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REPRESENTATIONS FOR THE DRAZIN INVERSE OF BOUNDED OPERATORS ON BANACH SPACE*

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Abstract. In this paper a representation is given for the Drazin inverse of a 2×2 operator matrix, extending to Banach spaces results of Hartwig, Li and Wei [SIAM J. Matrix Anal. Appl., 27 (2006) pp. 757–771]. Also, formulae are derived for the Drazin inverse of an operator matrix M under some new conditions.

Key words. Operator matrix, Drazin inverse, D-invertibility, GD-invertibility.

AMS subject classifications. 47A52, 47A62, 15A24.

1. Introduction. Throughout this paper $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces over the same field. We denote the set of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X},\mathcal{Y})$ and by $\mathcal{B}(\mathcal{X})$ when $\mathcal{X} = \mathcal{Y}$. For $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, let $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\sigma(A)$ and $r(A)$ be the range, the null space, the spectrum and the spectral radius of A, respectively. By $I_{\mathcal{X}}$ we denote the identity operator on $\mathcal{X}$.

In 1958, Drazin [16] introduced a pseudoinverse in associative rings and semigroups that now carries his name. When $\mathcal{A}$ is an algebra and $a \in \mathcal{A}$, then $b \in \mathcal{A}$ is the Drazin inverse of a if

$$ab = ba, \; b = bab \;\text{ and }\; a(1 - ba) \in \mathcal{A}^{nil}, \tag{1.1}$$

where $\mathcal{A}^{nil}$ is the set of all nilpotent elements of algebra $\mathcal{A}$.

Caradus [5], King [23] and Lay [25] investigated the Drazin inverse in the setting of bounded linear operators on complex Banach spaces. Caradus [5] proved that a bounded linear operator T on a complex Banach space has the Drazin inverse if and

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only if 0 is a pole of the resolvent $(\lambda I - T)^{-1}$ of T. The order of the pole is equal to the Drazin index of T which we shall denote by $\text{ind}(A)$ or i_A. In this case we say that A is D-invertible. If $\text{ind}(A) = k$, then Drazin inverse of A denoted by A^D satisfies

$$A^{k+1}A^D = A^k,\ A^DAA^D = A^D,\ AA^D = A^DA, \tag{1.2}$$

and k is the smallest integer such that (1.2) is satisfied. If $\text{ind}(A) \leq 1$, then A^D is known as the group inverse of A, denoted by $A^\sharp$. A is invertible if and only if $\text{ind}(A) = 0$ and in this case $A^D = A^{-1}$.

Harte [20] and Koliha [24] observed that in Banach algebra it is more natural to replace the nilpotent element in (1.1) by a quasinilpotent element. In the case when $a(1 - ba)$ in (1.1) is allowed to be quasinilpotent, we call b the generalized Drazin inverse (g-Drazin inverse) of a and say that a is GD-invertible. g-Drazin inverse was introduced in the paper of Koliha [24] and it has many applications in a number of areas. Harte [20] associated with each quasipolar operator T an operator $T^\times$, which is an equivalent to the generalized Drazin inverse. Nashed and Zhao [29] investigated the Drazin inverse for closed linear operators and applied it to singular evolution equations and partial differential operators. Drazin [17] investigated extremal definitions of generalized inverses that give a generalization of the original Drazin inverse.

Finding an explicit representation for the Drazin inverse of a general 2×2 block matrix, posed by Campbell in [4], appears to be difficult. This problem was investigated in many papers (see [21], [27], [14], [22], [33], [26], [8], [12]). In this paper we give a representation for the Drazin inverse of a 2×2 bounded operator matrix. We show that the results given by Hartwig, Li and Wei [22] are preserved when passing from matrices to bounded linear operators on a Banach space. Also, we derive formulae for the Drazin inverse of an operator matrix M under some new conditions.

