. \end{cases}$$

Proof. If G is not an $s\overset{\circ\circ}{\tilde{K}}_4$, then there exist two vertices of degree 3 that are adjacent in G. Call these a and b. Then G has the form

It is, of course, possible that $v_1 = c$, $w_1 = a$, $x_1 = a$, $y_1 = b$, or $z_1 = c$. Then $\{d, x_1, y_1\}$ is easily seen to be a zero forcing set for G.

If G is an $s\overset{\circ\circ}{\tilde{K}}_4$, then G has the form
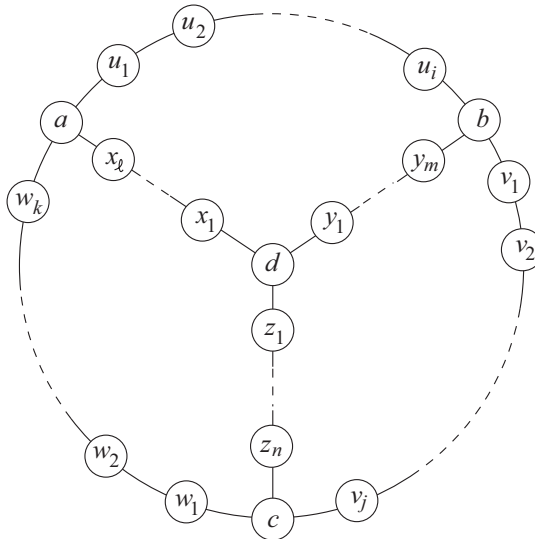
with i, j, k, ℓ, m, n all positive integers. In this case $\{d, x_1, y_1, u_1\}$ is a zero forcing set. $\square$

REMARK 3.8. The next result shows that equality holds in (3.1).

PROPOSITION 3.9. *Let F be any field and let G be an sK_4. Then*

$$M(F,G) = Z(G) = \begin{cases} 3 & \text{if } G \text{ is not an } s\overset{\approx}{K}_4 \\ 4 & \text{if } G \text{ is an } s\overset{\approx}{K}_4. \end{cases}$$

Proof. Successively applying the first inequality in (2.2) of Lemma 2.1, yields

$$3 = M(F, K_4) \le M(F, G).$$

But if G is not an $s\overset{\approx}{K}_4$, by Theorem 3.4 and Proposition 3.7,

$$M(F, G) \le Z(G) \le 3,$$

so that $M(F, G) = Z(G) = 3$.

Now consider the graph $\overset{\approx}{K}_4$ in Figure 3.1. Let

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$$
A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{array}
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 12 & 13 & 14 & 23 & 24 & 34 \\
\left[\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

which is in $S(F, \overset{\approx}{K}_4)$. Because the first four rows of A are linearly dependent (and consequently the first four columns), we have

$$\mathrm{mr}(F, \overset{\approx}{K}_4) \leq \mathrm{rank}\, A \leq 6.$$

By Theorem 3.4 and Proposition 3.7,

$$M(F, \overset{\approx}{K}_4) \leq Z(\overset{\approx}{K}_4) \leq 4.$$

Since $\mathrm{mr}(F, \overset{\approx}{K}_4) + M(F, \overset{\approx}{K}_4) = |\overset{\approx}{K}_4| = 10$,

$$\mathrm{mr}(F, \overset{\approx}{K}_4) = 6 \text{ and } M(F, \overset{\approx}{K}_4) = 4.$$

Then if G is an $s\overset{\approx}{K}_4$ by Corollary 2.6, Theorem 3.4, and Proposition 3.7, we have

$$4 = M(F, \overset{\approx}{K}_4) = M(F, G) \leq Z(G) \leq 4. \ \square$$

REMARK 3.10. It is interesting to compare this result with the more comprehensive result in the appendix of [JLS] which says that if F is an infinite field and G is a graph that contains an sK_4, then $M(F, G) \geq 3$.

REMARK 3.11. Instead of producing the matrix A in the last proof, we could have shown that $M(F, \overset{\approx}{K}_4) \geq 4$ by Theorem 2.4. However, the method of construction of A can be generalized to obtain a useful result.

PROPOSITION 3.12. *Let F be a field and let G be a bipartite graph with bipartite sets X, Y of cardinality n and m respectively. Assume that every vertex in Y has degree 2. Then $\mathrm{mr}(F, G) \leq 2n - 2$.*

Proof. We define a $(0, 1, -1)$ matrix A in $S(F, G)$ as follows. Assume all rows of X come before all rows of Y. Let $a_{ij} = 0$ if i, j are both in X or are both in Y. Each

j in Y has two neighbors i_1, i_2 in X. Let $a_{i_1,j} = 1$ and $a_{i_2 j} = -1$. If $i \in X$, $j \in Y$, and ij is not an edge of G, let $a_{ij} = 0$. Finally, if $i \in Y$, $j \in X$, let $a_{ij} = a_{ji}$. Then $A \in S(F, G)$ and

$$\operatorname{rank} A = \operatorname{rank} A[X \,|\, Y] + \operatorname{rank} A[Y \,|\, X] = 2 \operatorname{rank} A[X \,|\, Y].$$

But the rows of $A[X \,|\, Y]$ sum to 0, so

$$\operatorname{mr}(F, G) \leq \operatorname{rank} A \leq 2(|X| - 1) = 2n - 2. \qquad \square$$

COROLLARY 3.13. *Let F be a field and Let $G = (V, E)$ be a graph. Then*

$$\operatorname{mr}(F, \overset{\circ}{\tilde{G}}) \leq 2|G| - 2.$$

Proof. Let Y be the set of new vertices obtained by subdividing each edge in E. Then $\overset{\circ}{\tilde{G}}$ is a bipartite graph with bipartite sets V, Y with $|Y| = |E|$ and each vertex in Y of degree 2. By Proposition 3.12, $\operatorname{mr}(F, \overset{\circ}{\tilde{G}}) \leq 2|G| - 2$. $\square$

The method of proof of Proposition 3.12 can be adapted to prove a more general result that is needed in the last section.

PROPOSITION 3.14. *Let F be a field and let G be a bipartite graph with bipartite sets X, Y of cardinality n and m respectively. Assume that no vertex of Y has degree one. If $F = F_2$, assume further that each vertex of Y has even degree. Then $\operatorname{mr}(F, G) \leq 2n - 2$.*

Proof. We will create the required matrix A. Assume that all vertices in X precede all vertices in Y.

I. If $F = F_2$, let $a_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{if } i = j \text{ or } ij \notin E \end{cases}$

Then the rows of $A[X \,|\, Y]$ sum to zero and

$$\operatorname{mr}(F, G) \leq \operatorname{rank} A = 2 \operatorname{rank} A[X \,|\, Y] \leq 2(n - 1).$$

II. Assume $F \neq F_2$. Let c denote the characteristic of the field F. Let $a_{ij} = 0$ if i, j are both in X or are both in Y. For each non-isolated vertex v in Y, Let $d(v)$ be the degree of v, and let each of the first $d(v) - 2$ nonzero entries in the column corresponding to v be equal to 1. We then have 2 cases:

Case 1) c divides $d(v) - 1$. Then let the last two non-zero entries be $1 + a$ and $-a$, where $a \neq 0, -1$.

