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AN ALGORITHM FOR SOLVING THE ABSOLUTE VALUE EQUATION*

JIRI ROHN†

Abstract. Presented is an algorithm which for each $A, B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ in a finite number of steps either finds a solution of the equation $Ax + B|x| = b$, or states existence of a singular matrix S satisfying $|S - A| \leq |B|$ (and in most cases also constructs such an S).

1. Introduction. We consider here the equation

$$Ax + B|x| = b, \tag{1.1}$$

where $A, B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ (termed an *absolute value equation* by Mangasarian [3]). This equation, in a particular form suitable for solving interval linear equations, was first studied in [9] (and even earlier in report form in [7], [8]). In the general form (1.1) it was first introduced in [10] and has been since studied by Mangasarian [2], [3], [4], Prokopyev [6], and Schäfer [13]. Since the linear complementarity problem can be easily translated into the form (1.1) (see [9], [13]), this equation forms a common ground for the linear complementarity problem, linear programming and convex quadratic programming (Murty [5]).

As the main result of this paper we present an algorithm (Fig. 3.1 below) which for each $A, B \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ in a finite number of steps either finds a solution to (1.1), or states existence of a singular matrix S satisfying

$$|S - A| \leq |B| \tag{1.2}$$

(Theorem 3.1). The result is preceded by several auxiliary results in Section 2 and, as its consequence, a theorem of alternatives is proved in Section 4. Besides stating existence of a singular matrix S satisfying (1.2), the algorithm in most cases also constructs such a matrix. The algorithm proved to be surprisingly efficient, making on the average about $0.11 \cdot n$ iterations per example, where n is the problem size. An implementation of the algorithm is given in Section 5.

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We use the following notation. $A_{k\bullet}$ and $A_{\bullet k}$ denote the kth row and the kth column of a matrix A, respectively. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notations also apply to vectors that are considered one-column matrices. I is the unit matrix, e_k is the kth column of I, and $e = (1, \ldots, 1)^T$ is the vector of all ones. $Y_n = \{y \mid |y| = e\}$ is the set of all ± 1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $x \in \mathbb{R}^n$ we define its sign vector $\operatorname{sgn} x$ by

$$(\operatorname{sgn} x)_i = \begin{cases} 1 & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0 \end{cases} \qquad (i = 1, \ldots, n),$$

so that $\operatorname{sgn} x \in Y_n$. For each $y \in \mathbb{R}^n$ we denote

$$T_y = \operatorname{diag}(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n \end{pmatrix}.$$

2. Auxiliary results. In this section we give several auxiliary results to be used later. The first of them is the Sherman-Morrison formula ((iii) below) and the Sherman-Morrison determinant formula ((i) below), see [14]. As (i) is less known, we give a proof of it here, and we append a proof of (iii) for completeness as well.

PROPOSITION 2.1. *Let $A \in \mathbb{R}^{n \times n}$ be nonsingular, $b, c \in \mathbb{R}^n$, and let $\alpha = 1 + c^T A^{-1} b$. Then we have:*

(i) $\det(A + bc^T) = \alpha \det(A)$,
(ii) *if $\alpha = 0$, then $A + bc^T$ is singular,*
(iii) *if $\alpha \neq 0$, then $(A + bc^T)^{-1} = A^{-1} - \frac{1}{\alpha} A^{-1} bc^T A^{-1}$.*

Proof. (i) From the identities

$$\begin{pmatrix} I + A^{-1}bc^T & 0 \\ -c^T & 1 \end{pmatrix} = \begin{pmatrix} I & -A^{-1}b \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} I & A^{-1}b \\ -c^T & 1 \end{pmatrix},$$

$$\begin{pmatrix} I & A^{-1}b \\ 0^T & \alpha \end{pmatrix} = \begin{pmatrix} I & 0 \\ c^T & 1 \end{pmatrix} \begin{pmatrix} I & A^{-1}b \\ -c^T & 1 \end{pmatrix}$$

it follows that

$$\det(I + A^{-1}bc^T) = \det \begin{pmatrix} I & A^{-1}b \\ -c^T & 1 \end{pmatrix} = \det \begin{pmatrix} I & A^{-1}b \\ 0^T & \alpha \end{pmatrix} = \alpha,$$

hence

$$\det(A + bc^T) = \det(A) \cdot \det(I + A^{-1}bc^T) = \alpha \det(A).$$

(ii) If $\alpha = 0$, then $\det(A + bc^T) = 0$ by (i).