If $0 \notin \text{acc}\sigma(A)$, then the function $z \mapsto f(z)$ can be defined as $f(z) = 0$ in a neighborhood of 0 and $f(z) = 1/z$ in a neighborhood of $\sigma(A)\setminus\{0\}$. Function $z \mapsto f(z)$ is regular in a neighborhood of $\sigma(A)$ and the generalized Drazin inverse of A is defined using the functional calculus as $A^d = f(A)$. An operator $A \in \mathcal{B}(X)$ is GD-invertible, if $0 \notin \text{acc}\sigma(A)$ and in this case the spectral idempotent P of A corresponding to $\{0\}$ is given by $P = I - AA^d$ (see the well-known Koliha's paper [24]). If A is GD-invertible, then the resolvent function $z \mapsto (zI - A)^{-1}$ is defined in a punctured neighborhood of $\{0\}$ and the generalized Drazin inverse of A is the operator A^d such that

$$A^dAA^d = A^d,\ \ AA^d = A^dA \ \text{ and } \ A(I - AA^d) \text{ is quasinilpotent.}$$

It is well-known that if $A \in \mathcal{B}(\mathcal{X})$ is GD-invertible, then using the following

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decomposition

$$\mathcal{X} = \mathcal{N}(P) \oplus \mathcal{R}(P),$$

we have that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix},$$

where $A_1 : \mathcal{N}(P) \to \mathcal{N}(P)$ is invertible and $A_2 : \mathcal{R}(P) \to \mathcal{R}(P)$ is quasinilpotent operator.

In this case, the generalized Drazin inverse of A has the following matrix decomposition:

$$A^d = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}.$$

For other important properties of Drazin inverses see ([1], [2], [3], [5], [7], [8], [9], [10], [13], [15], [19], [21], [26], [27], [30], [31], [32], [33], [34]).

2. Main results. Firstly, we will state a very useful result concerning the additive properties of Drazin inverses which is the main result proved in [6] with $a^\pi = 1 - aa^{\mathsf{d}}$.

THEOREM 2.1. *Let a, b be GD-invertible elements of algebra $\mathcal{A}$ such that*

$$a^\pi b = b,\ ab^\pi = a,\ b^\pi aba^\pi = 0.$$

Then $a+b$ is GD-invertible and

$$\begin{aligned}(a+b)^{\mathsf{d}} = {} & \left(b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a(a+b)^n\right) a^\pi + b^\pi a^d \\ & + \sum_{n=0}^{\infty} b^\pi (a+b)^n b(a^{\mathsf{d}})^{n+2} - \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a(a+b)^n ba^{\mathsf{d}} \\ & - \sum_{n=0}^{\infty} b^{\mathsf{d}} a(a+b)^n b(a^{\mathsf{d}})^{n+2} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} (b^{\mathsf{d}})^{k+2} a(a+b)^{n+k+1} b(a^{\mathsf{d}})^{n+2}.\end{aligned}$$

Next we extend [22, Lemma 2.4] to the linear operator.

LEMMA 2.2. *Let $M \in \mathcal{B}(\mathcal{X})$, $G \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $H \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ be operators such that $HG = I_{\mathcal{X}}$. If M is GD-invertible operator, then the operator GMH is GD-invertible and*

$$(GMH)^d = GM^dH. \tag{2.1}$$

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Proof. It is evident that

$$(GM^dH)(GMH)(GM^dH) = GM^dMM^dH = GM^dH$$

and

$$(GM^dH)(GMH) = GM^dMH = GMM^dH = (GMH)(GM^dH).$$

To prove that $GMH\Big(I - (GMH)(GM^dH)\Big)$ is a quasinilpotent, note that

$$GMH\Big(I - (GMH)(GM^dH)\Big) = GM(I - MM^d)H.$$

Since $M(I - MM^d)$ is quasinilpotent, we have

$$\begin{aligned}
&r\Big(GMH(I - (GMH)(GM^dH))\Big) = r\Big(GM(I - MM^d)H\Big) \\
&= \lim_{n\to\infty} \Big\|\Big(GM(I - MM^d)H\Big)^n\Big\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \Big\|G\left(M(I - MM^d)\right)^n H\Big\|^{\frac{1}{n}} \\
&\le \lim_{n\to\infty} \|G\|^{\frac{1}{n}} \cdot \Big\|\Big(M(I - MM^d)\Big)^n\Big\|^{\frac{1}{n}} \cdot \|H\|^{\frac{1}{n}} = 0.
\end{aligned}$$

Hence, (2.1) is valid. □

From now on, we will assume that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $\mathcal{Z} = \mathcal{X} \oplus \mathcal{Y}$. For $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{B}(\mathcal{Y})$, consider the operator $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{Z})$.