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Case 2) c does not divide $d(v) - 1$. Then let the last two non-zero entries be given by 1 and $-(d(v) - 1)$, where by convention, $d(v) - 1 = 1 + 1 + 1 + ... + 1$, $d(v) - 1$ times.

Following this method of construction for every vertex in Y defines a matrix A in $S(F, G)$, in which the first n rows of A (and, consequently, the first n columns) sum to 0. Then rank $A \leq 2n - 2$. $\square$

4. A universality result. Remarkably, for many graphs G, the minimum rank of its subdivision graph depends only on the number of vertices of G.

THEOREM 4.1. *Let F be any field and let G be a graph on n vertices that contains the subgraph P_n. Then $\mathrm{mr}(F, \overset{\approx}{G}) = 2n - 2$.*

Proof. By Corollary 3.13 it suffices to show that $\mathrm{mr}(F, \overset{\approx}{G}) \geq 2n - 2$. Label the vertices of P_n consecutively as $v_1, v_2, \ldots, v_n$ and let $e_{i\,i+1}$ be the edge incident to v_i and v_{i+1}. Let $y_{i\,i+1}$ be the new vertex in $\overset{\approx}{G}$ formed by the subdivision of $e_{i\,i+1}$. Then $v_1, y_{12}, v_2, y_{23}, \ldots, v_{n-1}, y_{n-1\,n}, v_n$ induce P_{2n-1} in $\overset{\approx}{G}$. By Observation 1.8(c),

$$\mathrm{mr}(F, \overset{\approx}{G}) \geq \mathrm{mr}(F, P_{2n-1}) = 2n - 2. \quad \square$$

REMARK 4.2. In graph theory terminology the hypothesis that P_n is a subgraph of G is referred to by saying that G has a Hamiltonian path.

REMARK 4.3. The converse of Theorem 4.1 is false. Let G be the folding stool . Then P_5 is not a subgraph of G, but by Proposition 2.8, $M(F, \overset{\approx}{G}) = 2$. So

$$\mathrm{mr}(F, \overset{\approx}{G}) = |\overset{\approx}{G}| - 2 = 10 - 2 = 2|G| - 2.$$

From Theorem 4.1 we see that all $\overset{\approx}{G}$ arising from a graph G on n vertices with P_n as a subgraph have some surprising common features.

1. The number of vertices of G completely determine $\mathrm{mr}(F, \overset{\approx}{G})$. This is the universality feature.
   For example, consider the four graphs K_5, W_5, C_5, P_5. For any field F, we have

   $$\mathrm{mr}(F, K_5) = 1, \ \mathrm{mr}(F, W_5) = 2, \ \mathrm{mr}(F, C_5) = 3, \ \text{and} \ \mathrm{mr}(F, P_5) = 4.$$

   But since each has 5 vertices and P_5 is a subgraph of each,

   $$\mathrm{mr}(F, \overset{\approx}{K_5}) = \mathrm{mr}(F, \overset{\approx}{W_5}) = \mathrm{mr}(F, \overset{\approx}{C_5}) = \mathrm{mr}(F, \overset{\approx}{P_5}) = 8.$$

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2. Even if $\mathrm{mr}(F, G)$ depends on F, $\mathrm{mr}(F, \overset{\scriptscriptstyle\infty}{G})$ does not.

For example let G be the full house graph, ⧖. The following is known from [BGL] (see page 891):

$\mathrm{mr}(F_2, \text{full house}) = 3$, but $\mathrm{mr}(F, \text{full house}) = 2$ for every field $F \neq F_2$.

But since P_5 is a subgraph of the full house, $\mathrm{mr}(F, \overline{\text{full house}}) = 8$ for every field F.

COROLLARY 4.4. *Let F be any field and let G be any graph on n vertices and m edges that contains the subgraph P_n. Then $M(F, \overset{\scriptscriptstyle\infty}{G}) = Z(\overset{\scriptscriptstyle\infty}{G}) = 2 + m - n$.*

Proof. Since $\overset{\scriptscriptstyle\infty}{G}$ has $n + m$ vertices by Theorem 4.1,

$$M(F, \overset{\scriptscriptstyle\infty}{G}) = n + m - (2n - 2) = 2 + m - n.$$

By Theorem 3.4, $Z(\overset{\scriptscriptstyle\infty}{G}) \geq 2 + m - n$.

As we saw in the proof of Theorem 4.1, P_{2n-1} is an induced subgraph of $\overset{\scriptscriptstyle\infty}{G}$. Clearly, the set Z of all vertices in $\overset{\scriptscriptstyle\infty}{G}$ not in this P_{2n-1} and one of its pendant vertices is a zero forcing set for $\overset{\scriptscriptstyle\infty}{G}$. Then $Z(\overset{\scriptscriptstyle\infty}{G}) \leq |Z| = m + n - (2n - 1) + 1 = 2 + m - n$. This concludes the proof. □

5. Main Results. Let G be any graph on n vertices and m edges. We now explain a procedure for determining $M(F, H)$ for every graph $H \in \mathcal{SC}(G)$ in terms of the finitely many graphs in $\mathcal{SC}(G)$ that are intermediate between G and $\overset{\scriptscriptstyle\infty}{G}$. Throughout this section it will be convenient to assume that all graphs are labeled.

DEFINITION 5.1. Given a graph $G = (V, E)$ on m edges and any subset B of E, the graph obtained by subdividing each edge in B once is called an *intermediate subdivision* graph of G. We will denote the set of all such graphs by $\mathcal{IS}(G)$. Then $\mathcal{IS}(G) \subset \mathcal{SC}(G)$ and $|\mathcal{IS}(G)| = 2^m$. It is convenient to describe the graphs in $\mathcal{IS}(G)$ as follows. Given a spanning subgraph H of G, let $G(\overset{\scriptscriptstyle\infty}{H})$ be the graph obtained by subdividing each edge of G that belongs to H once.

EXAMPLE 5.2. *Let G be the paw*
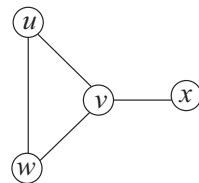
If H_1 is (u)——(v)——(x) (w), *then $G(\overset{\scriptscriptstyle\infty}{H}_1)$ is the graph*
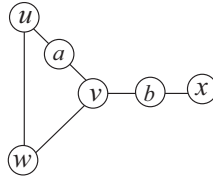
while if H_2 is *, then $G(\vec{\tilde{H}}_2)$ is*
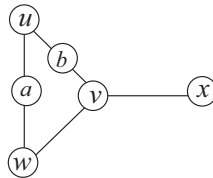
We note that even though H_1 and H_2 as unlabeled graphs are both isomorphic to $P_3 \cup K_1$, $G(\vec{\tilde{H}}_1)$ and $G(\vec{\tilde{H}}_2)$ are not isomorphic.