(iii) If $\alpha \neq 0$, then direct computation shows that

$$(A + bc^T)(A^{-1} - \tfrac{1}{\alpha}A^{-1}bc^T A^{-1}) = I - \tfrac{1}{\alpha}bc^T A^{-1} + bc^T A^{-1} - \tfrac{1}{\alpha}b(c^T A^{-1}b)c^T A^{-1}$$
$$= I + (-\tfrac{1}{\alpha} + 1 - \tfrac{c^T A^{-1}b}{\alpha})bc^T A^{-1} = I,$$

since the last term in parentheses equals zero. This implies that

$$(A + bc^T)^{-1} = A^{-1} - \tfrac{1}{\alpha}A^{-1}bc^T A^{-1},$$

which completes the proof. $\square$

The subsequent formulations will simplify if we use the notion of an interval matrix.

Definition. Given $A, B \in \mathbb{R}^{n \times n}$, the set of matrices

$$[A - |B|, A + |B|] := \{ S \mid |S - A| \leq |B| \} = \{ S \mid A - |B| \leq S \leq A + |B| \}$$

is called an interval matrix (with midpoint matrix A and radius matrix $|B|$).

Next we have this definition introducing an important distinction:

Definition. A square interval matrix $\mathbf{A}$ is called *regular* if each $S \in \mathbf{A}$ is nonsingular, and *singular* otherwise (i.e., if $\mathbf{A}$ contains a singular matrix).

PROPOSITION 2.2. *An interval matrix* $\mathbf{A} = [A - |B|, A + |B|]$ *is singular if and only if the inequality*

$$|Ax| \leq |B||x| \tag{2.1}$$

has a nontrivial solution.

Proof. If $\mathbf{A}$ contains a singular matrix S, then $Sx = 0$ for some $x \neq 0$, which implies

$$|Ax| = |(A - S)x| \leq |A - S||x| \leq |B||x|.$$

Conversely, let (2.1) hold for some $x \neq 0$. Define $y \in \mathbb{R}^n$ and $z \in Y_n$ by

$$y_i = \begin{cases} (Ax)_i/(|B||x|)_i & \text{if } (|B||x|)_i > 0, \\ 1 & \text{if } (|B||x|)_i = 0 \end{cases} \quad (i = 1, \ldots, n) \tag{2.2}$$

and

$$z = \operatorname{sgn} x.$$

Then $T_z x = |x|$, hence

$$((A - T_y|B|T_z)x)_i = (Ax)_i - y_i(|B||x|)_i = 0$$

for each i, so that $A - T_y|B|T_z$ is singular, and since $|y_i| \leq 1$ for each i due to (2.1), it follows that $|(A - T_y|B|T_z) - A| = |T_y|B|T_z| \leq |B|$, hence $A - T_y|B|T_z \in \mathbf{A}$ and $\mathbf{A}$ is singular. $\square$

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We have also proved the following constructive result which will be later used in the proof of Theorem 4.1.

COROLLARY 2.3. *If x is a nontrivial solution of (2.1), then the matrix*

$$S = A - T_y|B|T_z,$$

where y is given by (2.2) and $z = \operatorname{sgn} x$, is a singular matrix in $[A - |B|, A + |B|]$, and $Sx = 0$.

The last proposition will be used at the key point of the proof of the main theorem.

PROPOSITION 2.4. *Let $[A - |B|, A + |B|]$ be regular and let*

$$(A + BT_{z'})x' = (A + BT_{z''})x'' \tag{2.3}$$

hold for some $z', z'' \in Y_n$ and $x' \neq x''$. Then there exists a j satisfying $z_j' z_j'' = -1$ and $x_j' x_j'' > 0$.

Proof. Assume to the contrary that for each j, $z_j' z_j'' = -1$ implies $x_j' x_j'' \leq 0$, so that $|x_j' - x_j''| = |x_j'| + |x_j''|$. We shall prove that in this case

$$|T_{z'}x' - T_{z''}x''| \leq |x' - x''|, \tag{2.4}$$

i.e., that

$$|z_j' x_j' - z_j'' x_j''| \leq |x_j' - x_j''|$$

holds for each j. Since $|z_j' x_j' - z_j'' x_j''| = |z_j'(x_j' - z_j' z_j'' x_j'')| = |x_j' - z_j' z_j'' x_j''|$, this fact is obvious for $z_j' z_j'' = 1$. If $z_j' z_j'' = -1$, then

$$|z_j' x_j' - z_j'' x_j''| = |x_j' + x_j''| \leq |x_j'| + |x_j''| = |x_j' - x_j''|,$$

which together proves (2.4). Now, from (2.3) we have

$$|A(x' - x'')| = |B(T_{z'}x' - T_{z''}x'')| \leq |B||T_{z'}x' - T_{z''}x''| \leq |B||x' - x''|$$

due to (2.4), where $x' - x'' \neq 0$, hence $[A - |B|, A + |B|]$ is singular by Proposition 2.2, a contradiction. $\square$