Theorem 2.3. *If A and D are GD-invertible operators such that*

$$BC = 0 \quad \text{and} \quad DC = 0,$$

then M is GD-invertible and

$$M^d = \begin{bmatrix} A^d & X \\ C(A^d)^2 & Y + D^d \end{bmatrix},$$

where

$$X = X(A,B,D) = \sum_{n=0}^{\infty} (A^d)^{n+2} BD^n D^\pi + \sum_{n=0}^{\infty} A^\pi A^n B(D^d)^{n+2} - A^d BD^d \tag{2.2}$$

and $Y = CXD^d + CA^dX$.

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Proof. We rewrite $M = P + Q$, where $P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}$. By [14, Theorem 5.1], P^d is GD-invertible and

$$P^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix},$$

where $X = X(A, B, D)$ is defined by (2.2). Also, Q is GD-invertible and $Q^d = 0$. Now, we have that the condition $P^\pi Q = Q$ is equivalent to

$$\begin{aligned} &-(AX + BD^d)C = 0, \\ &D^\pi C = C \end{aligned} \tag{2.3}$$

whereas the condition $PQP^\pi = 0$ is equivalent to

$$\begin{aligned} &BCA^\pi = 0, DCA^\pi = 0, \\ &-BC(AX + BD^d) = 0, \\ &-DC(AX + BD^d) = 0. \end{aligned} \tag{2.4}$$

Since, $BC = 0$ and $DC = 0$, from (2.3) and (2.4) we get that $P^\pi Q = Q$ and $PQP^\pi = 0$, so by Theorem 2.1, we have that M is GD-invertible and

$$\begin{aligned} M^d &= P^d + \sum_{n=0}^{\infty} M^n Q (P^d)^{n+2} \\ &= \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} 0 & 0 \\ C(A^d)^{n+2} & \sum_{i=1}^{n+2} C(A^d)^{i-1} X (D^d)^{n+2-i} \end{bmatrix} \\ &= \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ C(A^d)^2 & \sum_{i=1}^{2} C(A^d)^{i-1} X (D^d)^{2-i} \end{bmatrix} \\ &= \begin{bmatrix} A^d & X \\ C(A^d)^2 & Y + D^d \end{bmatrix}, \end{aligned}$$

for $Y = CXD^d + CA^d X$. □

Remark 1. Theorem 2.3 is a strengthening of [14, Theorem 5.3], since it shows that one of the conditions of Theorem 2.3 ($BD = 0$) is actually redundant.

THEOREM 2.4. *If A and D are GD-invertible operators such that*

$$C(I - AA^d)B = 0, \quad A(I - AA^d)B = 0 \tag{2.5}$$

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and $S = D - CA^dB$ *is nonsingular, then* M *is GD-invertible and*

$$M^d = \left(I + \begin{bmatrix} 0 & (I - AA^d)B \\ 0 & 0 \end{bmatrix} R\right) R \left(I + \sum_{i=0}^{\infty} R^{i+1} \begin{bmatrix} 0 & 0 \\ C(I - AA^d)A^i & 0 \end{bmatrix}\right), \tag{2.6}$$

where

$$R = \begin{bmatrix} A^d + A^dBS^{-1}CA^d & -A^dBS^{-1} \\ -S^{-1}CA^d & S^{-1} \end{bmatrix}. \tag{2.7}$$

Proof. In [18] it is proved that $\sigma(A) \cup \sigma(M) = \sigma(A) \cup \sigma(D)$, so we conclude that $0 \notin \mathrm{acc}\sigma(M)$, i.e., M is GD-invertible.

Using that $\mathcal{X} = \mathcal{N}(P) \oplus \mathcal{R}(P)$, for $P = I - AA^d$, we have

$$M = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix},$$

where $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} : Y \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to Y$.

Now, we have

$$\begin{aligned} M_1 &= I_2 \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} I_1 \\ &= \begin{bmatrix} A_1 & B_1 & 0 \\ C_1 & D & C_2 \\ 0 & B_2 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ Y \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ Y \\ \mathcal{R}(P) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} I_2 &= \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ Y \\ \mathcal{R}(P) \end{bmatrix}, \\ I_1 &= \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ Y \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix}. \end{aligned}$$

Since $I_1 = I_2^{-1}$, using Lemma 2.2, we have that $M^d = I_1 M_1^d I_2$, so we proceed towards finding the Drazin inverse of M_1.