REMARK 5.3. *$G(\vec{\tilde{G}})$ is the subdivision graph $\vec{\tilde{G}}$ of G.*

DEFINITION 5.4. *Let G be a graph and let H be a spanning subgraph of G. A graph X is in the H-subdivision class $\mathcal{C}(G(\vec{\tilde{H}}))$ if X can be obtained from G by subdividing each edge of G that belongs to H at least once.*

EXAMPLE 5.5. *If G and H_1 are as in Example 5.2, then the graphs $C_4 \underset{v}{\oplus} P_6$ and $C_6 \underset{v}{\oplus} P_5$ are in $\mathcal{C}(G(\vec{\tilde{H}}_1))$.*

OBSERVATION 5.6. *Given a graph G, any sG is in $\mathcal{C}(G(\vec{\tilde{H}}))$ for some spanning subgraph H of G.*

THEOREM 5.7. *Let F be a field and let H be a spanning subgraph of G. If the graph $X \in \mathcal{C}(G(\vec{\tilde{H}}))$, then*

$$M(F, X) = M(F, G(\vec{\tilde{H}})).$$

Proof. X can be obtained from $G(\vec{\tilde{H}})$ by successively subdividing edges adjacent to a degree 2 vertex. The result follows from Theorem 2.5. $\square$

We can now establish the fact that for any graph G, the maximum nullity of any graph in $\mathcal{SC}(G)$ is constrained to belong to a finite set of integers.

THEOREM 5.8. *Let F be any field, let G be any graph, and let X be an sG. Then*

(5.1) $$M(F, G) \leq M(F, X) \leq M(F, \vec{\tilde{G}}).$$

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Moreover, if k is any integer in the set

$$\{M(F,G), M(F,G)+1, M(F,G)+2, \ldots, M(F,\overset{\approx}{\tilde{G}})\},$$

then there is a spanning subgraph H of G such that $M(F,G(\overset{\approx}{\tilde{H}})) = k$.

Proof. By Observation 5.6 and Theorem 5.7, $M(F,X) = M(F,G(\overset{\approx}{\tilde{H}}))$ for some spanning subgraph H of G. But $G(\overset{\approx}{\tilde{H}})$ is obtained from G by subdividing each edge of H once, and $\overset{\approx}{\tilde{G}}$ is obtained from $G(\overset{\approx}{\tilde{H}})$ by subdividing each edge of G that is not in H once. By Lemma 2.1,

$$M(F,G) \leq M(F,G(\overset{\approx}{\tilde{H}})) \leq M(F,\overset{\approx}{\tilde{G}})$$

which verifies (5.1).

To verify the second claim, let $e_1, e_2, \ldots, e_m$ be the edges of G and for $j = 1, \ldots, m$ let G_j be the graph obtained from G by subdividing each edge in $\{e_1, \ldots, e_j\}$ once. Then $G_o = G$, $G_m = \overset{\approx}{\tilde{G}}$, and by Lemma 2.1,

$$M(F,G_j) \leq M(F,G_{j+1}) \leq M(F,G_j) + 1.$$

It follows that $M(F,G_j) = k$ for some $j \in \{0, 1, \ldots, m\}$. Let H be the spanning subgraph of G induced by the edges of G_j. □

COROLLARY 5.9. *Let F be any field and let G be a graph on n vertices and m edges that contains the subgraph P_n. Then if X is any sG,*

$$M(F,G) \leq M(F,X) \leq M(F,\overset{\approx}{\tilde{G}}) = 2 + m - n.$$

Proof. Apply Corollary 4.4 and Theorem 5.8. □

COROLLARY 5.10. *Let F be a field. Then if X is any sK_n,*

$$n - 1 \leq M(F,X) \leq M(F,\overset{\approx}{\tilde{K}}_n) = 1 + \binom{n-1}{2}.$$

Proof. This follows from Observation 1.8, Corollary 5.9, and the fact that K_n has $\binom{n}{2}$ edges. □

Note that if $n = 4$, we have for any sK_4, $3 \leq M(F,X) \leq 4$ in agreement with Proposition 3.9. Although Proposition 3.9 gives $M(F,X)$ more precisely in this special case, Corollary 5.10 also gives the possible maximum nullities for any sK_n, $n > 4$.

Our final aim is to show, given a field F and a graph G, how to determine $M(F,X)$ for all $X \in \mathcal{IS}(G)$ and consequently, via Theorem 5.7, for all $X \in \mathcal{SC}(G)$. Results obtained will resemble Proposition 3.9 but will in general be more complicated.

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DEFINITION 5.11. Let F be a field and let G be a graph. We say a spanning subgraph H of G is *M-critical* for (F, G) if for each subgraph H' of H with one less edge than H,

$$M(F, G(\overset{\circ}{\tilde{H}}')) < M(F, G(\overset{\circ}{\tilde{H}})).$$

For $k = M(F, G) + 1, \ M(F, G) + 2, \ldots, M(F, \overset{\circ}{\tilde{G}})$, let $\mathcal{M}_k(F, G)$ be the set of all M-critical graphs H satisfying $M(F, G(\overset{\circ}{\tilde{H}})) = k$.

THEOREM 5.12. *Let F be a field, let G be a graph, and let*

$$k \in \{M(F, G) + 1, M(F, G) + 2, \ldots, M(F, \overset{\circ}{\tilde{G}})\}.$$

Let $H \in \mathcal{IS}(G)$ satisfying

  i) There is a subgraph X of H in $\mathcal{M}_k(F, G)$.
  ii) No $Y \in \mathcal{M}_{k+1}(F, G)$ is a subgraph of H.

Then $M(F, G(\overset{\circ}{\tilde{H}})) = k$.

Proof. Since $G(\overset{\circ}{\tilde{H}})$ can be obtained from $G(\overset{\circ}{\tilde{X}})$ by subdividing the edges of H that are not in X, by Lemma 2.1

$$k = M(F, G(\overset{\circ}{\tilde{X}})) \leq M(F, G(\overset{\circ}{\tilde{H}})).$$

Now suppose $M(F, G(\overset{\circ}{\tilde{H}})) > k$. Let $e_1, \ldots, e_m$ be the edges of H. For $j = 1, \ldots, m$, let G_j be the graph obtained from G by subdividing the edges $e_1, \ldots, e_j$ of G once. Then,

$$M(F, G_m), M(F, G_{m-1}), \ldots, M(F, G_1), M(F, G)$$

is a decreasing sequence of integers beginning above k, ending at an integer less than or equal to k and with no gaps by Lemma 2.1. So for some ℓ, $M(F, G_\ell) = k+1$. Let Y_ℓ be the subgraph of H induced by $e_1, \ldots, e_\ell$. By definition, $M(F, G(\overset{\circ}{\tilde{Y}}_\ell)) = M(F, G_\ell) = k + 1$. Since Y_ℓ is one such subgraph, there is a subgraph Y of H with a minimum number of edges satisfying $M(F, G(\overset{\circ}{\tilde{Y}})) = k + 1$. Necessarily, $Y \in \mathcal{M}_{k+1}(F, H)$ contradicting the hypothesis. Therefore, $M(F, G(\overset{\circ}{\tilde{H}})) \leq k$. □

6. Examples. We now give two moderately complex examples of graphs G for which we determine $M(F, X)$ for every $X \in \mathcal{SC}(G)$. There is no intrinsic difficulty in working out the values of these maximum nullities, but because of the number of intermediate subdivision graphs that must be examined, it takes a few pages to determine all the possibilities for each graph.