3. The sign accord algorithm. The following theorem is the main result of this paper.

THEOREM 3.1. *For each $A, B \in \mathbb{R}^{n \times n}$ and each $b \in \mathbb{R}^n$, the sign accord algorithm (Fig. 3.1) in a finite number of steps either finds a solution of the equation*

$$Ax + B|x| = b, \tag{3.1}$$

or states singularity of the interval matrix $[A - |B|, A + |B|]$ (and, in most cases, also finds a singular matrix $S \in [A - |B|, A + |B|]$).

Comment 1. For better understandability, we first describe the basic idea behind the algorithm (Fig. 3.1). If we knew the sign vector $z = \operatorname{sgn} x$ of the solution x of

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```
function [x, S, flag] = signaccord (A, B, b)
% Finds a solution to Ax + B|x| = b or states
% singularity of [A − |B|, A + |B|].
x = [ ]; S = [ ]; flag = 'singular';
if A is singular, S = A; return, end
p = 0 ∈ ℝⁿ;
z = sgn (A⁻¹b);
if A + BT_z is singular, S = A + BT_z; return, end
x = (A + BT_z)⁻¹b;
C = −(A + BT_z)⁻¹B;
while z_j x_j < 0 for some j
    k = min{j | z_j x_j < 0};
    if 1 + 2z_k C_kk ≤ 0
        S = A + B(T_z + (1/C_kk)e_k e_k^T);
        x = [ ];
        return
    end
    p_k = p_k + 1;
    if log₂ p_k > n − k, x = [ ]; return, end
    z_k = −z_k;
    α = 2z_k/(1 − 2z_k C_kk);
    x = x + αx_k C_•k;
    C = C + αC_•k C_k•;
end
flag = 'solution';
```

$$\text{function } [x, S, flag] = \textbf{signaccord}\,(A, B, b)$$

FIG. 3.1. *The sign accord algorithm.*

(3.1), we could rewrite (3.1) as $(A+BT_z)x = b$ and solve it for x as $x = (A+BT_z)^{-1}b$. The problem is, we know neither x, nor z; but we do know that they should satisfy $T_z x = |x| \geq 0$, i.e., $z_j x_j \geq 0$ for each j (a situation we call a sign accord of z and x). In its kernel form (Fig. 3.2) the sign accord algorithm computes the z's and x's repeatedly until a sign accord occurs. A combinatorial argument (parts 3.1 and 3.2