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In order to get an explicit formula for M_1^d, we partition M_1 as a 2×2 block-matrix, i.e.,

$$M_1 = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

where

$$A_3 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D \end{bmatrix},\ B_3 = \begin{bmatrix} 0 \\ C_2 \end{bmatrix},\ C_3 = \begin{bmatrix} 0 & B_2 \end{bmatrix}, D_3 = A_2.$$

From (2.5), we get $C_2B_2 = 0$ and $A_2B_2 = 0$, so $B_3C_3 = 0$ and $D_3C_3 = 0$. Also, by $\sigma(A_3)\cup\sigma(A_1) = \sigma(A_1)\cup\sigma(D)$, it follows that A_3 is GD-invertible. Applying Theorem 2.3 we get that

$$\begin{aligned} M_1^d &= \begin{bmatrix} A_3^d & \sum_{i=0}^{\infty}(A_3^d)^{i+2}B_3D_3^i \\ C_3(A_3^d)^2 & \sum_{i=0}^{\infty}C_3(A_3^d)^{i+3}B_3D_3^i \end{bmatrix} \\ &= \begin{bmatrix} I \\ C_3A_3^d \end{bmatrix} A_3^d \begin{bmatrix} I & \sum_{i=0}^{\infty}(A_3^d)^{i+1}B_3D_3^i \end{bmatrix}. \end{aligned}$$

For the operator matrix A_3 we have that its upper left block, the operator A_1 is nonsingular and its Schur complement

$$S(A_3) = D - C_1A_1^{-1}B_1 = D - CA^dB$$

is nonsingular, which implies that the operator A_3 is nonsingular and

$$A_3^{-1} = \begin{bmatrix} A_1^{-1} + A_1^{-1}B_1S^{-1}C_1A_1^{-1} & A_1^{-1}B_1S^{-1} \\ S^{-1}C_1A_1^{-1} & S^{-1} \end{bmatrix}.$$

Now,

$$\begin{aligned} M^d &= I_1M_1^dI_2 \\ &= \left(I_3 + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} C_3A_3^d\right) A_3^d \left(I_4 + \sum_{i=0}^{\infty}(A_3^d)^{i+1}B_3D_3^i \begin{bmatrix} 0 & I & 0 \end{bmatrix}\right) \end{aligned}$$

where

$$I_3 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ Y \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix},$$

$$I_4 = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ Y \end{bmatrix}.$$

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It is obvious that $I_4 I_3 = I_{\mathcal{N}(P)\oplus Y}$. Let us denote by $R = I_3 A_3^d I_4$,

$$I_5 = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} : \mathcal{R}(P) \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix},$$

$$I_6 = \begin{bmatrix} 0 & I & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix} \to \mathcal{R}(P).$$

Obviously, R is given by (2.7). Now,

$$M^d = \left(I_{\mathcal{Z}} + I_5 C_3 A_3^d I_4\right) R \left(I_{\mathcal{Z}} + I_3 \sum_{i=0}^{\infty} (A_3^d)^{i+1} B_3 D_3^i I_6\right).$$

By computation, we get that

$$\begin{aligned} I_5 C_3 A_3^d I_4 &= \begin{bmatrix} 0 & (I - AA^d)B \\ 0 & 0 \end{bmatrix} R, \\ I_3 (A_3^d)^{i+1} B_3 D_3^i I_6 &= I_3 (A_3^d)^i I_4 (I_3 A_3^d B_3 I_6)(I_5 D_3^i I_6) \\ &= R^i R \begin{bmatrix} 0 & 0 \\ C(I - AA^d) & 0 \end{bmatrix} \begin{bmatrix} (I - AA^d)A^i & 0 \\ 0 & 0 \end{bmatrix} \\ &= R^{i+1} \begin{bmatrix} 0 & 0 \\ C(I - AA^d)A^i & 0 \end{bmatrix}, \end{aligned}$$

so, (2.6) is valid. □

Remark 2. Theorem 2.3 generalizes [22, Theorem 3.1] to the bounded linear operator.

Taking conjugate operator of M in Theorem 2.4, we derived the following corollary:

COROLLARY 2.5. *If A and D are GD-invertible operators such that*

$$C(I - AA^d)B = 0, \quad C(I - AA^d)A = 0$$

and $S = D - CA^d B$ is nonsingular, then M is GD-invertible and

$$M^d = \left(I + \begin{bmatrix} 0 & \sum_{i=0}^{\infty} A^i (I - AA^d)B \\ 0 & 0 \end{bmatrix} R^{i+1}\right) R \left(I + R \begin{bmatrix} 0 & 0 \\ C(I - AA^d) & 0 \end{bmatrix}\right),$$

where R is defined by (2.7).