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$\mathbf{K_5}$: For simplicity, we depict K_5 as embedded on a torus
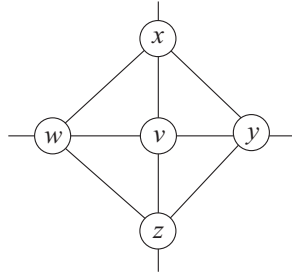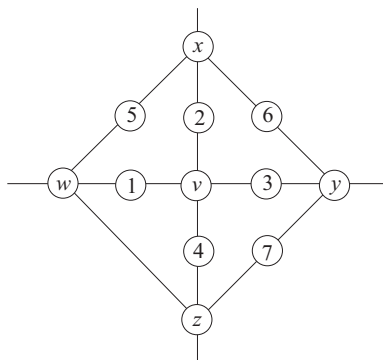

Let F be any field. By Corollary 5.10, if X is any sK_5, $4 \leq M(F, X) \leq 7$. The problem of determining $M(F, X)$ thus reduces to determining $\mathcal{M}_k(F, K_5)$ for $k = 5, 6, 7$.

We begin by finding $M(F, K_5(\overset{\leftrightarrow}{H}))$ if H is either of the two 2-trees

gem:
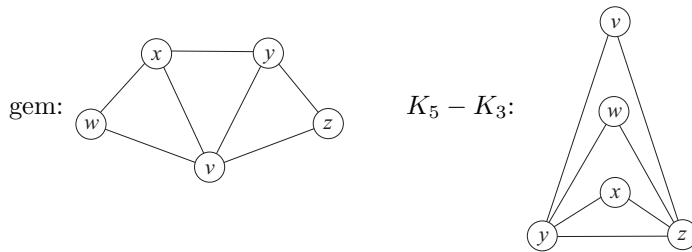
$K_5 - K_3$:

$K_5(\overset{\leftrightarrow}{\text{gem}})$ is the graph

The set $\{w, 1, 3, 6\}$ is a zero forcing set for $K_5(\overset{\leftrightarrow}{\text{gem}})$; one possible forcing sequence is v, y, 7, z, x, 4, 2, 5. Therefore

$$4 = M(F, K_5) \leq M(F, K_5(\overset{\leftrightarrow}{\text{gem}})) \leq Z(K_5(\overset{\leftrightarrow}{\text{gem}})) = 4$$

and $M(F, K_5(\overset{\leftrightarrow}{\text{gem}})) = 4$.

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$K_5(\overline{K_5 \overset{\circ}{-} K_3})$ is the graph
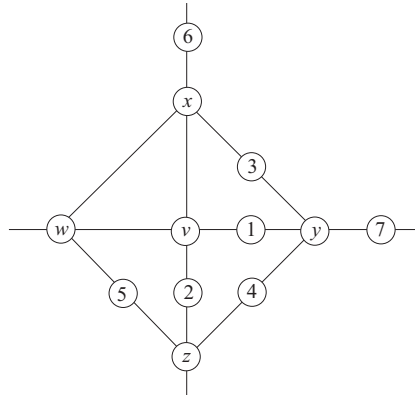
It has $\{v, 1, 3, 7\}$ as a zero forcing set so, as before, $M(F, K_5(\overline{K_5 \overset{\circ}{-} K_3})) = 4$.

By Lemma 2.1, if H is a partial 2-tree (i.e., any subgraph of a 2-tree), we have $M(F, K_5(\overset{\approx}{H})) = 4$. Therefore, no partial 2-tree is M-critical for (F, K_5).

If H is not a partial 2-tree, by Theorem 11.2.3 in [BLS] it contains an sK_4, so we consider subgraphs of K_5 containing an sK_4. (One may reach the same conclusion by examining a table of all graphs on 5-vertices; see for example page 8 of [RW].) There are 7 such graphs: K_5, $K_5 - e$, W_5, full house, $(K_4)_e$, $K_4 \underset{v}{\oplus} K_2$, $K_4 \cup K_1$. (Here $(K_4)_e$ means the graph obtained from K_4 by subdividing one edge.) We consider $K_5(\overset{\approx}{H})$ for each of these.

1. $K_5(\overset{\approx}{K_5}) \equiv \overset{\approx}{K_5}$:

By Corollary 5.10, $M(F, \overset{\approx}{K_5}) = 1 + \binom{4}{2} = 7$.

2. $K_5(\overline{K_5 \overset{\circ}{-} e})$:
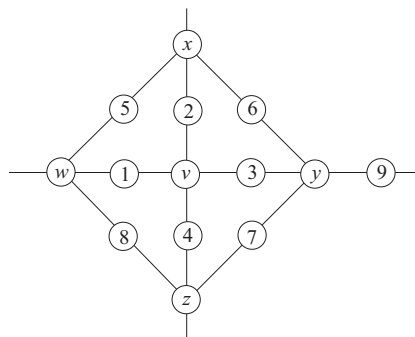
Then $Z = \{x, 1, 2, 3, 5, 9\}$ is a zero forcing set (one possible forcing sequence is

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$w, 8, z, v, 4, 7, y, 6)$, so

$$M(F, K_5(\overline{K_5 \overset{\circ}{-} e})) \le 6.$$

By Lemma 2.1,

$$7 = M(F, \overset{\circ\circ}{\bar{K}}_5) \le M(F, K_5(\overline{K_5 \overset{\circ}{-} e})) + 1.$$

Thus we have $M(F, K_5(\overline{K_5 \overset{\circ}{-} e})) = 6$.