```
z = sgn (A⁻¹b);
x = (A + BT_z)⁻¹b;
while z_j x_j < 0 for some j
    k = min{j | z_j x_j < 0};
    z_k = −z_k;
    x = (A + BT_z)⁻¹b;
end
```

FIG. 3.2. *The kernel of the sign accord algorithm.*

of the proof) based on Proposition 2.4 is used to prove that in case of regularity of

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$[A - |B|, A + |B|]$, a sign accord is achieved within a prespecified number of steps, so that crossing this number indicates singularity of $[A - |B|, A + |B|]$.

Comment 2. The algorithm (Fig. 3.1) may state singularity of the interval matrix $[A-|B|, A+|B|]$ without actually finding a singular matrix in $[A-|B|, A+|B|]$ (this is the case of the last **if** statement in its description), but such a situation occurs rarely; it finds a singular matrix "almost always", as it will be shown in Section 5. Once singularity has been established, the algorithm stops; the equation (1.1) may possess a solution, but it has not been found.

Proof. The proof consists of several steps.

1. Termination. The algorithm starts with the vector $p = 0$ and during each pass through the **while** loop it increases some p_k by 1. This means that after a finite number of steps p_k will become greater than 2^{n-k} for some k, and the algorithm will terminate in the fourth **if** statement[1] (if not earlier).

2. Simplification. Next we shall simplify the description of the algorithm by proving by induction that after each updating of C at the end of the **while** loop, the current values of z, x and C satisfy

$$x = (A + BT_z)^{-1}b, \tag{3.2}$$
$$C = -(A + BT_z)^{-1}B. \tag{3.3}$$

This is obviously so for the initial values of z, x and C. Thus let (3.2), (3.3) hold true at some step. Then for each real t the matrix

$$A + B(T_z - 2tz_k e_k e_k^T) = A + BT_z - (2tz_k Be_k)e_k^T$$

is a rank one update of the matrix $A + BT_z$, which is nonsingular by the induction hypothesis because (3.2) holds, hence by the Sherman-Morrison determinant formula we have

$$\det(A + B(T_z - 2tz_k e_k e_k^T)) = (1 - 2tz_k e_k^T(A + BT_z)^{-1}Be_k)\det(A + BT_z)$$
$$= (1 + 2tz_k C_{kk})\det(A + BT_z). \tag{3.4}$$

Now two possibilities may occur.

2.1. Case of $1 + 2z_k C_{kk} \leq 0$. Then $C_{kk} \neq 0$ and the real function $\varphi(t) = 1 + 2tz_k C_{kk}$ satisfies $\varphi(0)\varphi(1) = 1 + 2z_k C_{kk} \leq 0$, hence $\varphi(\tau) = 0$ for $\tau = (-1)/(2z_k C_{kk}) \in [0, 1]$ and

$$\det(A + B(T_z - 2\tau z_k e_k e_k^T)) = 0.$$

Because of $\tau \in [0, 1]$ we have $|T_z - 2\tau z_k e_k e_k^T| \leq I$, so that the matrix $A + B(T_z - 2\tau z_k e_k e_k^T)$ belongs to $[A - |B|, A + |B|]$ and is singular. This is the case of the first

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[1] We prefer to write the condition as $\log_2 p_k > n - k$ instead of $p_k > 2^{n-k}$ to avoid a possibly large number 2^{n-k}.

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if statement in the **while** loop. In this case the algorithm terminates with a singular matrix $S = A + B(T_z - 2\tau z_k e_k e_k^T) = A + B(T_z + (1/C_{kk})e_k e_k^T) \in [A - |B|, A + |B|]$.

2.2. *Case of* $1 + 2z_k C_{kk} > 0$. Here the first **if** statement of the **while** loop is not in effect and provided this is also the case for the second one, the algorithm constructs the updated values $\widetilde{z}$, $\widetilde{x}$ and $\widetilde{C}$ along the formulae

$$\widetilde{z}_k = -z_k,$$
$$\alpha = 2\widetilde{z}_k/(1 - 2\widetilde{z}_k C_{kk}) = -2z_k/(1 + 2z_k C_{kk}),$$
$$\widetilde{x} = x + \alpha x_k C_{\bullet k},$$
$$\widetilde{C} = C + \alpha C_{\bullet k} C_{k\bullet}.$$

Then the matrix

$$A + BT_{\widetilde{z}} = A + B(T_z - 2z_k e_k e_k^T) = A + BT_z - (2z_k Be_k)e_k^T$$

is nonsingular due to (3.4) (with $t = 1$), hence by the Sherman-Morrison formula there holds

$$(A + BT_{\widetilde{z}})^{-1} = (A + BT_z)^{-1} + \frac{(A + BT_z)^{-1} 2z_k Be_k e_k^T (A + BT_z)^{-1}}{1 + 2z_k C_{kk}}$$
$$= (A + BT_z)^{-1} + \alpha C_{\bullet k} e_k^T (A + BT_z)^{-1}.$$

Then we have

$$(A + BT_{\widetilde{z}})^{-1}b = (A + BT_z)^{-1}b + \alpha C_{\bullet k} e_k^T (A + BT_z)^{-1}b = x + \alpha x_k C_{\bullet k} = \widetilde{x}$$

and

$$-(A + BT_{\widetilde{z}})^{-1}B = -(A + BT_z)^{-1}B - \alpha C_{\bullet k} e_k^T (A + BT_z)^{-1}B = C + \alpha C_{\bullet k} C_{k\bullet} = \widetilde{C},$$

which proves (3.2), (3.3) by induction. Hence we can see that the matrix C plays a purely auxiliary role, helping to avoid an explicit computation of $x = (A + BT_z)^{-1}b$ at each step.

3. *Correctness.* If the condition of the **while** loop is not satisfied at some step, then $z_j x_j \geq 0$ for each j, hence $T_z x \geq 0$, so that $T_z x = |x|$. Because $x = (A + BT_z)^{-1}b$ by (3.2), we have that $Ax + B|x| = (A + BT_z)x = b$, so that x solves the equation (3.1). Next there are four possible terminations in the four **if** statements. In the first three of them singularity is clearly detected (this is obvious with the first two of them, and the fact that the matrix S constructed in the third **if** statement is singular has been proved in part 2.1). Thus it remains to be shown that if the condition of the fourth **if** statement is satisfied, i.e., if $\log_2 p_k > n - k$ for some k, then $[A - |B|, A + |B|]$ is singular. This will be proved if we demonstrate that if $[A - |B|, A + |B|]$ is regular, then

$$p_k \leq 2^{n-k} \tag{3.5}$$

holds throughout the algorithm for each k, which will exclude the possibility of $\log_2 p_k > n - k$. Thus let $[A - |B|, A + |B|]$ be regular, and consider the sequence of