If an additional condition $C(I - AA^d)A = 0$ is satisfied in Theorem 2.4, we get a simpler formula for M^d:

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Corollary 2.6. *If A and D are GD-invertible operators such that*

$$C(I - AA^d)B = 0, \quad A(I - AA^d)B = 0, \quad C(I - AA^d)A = 0$$

and $S = D - CA^dB$ is nonsingular, then M is GD-invertible and

$$M^d = \left(I + \begin{bmatrix} 0 & (I - AA^d)B \\ 0 & 0 \end{bmatrix} R \right) R \left(I + R \begin{bmatrix} 0 & 0 \\ C(I - AA^d) & 0 \end{bmatrix} \right),$$

where R is defined by (2.7)*.*

In the paper of Miao [28] a representation of the Drazin inverse of block-matrices M is given under the conditions:

$$C(I - AA^D) = 0, \quad (I - AA^D)B = 0 \text{ and } S = D - CA^DB = 0.$$

Hartwig et al. [22] generalized this result in Theorem 4.1 and gave a representation of the Drazin inverse of block-matrix M under the conditions:

$$C(I - AA^D)B = 0, \quad A(I - AA^D)B = 0 \text{ and } S = D - CA^DB = 0.$$

In the following theorem we generalized Theorem 4.1 from [22] to the linear bounded operator.

Theorem 2.7. *If A and D are GD-invertible operators such that*

$$C(I - AA^d)B = 0, \quad A(I - AA^d)B = 0, \ S = D - CA^dB = 0$$

and the operator AW is GD-invertible, then M is GD-invertible and

$$M^d = \left(I + \begin{bmatrix} 0 & (I - AA^d)B \\ 0 & 0 \end{bmatrix} R_1 \right) R_1 \left(I + \sum_{i=0}^{\infty} R_1^{i+1} \begin{bmatrix} 0 & 0 \\ C(I - AA^d)A^i & 0 \end{bmatrix} \right), \tag{2.8}$$

where

$$R_1 = \begin{bmatrix} I \\ CA^d \end{bmatrix} A^{d,w} \begin{bmatrix} I & A^dB \end{bmatrix}, \tag{2.9}$$

and $A^{d,w} = [(AW)^d]^2 A$ is the weighted Drazin inverse [11] of A with weight operator $W = AA^d + A^dBCA^d$.

Proof. Using the notations and method from the proof of Theorem 2.4, we have that

$$\begin{aligned} M_1^d &= \begin{bmatrix} A_3^d & \sum\limits_{i=0}^{\infty} (A_3^d)^{i+2} B_3 D_3^i \\ C_3 (A_3^d)^2 & \sum\limits_{i=0}^{\infty} C_3 (A_3^d)^{i+3} B_3 D_3^i \end{bmatrix} \\ &= \begin{bmatrix} I \\ C_3 A_3^d \end{bmatrix} A_3^d \begin{bmatrix} I & \sum\limits_{i=0}^{\infty} (A_3^d)^{i+1} B_3 D_3^i \end{bmatrix}. \end{aligned}$$

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Now, prove that the generalized Drazin inverse of A_3 is given by

$$F = \begin{bmatrix} I \\ C_1A_1^{-1} \end{bmatrix} ((A_1H)^2)^d A_1 \begin{bmatrix} I & A_1^{-1}B_1 \end{bmatrix},$$

where $H = I + A_1^{-1}B_1C_1A_1^{-1}$. Remark that from the fact that AW is GD-invertible, it follows that A_1H is GD-invertible. By computation we check that

$$A_3F = FA_3 \quad \text{and} \quad FA_3F = F.$$

To prove that the operator $A_3(I - FA_3)$ is a quasinilpotent, we will use the fact that for bounded operators A and B on Banach spaces, $r(AB) = r(BA)$. First note that

$$A_3 = \begin{bmatrix} I \\ C_1A_1^{-1} \end{bmatrix} A_1 \begin{bmatrix} I & A_1^{-1}B_1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} I & A_1^{-1}B_1 \end{bmatrix} \begin{bmatrix} I \\ C_1A_1^{-1} \end{bmatrix}.$$

Since

$$A_3(I - FA_3) = \begin{bmatrix} I \\ C_1A_1^{-1} \end{bmatrix} \Big(I - (A_1H)(A_1H)^d\Big) A_1 \begin{bmatrix} I & A_1^{-1}B_1 \end{bmatrix},$$

it follows that

$$r\Big(A_3(I - FA_3)\Big) = r\Big(\Big(I - (A_1H)(A_1H)^d\Big)A_1H\Big) = 0,$$

so $A_3(I - FA_3)$ is a quasinilpotent. Hence, $A_3^d = F$.