Since $K_5 - e$ is the only subgraph of K_5 with one less edge, by definition, $K_5 \in \mathcal{M}_7(F, K_5)$.
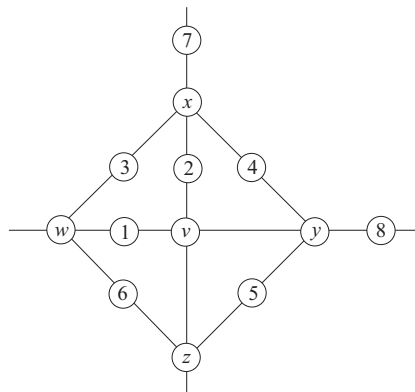
3. $K_5(\overset{\circ\circ}{\bar{W}}_5)$:
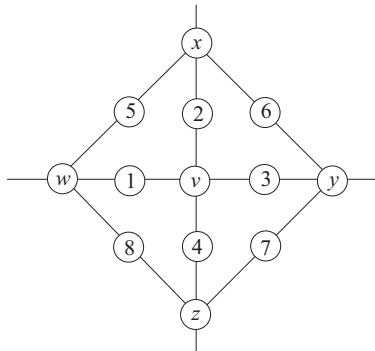
Since $\{v, 1, 2, 3, 5\}$ is a zero forcing set, $M(F, K_5(\overset{\circ\circ}{\bar{W}}_5)) \le 5$. But $K_5(\overline{K_5 \overset{\circ}{-} e})$ is an edge subdivision of $K_5(\overset{\circ\circ}{\bar{W}}_5)$, so by Lemma 2.1,

$$M(F, K_5(\overset{\circ\circ}{\bar{W}}_5)) \ge 6 - 1$$

and we conclude that $M(F, K_5(\overset{\circ\circ}{\bar{W}}_5)) = 5$.

4. $K_5(\overline{\text{full } \overset{\circ}{\text{house}}})$:

Note that $\{v, 1, 5, 6, 8\}$ is a zero forcing set, so $M(F, K_5(\overline{\text{full house}})) \leq 5$. Since $K_5(\overline{K_5 - e})$ is an edge subdivision of $K_5(\overline{\text{full house}})$, $M(F, K_5(\overline{\text{full house}})) \geq 6 - 1$, as in the previous case, and we have $M(F, K_5(\overline{\text{full house}})) = 5$.

Now note that W_5 and full house are the only subgraphs of $K_5 - e$ with one less edge. Since $M(F, K_5(\overline{K_5 - e})) = 6$ while $M(F, K_5(\overline{W_5})) = M(F, K_5(\overline{\text{full house}})) = 5$, we have $K_5 - e \in \mathcal{M}_6(F, K_5)$.

5. $K_5(\overline{K_4 \cup K_1})$:
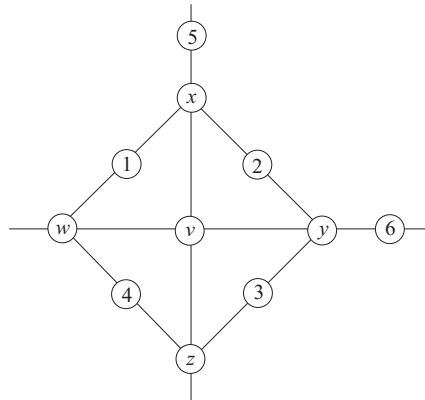
We apply Proposition 3.14 with $X = \{w, x, y, z\}$ and $Y = \{v, 1, 2, 3, 4, 5, 6\}$. Since every vertex of Y has even degree, we have $\mathrm{mr}(F, K_5(\overline{K_4 \cup K_1})) \leq 2 \cdot 4 - 2 = 6$ for every field F. Since $\{v, w, 1, 2, 6\}$ is a zero forcing set, $M(F, K_5(\overline{K_4 \cup K_1})) \leq 5$.

Since $K_5(\overline{K_4 \cup K_1})$ has 11 vertices, both of these inequalities are equalities. So $M(F, K_5(\overline{K_4 \cup K_1})) = 5$. Any subgraph H of $K_4 \cup K_1$ with one less edge is a partial 2-tree, so $M(F, K_5(\overline{H})) = 4$. It follows that $K_4 \cup K_1 \in \mathcal{M}_5(F, K_5)$.

6. $K_5(\overline{K_4 \underset{v}{\oplus} K_2})$:

Note that $K_5(\overline{K_4 \underset{v}{\oplus} K_2})$ is an edge subdivision of $K_5(\overline{K_4 \cup K_1})$ and $K_5(\overline{\text{full house}})$ is an edge subdivision of $K_5(\overline{K_4 \underset{v}{\oplus} K_2})$. By Lemma 2.1,

$$5 = M(F, K_5(\overline{K_4 \cup K_1}) \leq M(F, \overline{K_4 \underset{v}{\oplus} K_2}) \leq M(F, K_5(\overline{\text{full house}})) = 5.$$

It follows that neither $K_4 \underset{v}{\oplus} K_2$ nor full house is an M critical graph for (F, K_5).

7. $K_5((\overrightarrow{\overline{K_4})_e}})$:
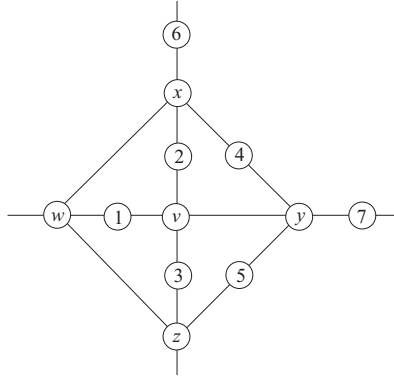
$\{v, 1, 3, 5\}$ is a zero forcing set, so $M(F, K_5((\overrightarrow{\overline{K_4})_e}})) = 4$ and $(K_4)_e$ is not M-critical.

Finally W_5 has two subgraphs with one less edge, the gem and $(K_4)_e$. Since $M(F, K_5(\overrightarrow{\overline{W_5}})) = 5$ while

$$M(F, K_5(\overrightarrow{\overline{\text{gem}}})) = M(F, K_5((\overrightarrow{\overline{K_4})_e}})) = 4,$$

we conclude that $W_5 \in \mathcal{M}_5(F, K_5)$.

We summarize these calculations as

THEOREM 6.1. *Let F be any field. Then there are exactly four M-critical graphs for (F, K_5): $K_4 \cup K_1$, W_5, $K_5 - e$, and K_5. More precisely:*

$$\mathcal{M}_5(F, K_5) = \{K_4 \cup K_1, W_5\}$$

$$\mathcal{M}_6(F, K_5) = \{K_5 - e\}$$

$$\mathcal{M}_7(F, K_5) = \{K_5\}.$$

Applying Theorem 5.12 to this case, we have the following result which is analogous to Proposition 3.9.

THEOREM 6.2. *Let F be any field and let G be an sK_5 so that $G \in \mathcal{C}(K_5(\overrightarrow{\overline{H}}))$ for some $H \in \mathcal{IS}(K_5)$. Then*

$$M(F, G) = \begin{cases} 4 & \text{if neither } K_4 \cup K_1 \text{ nor } W_5 \text{ is a subgraph of } H \\ 5 & \text{if either } K_4 \cup K_1 \text{ or } W_5 \text{ is a subgraph of } H, \text{ but } K_5 - e \text{ is not} \\ 6 & \text{if } H = K_5 - e \\ 7 & \text{if } H = K_5. \end{cases}$$

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Two rather surprising features of this example is that $M(F, X)$ is field indepen-
dent for every graph in $\mathcal{SC}(K_5)$, and furthermore, it can be checked that $M(F, X) =
Z(X)$ for all these graphs. In our next example we will see that both of these can fail.