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k's generated by the **while** loop of the algorithm. We shall prove by induction that each k can appear there at most 2^{n-k} times ($k = n, \ldots, 1$).

3.1. Case $k = n$. Assume that n appears at least twice in the sequence, and let z', x' and z'', x'' correspond to any two nearest occurrences of it (i.e., there is no other occurrence of n between them). Then $z'_j x'_j \geq 0$, $z''_j x''_j \geq 0$ for $j = 1, \ldots, n-1$, and $z'_n x'_n < 0$, $z''_n x''_n < 0$, $z'_n z''_n = -1$, which implies $z'_n x'_n z''_n x''_n > 0$ and $x'_n x''_n < 0$. Hence, $z'_j x'_j z''_j x''_j \geq 0$ for each j. But since

$$(A + BT_{z'})x' = b = (A + BT_{z''})x'' \tag{3.6}$$

holds due to (3.2) and $x' \neq x''$ (because $x'_n x''_n < 0$), it follows from Proposition 2.4 that there exists a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, implying $z'_j x'_j z''_j x''_j < 0$, a contradiction; hence n occurs at most once in the sequence.

3.2. Case $k < n$. Again, let z', x' and z'', x'' correspond to any two nearest occurrences of k, so that $z'_j x'_j \geq 0$, $z''_j x''_j \geq 0$ for $j = 1, \ldots, k-1$, $z'_k x'_k < 0$, $z''_k x''_k < 0$ and $z'_k z''_k = -1$. This implies that $z'_j x'_j z''_j x''_j \geq 0$ for $j = 1, \ldots, k-1$, $z'_k x'_k z''_k x''_k > 0$ and $x'_k x''_k < 0$. Since (3.6) holds due to (3.2), and $x' \neq x''$ because of $x'_k x''_k < 0$, Proposition 2.4 implies existence of a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, hence $z'_j x'_j z''_j x''_j < 0$, so that $j > k$. Since $z'_j z''_j = -1$, j must have entered the sequence between the two occurrences of k. Hence between any two nearest occurrences of k there is an occurrence of some $j > k$ in the sequence; this means by the induction hypothesis that k cannot occur there more than $(2^{n-k-1} + \ldots + 2 + 1) + 1 = 2^{n-k}$ times.

3.3. Conclusion. We have proved that in case of regularity (3.5) holds for each k, hence a situation of $\log_2 p_k > n - k$ indicates singularity of $[A - |B|, A + |B|]$. This justifies the last possible termination, and thereby also the whole algorithm. □

As the reader might have noticed, we have never used the fact that $z = \text{sgn}(A^{-1}b)$ is set at the outset. This is only a heuristic step, supported by computational experience, aimed at diminishing the number of steps of the algorithm. The finiteness of the algorithm will remain unaffected if we start from an arbitrary $z \in Y_n$.

The sign accord algorithm was first given in [9], albeit only in its kernel form (Fig. 3.2) and for a very special case of the equation (1.1) arising in the process of solving interval linear equations ([9], Algorithm 3.1 and Theorem 3.1). In its present form it was formulated in the internet text [11], but without proof.

4. Theorem of the alternatives. The following theorem could be inferred from Theorems 1 and 2 in [10]. We give here a direct proof based on Theorem 3.1 which, in contrast to [10], does not use properties of P-matrices or of the linear complementarity problem.

THEOREM 4.1. *For each $A, B \in \mathbb{R}^{n \times n}$, exactly one of the two alternatives holds:*

(i) for each B' with $|B'| \leq |B|$ and for each $b \in \mathbb{R}^n$ the equation

$$Ax + B'|x| = b \tag{4.1}$$

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has a unique solution,

(ii) the inequality

$$|Ax| \leq |B||x| \tag{4.2}$$

has a nontrivial solution.

Proof. Given $A, B \in \mathbb{R}^{n \times n}$, the interval matrix $[A - |B|, A + |B|]$ is either regular, or singular. In the latter case the assertion (ii) holds due to Proposition 2.2. To prove that regularity implies (i), we first note that each B' with $|B'| \leq |B|$ satisfies $[A - |B'|, A + |B'|] \subseteq [A - |B|, A + |B|]$, hence regularity of $[A - |B|, A + |B|]$ implies that of $[A - |B'|, A + |B'|]$ and the sign accord algorithm as applied to the equation (4.1) with arbitrary $b \in \mathbb{R}^n$ cannot state singularity, hence according to Theorem 3.1 it finds in a finite number of steps a solution of the equation (4.1). To prove uniqueness, assume to the contrary that (4.1) has solutions x' and x'', $x' \neq x''$. Put $z' = \operatorname{sgn} x'$, $z'' = \operatorname{sgn} x''$, then $T_{z'}x' \geq 0$, $T_{z''}x'' \geq 0$ and $(A + B'T_{z'})x' = b = (A + B'T_{z''})x''$ holds, hence by Proposition 2.4 there exists a j with $z'_j z''_j = -1$ and $x'_j x''_j > 0$, implying $z'_j x'_j z''_j x''_j < 0$ contrary to $z'_j x'_j \geq 0$ and $z''_j x''_j \geq 0$, a contradiction. Hence the solution of (4.1) is unique.

We have proved that either (i), or (ii) always holds. Assume to the contrary that both of them hold together. Then according to Corollary 2.3 there exists a singular matrix S of the form $S = A - T_y|B|T_z$ such that $|T_y| \leq I$, $Sx = 0$ and $T_z x = |x|$, where x is a nontrivial solution of (4.2). Put $B' = -T_y|B|$. Then $|B'| \leq |B|$ and the equation

$$Ax + B'|x| = 0$$

has at least two different solutions (namely, x and 0), which contradicts (i). Hence, (i) and (ii) cannot hold simultaneously; this means that exactly one of them holds. $\square$

Returning back to the single equation (1.1), we have the following consequence.

PROPOSITION 4.2. *If the interval matrix $[A - |B|, A + |B|]$ is regular, then for each right-hand side b the equation*

$$Ax + B|x| = b$$

has a unique solution which can be found by the sign accord algorithm (Fig. 3.1) in a finite number of steps.

5. Implementation. The sign accord algorithm (Fig. 3.1) has been implemented (with minor modifications) in the MATLAB function EK.P which is a part of the free software package VERSOFT [1]. Its syntax is