Now, for $R_1 = I_3A_3^dI_4$, we get that (2.8) holds. By computation we obtain that $R_1 = I_3A_3^dI_4 = \begin{bmatrix} I \\ CA^d \end{bmatrix} A^{d,w} \begin{bmatrix} I & A^dB \end{bmatrix}$, where $W = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} = AA^d + A^dBCA^d$. $\square$

We obtain the following corollary by taking conjugate operator:

Corollary 2.8. *If A and D are GD-invertible operators such that*

$$C(I - AA^d)B = 0, \quad C(I - AA^d)A = 0, \; S = D - CA^dB = 0$$

and the operator AW is GD-invertible, then M is GD-invertible and

$$M^d = \left(I + \sum_{i=0}^{\infty} \begin{bmatrix} 0 & A^i(I - AA^d)B \\ 0 & 0 \end{bmatrix} R_1^{i+1}\right) R_1 \left(I + R_1 \begin{bmatrix} 0 & 0 \\ C(I - AA^d) & 0 \end{bmatrix}\right),$$

where R_1 is given by (2.9) *in Theorem 2.7.*

If the condition $C(I - AA^d)A = 0$ is added to Theorem 2.7, we have a simpler formula for M^d.

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COROLLARY 2.9. *If A and D are GD-invertible operators such that*

$$C(I-AA^d)B=0, \quad C(I-AA^d)B=0, \quad A(I-AA^d)B=0, \; S=D-CA^dB=0$$

and the operator AW is GD-invertible, then M is GD-invertible and

$$M^d = \left(I + \begin{bmatrix} 0 & (I-AA^d)B \\ 0 & 0 \end{bmatrix} R_1 \right) R_1 \left(I + R_1 \begin{bmatrix} 0 & 0 \\ C(I-AA^d) & 0 \end{bmatrix} \right),$$

where R_1 is given by (2.9) *in Theorem 2.7.*

The next theorem presents new conditions under which we give a representation of M^d in terms of the block-operators of M.

THEOREM 2.10. *If A and D are GD-invertible operators and*

$$AA^dB=0 \quad \textit{and} \quad C(I-AA^d)=0, \tag{2.10}$$

then M is GD-invertible and

$$M^d = R^d \left(I + \begin{bmatrix} 0 & 0 \\ CA^d & 0 \end{bmatrix} \right) + R^\pi \sum_{i=0}^{\infty} R^i \begin{bmatrix} 0 & 0 \\ C(A^d)^{i+2} & 0 \end{bmatrix} + \begin{bmatrix} A^d & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$R = \begin{bmatrix} (I-AA^d)A & B \\ 0 & D \end{bmatrix} \quad \text{and} \quad R^d = \begin{bmatrix} 0 & \sum_{i=0}^{\infty}(I-AA^d)A^iB(D^d)^{i+2} \\ 0 & D^d \end{bmatrix}.$$

Proof. As in the proof of the Theorem 2.4, we conclude that M is GD-invertible. Using that $\mathcal{X}=\mathcal{N}(P)\oplus\mathcal{R}(P)$, for $P=I-AA^d$, we have

$$M = \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix},$$

where $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} : Y \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to Y$.

Now,

$$\begin{aligned} M_1 &= J_2MJ_1 \\ &= \begin{bmatrix} A_2 & B_2 & 0 \\ C_2 & D & C_1 \\ 0 & B_1 & A_1 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(P) \\ Y \\ \mathcal{N}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(P) \\ Y \\ \mathcal{N}(P) \end{bmatrix}, \end{aligned}$$

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where $J_2 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(P) \\ Y \\ \mathcal{N}(P) \end{bmatrix}$ and $J_1 = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} :$ $\begin{bmatrix} \mathcal{R}(P) \\ Y \\ \mathcal{N}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \\ Y \end{bmatrix}$.