$\mathbf{W_5}$: We label the wheel on 5 vertices, W_5, in the following way:
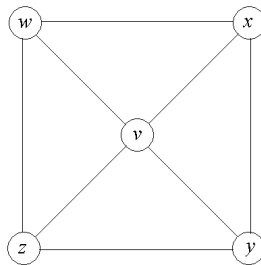
and note that by Theorem 4.1, $M(F, \overset{\circ}{\tilde{W}}_5) = 5$, and by Corollary 5.9, if G is any sW_5
then $3 \leq M(F, G) \leq 5$.

We first consider the 2 subgraphs of W_5 with one less edge, the gem and $(K_4)_e$.
Since $\overset{\circ}{\tilde{W}}_5$ can be obtained from $W_5(\overline{\text{gem}})$ or $W_5((\overline{(K_4)_e}))$ by exactly one subdivision,
Lemma 2.1 implies $4 \leq M(F, W_5(\overset{\circ}{\tilde{G}}))$ when G is either of these two graphs. First we
consider $W_5(\overline{\text{gem}})$.
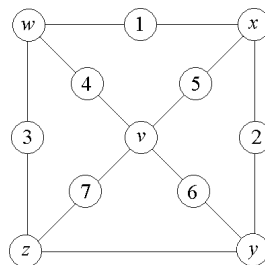
The set $\{v, 1, 6, 7\}$ forms a zero forcing set for this graph, so

$$\mathrm{M}(F, W_5(\overline{\text{gem}})) \leq Z(W_5(\overline{\text{gem}})) \leq 4.$$

Therefore $\mathrm{M}(F, W_5(\overline{\text{gem}})) = 4$.

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Now, we consider $W_5((\overset{\circ}{\overline{K_4)_e}})$.
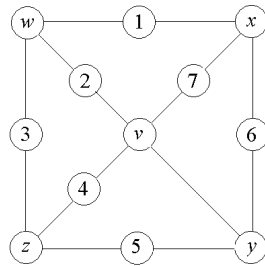
The set $\{w, 1, 2, 7\}$ is a zero forcing set, so similarly, $M(F, W_5((\overset{\circ}{\overline{K_4)_e}})) = 4$. Since the gem and $(K_4)_e$ are the only two subgraphs of W_5 with one less edge, it follows that $W_5 \in \mathcal{M}_5(F, W_5)$ for any field F.

There are six spanning subgraphs of W_5 with exactly 6 edges. We presently consider four of these and will save discussion of the other two for later.

1. Bowtie: $W_5(\overset{\circ}{\overline{\text{bowtie}}})$ is the graph:
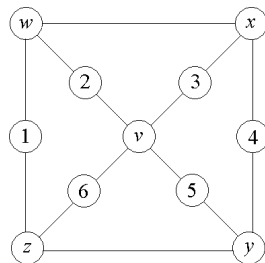
This has a zero forcing set of size 3, namely $\{v, 2, 3\}$ Hence

$$3 \leq M(F, W_5(\overset{\circ}{\overline{\text{bowtie}}}) \leq Z(W_5(\overset{\circ}{\overline{\text{bowtie}}}) \leq 3.$$

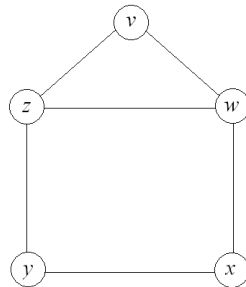So $M(F, W_5(\overset{\circ}{\overline{\text{bowtie}}})) = 3$.

2. House1:
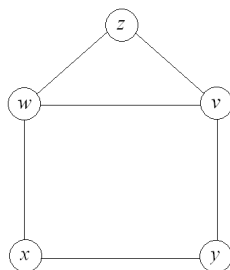
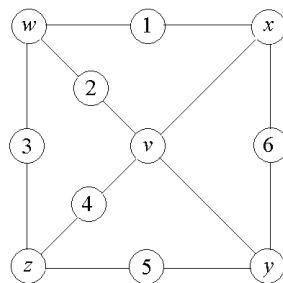Then $W_5(\overline{\text{house1}})$ is the graph:
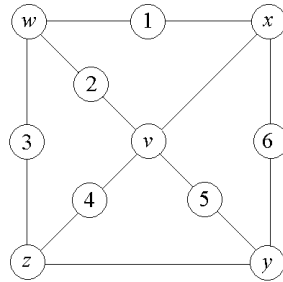
$\{w,\, 1,\, 2\}$ is a zero forcing set, so similarly $M(W_5(\overline{\text{house1}})) = 3$
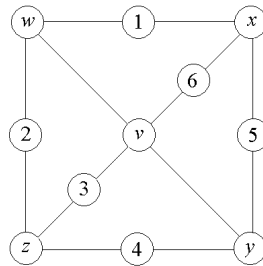
3. House2:

$W_5(\overline{\text{house2}})$ is the graph:

Notice that $W_5(\overline{\text{house1}})$ and $W_5(\overline{\text{house2}})$ are not isomorphic. One zero forcing set for $W_5(\overline{\text{house2}})$ is $\{w, 1, 2\}$, so $M(W_5(\overline{\text{house2}})) = 3$.

4. $K_{2,3}$: $W_5(\overline{\overset{\circ}{K_{2,3}}})$ is the graph:

A zero forcing set for this graph is $\{z, 2, 3\}$, and $M(W_5(\overline{\overset{\circ}{K_{2,3}}})) = 3$

Therefore, $M(F, W_5(\overset{\circ}{G})) = 3$ for any field F and any subgraph G of the bowtie, house1, house2, or $K_{2,3}$. By examining a table of graphs we see that we have now determined $M(F, W_5(\overset{\circ}{H}))$ for all subgraphs of W_5 except diamond $\cup K_1$ and the two graphs
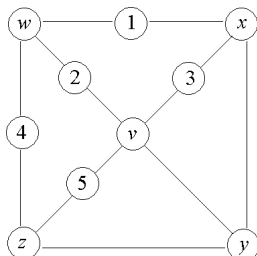

kite   dart
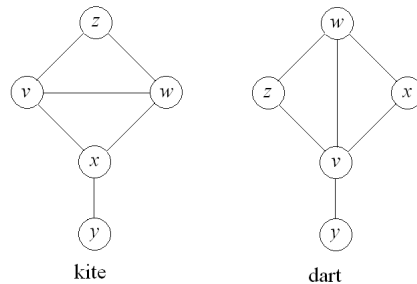
(we have chosen one among many equivalent labelings for these two graphs). Note that, for all of the graphs G considered thus far in this example, we have found $M(F, G)$ independent of our field F. Such is not the case for the remaining graphs.