```
[x,y,C]=ek(A,B,b)
```

where A, B, b are the data of (1.1), and (if applicable) x is a solution of (1.1), $C.As$ is a singular matrix in $[A - |B|, A + |B|]$, y is a nonzero vector satisfying (2.1), $C.iter$ is the number of iterations (i.e., the number of k's generated by the **while** loop), and

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C.flag contains a verbal description of the output. Contrary to the description of the algorithm (Fig. 3.1) a vector (or matrix) of *NaN*'s is used instead of an empty output here (thus complying to the INTLAB [12] standard). We have run the following test with 1000 randomly generated examples of size 500×500 on a laptop with Mobile AMD Sempron(tm) Processor 3500+ 1.80 GHz and 1.00 GB RAM:

```
tic
n=500; m=1000;
sols=0; sing=0; iter=0;
for j=1:m
    rand('state',j);
    A=2*rand(n,n)-1;
    B=0.01*(2*rand(n,n)-1);
    b=2*rand(n,1)-1;
    [x,y,C]=ek(A,B,b);
    if ~isnan(x(1)), sols=sols+1; end
    if ~isnan(C.As(1,1)), sing=sing+1; end
    iter=iter+C.iter;
end
sols, sing, averiter=iter/m, avertime=toc/m
sols =
   877
sing =
   123
averiter =
   60.6610
avertime =
    3.7141
```

As it can be seen, a solution has been found in 877 cases, and a singular matrix *C.As* has been found in *all* the remaining 123 singularity cases. The average number of iterations *averiter* corresponds well to the results of the author's test on 100,000 various-size examples done back in 2005 which showed that the average number of iterations is about $0.11 \cdot n$, where n is the matrix size. The average running time for a 500×500 example is 3.7141 seconds.

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