Using Lemma 2.2, we deduce that $M^d = J_1 M_1^d J_2$. In order to compute M^d it suffices to find the Drazin inverse of M_1. To derive an explicit formula for M_1^d, we partition M_1 as a 2×2 block-matrix, i.e.,

$$M_1 = \begin{bmatrix} A_3 & B_3 \\ C_3 & D_3 \end{bmatrix}$$

where

$$A_3 = \begin{bmatrix} A_2 & B_2 \\ C_2 & D \end{bmatrix},\ B_3 = \begin{bmatrix} 0 \\ C_1 \end{bmatrix},\ C_3 = \begin{bmatrix} 0 & B_1 \end{bmatrix},\ D_3 = A_1.$$

Since

$$B_3C_3 = 0 \Leftrightarrow C_1B_1 = 0 \Leftrightarrow CAA^dB = 0$$

and

$$D_3C_3 = 0 \Leftrightarrow A_1B_1 = 0 \Leftrightarrow AA^dB = 0.$$

by (2.10) we have $B_3C_3 = 0$, $D_3C_3 = 0$ and $B_1 = 0$.

Similarly as in the proof of the Theorem 2.4, we conclude that A_3 is GD-invertible operator. Now, by Theorem 2.3,

$$\begin{aligned} M_1^d &= \begin{bmatrix} A_3^d & \sum\limits_{i=0}^{\infty} A_3^{\pi} A_3^i B_3 (A_1^{-1})^{i+2} - A_3^d B_3 A_1^{-1} \\ 0 & (A_1)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I \\ 0 \end{bmatrix} A_3^d \begin{bmatrix} I & -B_3A_1^{-1} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} A_3^{\pi} \begin{bmatrix} 0 & \sum\limits_{i=0}^{\infty} A_3^i B_3 (A_1^{-1})^{i+2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A_1^{-1} \end{bmatrix}. \end{aligned}$$

By the second condition of (2.10), we obtain that $C_2 = 0$, as for the operator A_3 we have that

$$B_2C_2 = 0 \quad \text{and} \quad DC_2 = 0.$$

Applying Theorem 2.3 to A_3, we get

$$A_3^d = \begin{bmatrix} 0 & \sum\limits_{i=0}^{\infty} A_2^i B_2 (D^d)^{i+2} \\ 0 & D^d \end{bmatrix}.$$

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Now,

$$\begin{aligned} M^d &= J_1 M_1^d J_2 \\ &= J_3 A_3^d \left(J_4 + B_3 A_1^{-1} J_5\right) + J_3 A_3^\pi \left(\sum_{i=0}^{\infty} A_3^i B_3 (A_1^{-1})^{i+2} J_5\right) + \begin{bmatrix} A^d & 0 \\ 0 & 0 \end{bmatrix} \\ &= R^d \left(I + J_3 B_3 A_1^{-1} J_5\right) + R^\pi J_3 \sum_{i=0}^{\infty} A_3^i B_3 (A_1^{-1})^{i+2} J_5 + \begin{bmatrix} A^d & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where $R = J_3 A_3 J_4,\ J_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$, $J_4 = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ and $J_5 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

It is evident that $J_4 J_3 = I$. By computation, we get that

$$\begin{aligned} J_3 B_3 A_1^{-1} J_5 &= \begin{bmatrix} 0 & 0 \\ CA^d & 0 \end{bmatrix}, \\ J_3 A_3^i B_3 (A_1^{-1})^{i+2} J_5 &= R^i \begin{bmatrix} 0 & 0 \\ C(A^d)^{i+2} & 0 \end{bmatrix}. \end{aligned}$$

Also, from the definition of R, we have that

$$R = \begin{bmatrix} (I - AA^d)A & B \\ 0 & D \end{bmatrix}$$

and by [14, Theorem 5.1]

$$R^d = \begin{bmatrix} 0 & \sum_{i=0}^{\infty} (I - AA^d) A^i B (D^d)^{i+2} \\ 0 & D^d \end{bmatrix}. \qquad \square$$

3. Concluding remarks. The whole paper would appear to be valid in general Banach algebras, not just algebras of operators. Whenever $P = P^2 \in G$, for a Banach algebra G, there is an induced block structure

$$G = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$$

in which A and B are Banach algebras and M and N are bimodules over A and B.

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