Now we consider diamond $\cup K_1$, labeled as follows:
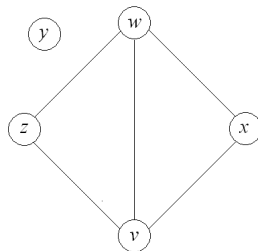
and we will assume for now that $F \neq F_2$. $W_5(\overline{\overset{\circ}{\text{diamond} \cup K_1}})$ is the graph:

Since $W_5(\overset{\circ}{\overrightarrow{\text{gem}}})$ is a subdivision of $W_5(\overline{\text{diamond} \cup K_1})$,

$$3 \leq M(F, W_5(\overline{\text{diamond} \cup K_1})) \leq 4.$$

Applying Proposition 3.14 with $F \neq F_2$ and $X = \{v, w, x, z\}$ gives

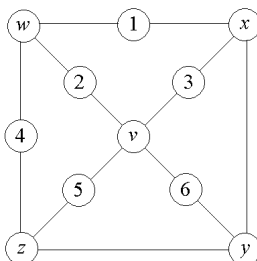$$\mathrm{mr}(F, W_5(\overline{\text{diamond} \cup K_1})) \leq 6.$$

Hence $\mathrm{M}(F, W_5(\overline{\text{diamond} \cup K_1})) \geq 4$. So $\mathrm{M}(F, W_5(\overline{\text{diamond} \cup K_1})) = 4$.

Remembering that $M(F, W_5(\overset{\approx}{\tilde{H}})) = 3$ for every proper spanning subgraph H of diamond $\cup K_1$, it follows that diamond $\cup K_1$ must be M-critical for (F, W_5) for all fields other than F_2. Since $M(F, W_5(\overset{\approx}{\tilde{K}})) = 3$, for all graphs K that do not contain diamond $\cup K_1$, it also follows that no other graph can be an element of $\mathcal{M}_4(F, W_5)$. Thus, we have the following proposition:

PROPOSITION 6.3. *Let F be a field, $F \neq F_2$. Then there are exactly 2 M-critical graphs for (F, W_5): W_5 and diamond $\cup K_1$. In other words,*

$$\mathcal{M}_5(F, W_5) = \{W_5\}$$
$$\mathcal{M}_4(F, W_5) = \{\text{diamond} \cup K_1\}$$

Now, for the case $F = F_2$: we have examined 4 of the possible 6 graphs in $\mathcal{IS}(W_5)$ that are obtained from W_5 by subdividing exactly 6 edges. Now we examine the other two in turn. First, we look at $W_5(\overrightarrow{\overset{\circ\circ}{\text{dart}}})$:

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Using Theorem 2.4, with u equal to the vertex 3,

$$M(F_2, W_5(\overset{\circ}{\overline{\text{dart}}})) = \max\{M(F_2, (W_5(\overset{\circ}{\overline{\text{dart}}}) - 3) + vx),$$
$$M(F_2, (W_5(\overset{\circ}{\overline{\text{dart}}}) - 3)/vx)\}$$
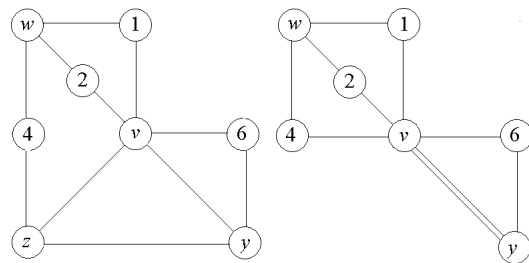
These two graphs are the following:
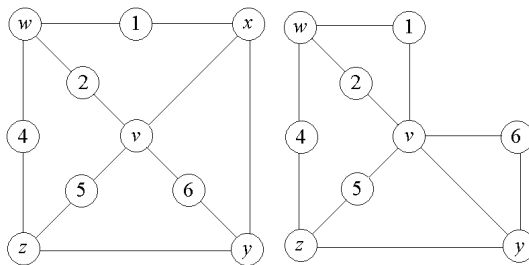
Notice that subdividing the edge xy in the first graph yields $W_5(\overset{\circ}{\overline{\text{house1}}})$, so we have $M(F_2, (W_5(\overset{\circ}{\overline{\text{dart}}}) - 3) + vx) = 3$. We use Theorem 2.4 again on the second graph, with u equal to vertex 5.
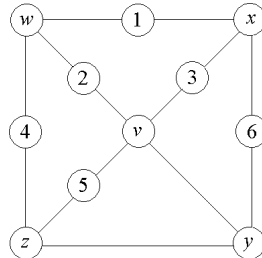
$$M(F_2, W_5(\overset{\circ}{\overline{\text{dart}}})) = \max\{3, M(F_2, ((W_5(\overset{\circ}{\overline{\text{dart}}}) - 3)/vx) - 5 + vz),$$
$$M(F_2, ((W_5(\overset{\circ}{\overline{\text{dart}}}) - 3)/vx - 5)/vz)\}.$$

These two graphs are the following:

The first has a zero forcing set of $\{w, 1, 2\}$, and hence has maximum nullity at most 3. Since we are working over F_2, we can replace the double edge from v to y in the second graph by no edge. Then $\{w, 1, 2\}$ is again a zero forcing set, and the maximum nullity over F_2 is at most 3. Therefore, $M(F_2, W_5(\overset{\circ}{\overline{\text{dart}}})) = 3$.
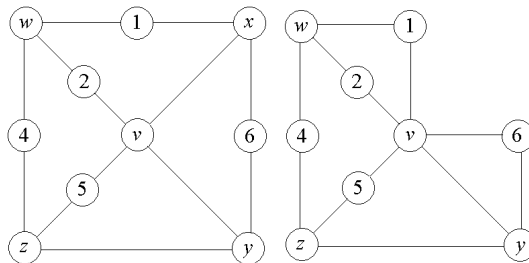
The final graph we examine is the kite. $W_5(\overset{\circ}{\overline{\text{kite}}})$ is the graph:

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Again using Theorem 2.4, with u equal to the vertex 3,

$$M(F_2, W_5(\overset{\rightharpoonup\circ}{\text{kite}})) = \max\{M(F_2, (W_5(\overset{\approx}{H}) - 3) + vx),$$

$$M(F_2, (W_5(\overset{\approx}{H}) - 3)/vx)\}$$

These two graphs are the following:

We recognize that the second of these is isomorphic to $(W_5(\overset{\rightharpoonup\circ}{\text{dart}}) - 3)/vx$ from above, so we only need consider the first. The first graph has a zero forcing set of $\{w, 1, 2\}$, so its maximum nullity is at most 3. Therefore,

$$\mathrm{M}(F_2, W_5(\overset{\rightharpoonup\circ}{\text{kite}})) = 3$$

It follows that $W_5(\overset{\rightharpoonup\circ}{\text{gem}})$ and $W_5((\overline{K_4)_e})$ are the only graphs in $\mathcal{IS}(W_5)$ with maximum nullity 4 over F_2, and hence both gem and $(K_4)_e$ are M-critical over F_2. We summarize our findings over F_2 as follows:

PROPOSITION 6.4. *There are exactly 3 M-critical graphs for (F_2, W_5):*

$$\mathcal{M}_5 = \{W_5\}$$
$$\mathcal{M}_4 = \{\text{gem}, (K_4)_e\}$$

We summarize the preceding two propositions in the following theorem:

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THEOREM 6.5. *Let F be a field, and let G be an sW_5 so that $G \in \mathcal{C}(W_5(\overset{\circ}{\tilde{H}}))$ for some $H \in \mathcal{IS}(W_5)$. Then*

$$
M(F,G) = \begin{cases}
5 & \text{if } H = W_5 \\[4pt]
4 & \text{if one of the following holds:} \\
& \quad \text{a. } H = \text{gem or } H = (K_4)_e \\
& \quad \text{b. } F \neq F_2, \ H \neq W_5, \text{ and } \text{diamond} \cup K_1 \text{ is a subgraph of } H. \\[4pt]
3 & \text{otherwise}
\end{cases}
$$

For all graphs X in $\mathcal{SC}(K_5)$ we have $M(F,X) = Z(X)$, and for all X in $\mathcal{SC}(W_5)$ we have $M(F,X) = Z(X)$ as long as $F \neq F_2$. Such examples as these and the ones encountered earlier in the paper may lead one to believe that if G is any graph for which $M(F,G) = Z(G)$ and $F \neq F_2$, then $M(F,X) = Z(X)$ for every $X \in \mathcal{SC}(G)$. However, this is not the case.

Let $G = W_6$ be labeled as follows.
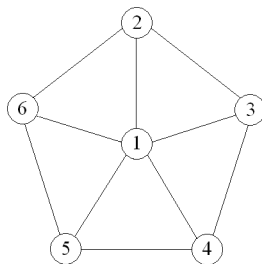
For any field $F \neq F_2$ and any $a \neq 0, -1$, the matrix

$$
\begin{bmatrix}
a^2 - 1 & a & 1+a & -1 & 1 & -a \\
a & 1 & 1 & 0 & 0 & -1 \\
1+a & 1 & 1 & 1 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
-a & -1 & 0 & 0 & 1 & 1+1
\end{bmatrix}
\in S(F, W_6)
$$

and it is straightforward to check that $\operatorname{rank} A = 3$. Therefore, whenever $F \neq F_2$, $\operatorname{mr}(F, W_6) \leq 3$ and $M(F, W_6) \geq 3$. It is known [BGL] that $\operatorname{mr}(F_2, W_6) = 4$ (see the table accompanying Theorem 49).

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Now subdivide each spoke of W_6 to obtain the graph W_6^*.
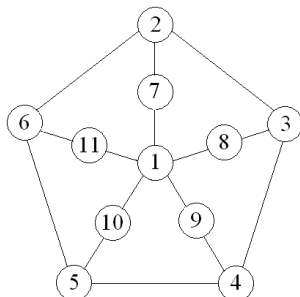
We verified with a computer program that $Z(W_6^*) > 3$, so it is 4 since $\{1, 7, 8, 9\}$ is a zero forcing set. But the graph $W_6^* - 1$ is the 5-sun and $\mathrm{mr}(F, 5\text{-sun}) = 8$ for every field $[BFH]$. Therefore $\mathrm{mr}(F, W_6^*) \geq 8$ and $M(F, W_6^*) \leq 3$. By Lemma 2.1, for $F \neq F_2$, we have $M(F, W_6^*) \geq M(F, W_6) \geq 3$, so $M(F, W_6^*) = M(F, W_6) = 3$. Finally, by Corollary 4.4, $M(F, \overset{\circ}{\overline{W}}_6) = Z(\overset{\circ}{\overline{W}}_6) = 6$.

In summary:

$$M(F, W_6) = 3 = Z(W_6), \qquad F \neq F_2$$

$$M(F, W_6^*) = 3 < Z(W_6^*) = 4, \qquad F \neq F_2$$

$$M(F, \overset{\circ}{\overline{W}}_6) = 6 = Z(\overset{\circ}{\overline{W}}_6) \qquad \text{for every field F}$$

Thus as edges are subdivided in a graph the equality $M(F, G) = Z(G)$ may be lost and then regained.

We end this section by mentioning that the procedure in the examples above may be simplified for some graphs. Since the minimum degree of K_5 and W_5 is at least 3, it was necessary to consider all intermediate subdivision graphs for each. But we saw in Proposition 2.11 that everything is determined by whether or not the one edge adjacent to the degree 3 vertices is subdivided.

More generally, in view of Theorem 2.5, it suffices to consider only the intermediate subdivision graphs obtained by subdividing those edges incident to vertices whose degrees are at least 3. For example in the gem
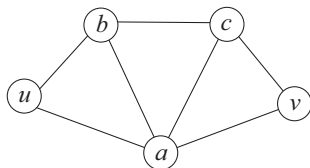
we need only concern ourselves with the edges ab, ac, bc, and the eight intermediate subdivision graphs X obtained by either subdividing once or not subdividing each of these edges. Once $M(F, X)$ is known for each of these graphs, it is known for all $X \in \mathcal{SC}(\text{gem})$.

7. Conclusion and Open Questions. Given a field F and a graph G, we have considered the problem of finding the minimum rank (maximum nullity) of any graph obtained from G by subdividing edges. Theorem 2.5 enables us to reduce this problem to finding the minimum rank of the intermediate subdivision graphs (those graphs obtained from G by subdividing each edge at most once). Moreover, in any minimum rank problem whatsoever, we need not concern ourselves with any graphs in which two degree two vertices are adjacent, or a degree two vertex is adjacent to a degree one vertex, as these problems reduce to a minimum rank problem for a graph on fewer vertices.

We have also elucidated the special role of the subdivision graph $\overset{\approx}{G}$ showing that $M(F, \overset{\approx}{G})$ is an upper bound for $M(F, X)$ for any $X \in \mathcal{SC}(G)$. Since $M(F, G)$ is a lower bound, $M(F, X)$ for $X \in \mathcal{SC}(G)$ can take on only finitely many values. In the penultimate section we gave examples to show how all of these maximum nullities may be determined. Moreover, if G contains a Hamiltonian path, then $M(F, \overset{\approx}{G})$ depends only on the number of vertices in G.

We conclude with the following questions.

1. Suppose $e = vw$ is an edge in a graph G, and that $\deg(v), \deg(w) \geq 3$. When is $M(F, G_e) = M(F, G)$?
2. Suppose G is any graph in which each vertex has degree at least 3 and that H is a graph which has one less edge subdivision than $\overset{\approx}{G}$. Is it always the case that $M(F, H) < M(F, \overset{\approx}{G})$?
3. Is $M(F, \overset{\approx}{G}) = Z(\overset{\approx}{G})$ for every field F and graph G?
   This is true if G is a tree or if G contains P_n as a subgraph.
4. For which graphs G is $M(F, X)$ field independent for all $X \in \mathcal{SC}(G)$?

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