ELA

http://math.technion.ac.il/iic/ela

EXPLICIT POLAR DECOMPOSITIONS OF COMPLEX MATRICES[*]

ROGER A. HORN[†], GIUSEPPE PIAZZA[‡], AND TIZIANO POLITI[§]

Abstract. In [F. Uhlig, Explicit polar decomposition and a near-characteristic polynomial: The 2×2 case, *Linear Algebra Appl.*, 38:239–249, 1981], the author gives a representation for the factors of the polar decomposition of a nonsingular real square matrix of order 2. Uhlig's formulae are generalized to encompass all nonzero complex matrices of order 2 as well as all order n complex matrices with rank at least $n - 1$.

Key words. Polar decomposition.

AMS subject classifications. 15A23.

1. Introduction and notation. A polar decomposition of $A \in \mathbb{C}^{n \times n}$ is a factorization

$$(1.1) \qquad\qquad\qquad A = UH$$

in which $U \in \mathbb{C}^{n \times n}$ is unitary and $H \in \mathbb{C}^{n \times n}$ is positive semidefinite (and therefore also Hermitian). The factor H is always uniquely determined as the unique positive semidefinite square root of A^*A: $H = (A^*A)^{1/2}$. If A is nonsingular, then the unitary factor is also unique: $U = AH^{-1}$. The polar factors of A can be found by exploiting its singular value decomposition, even if A is singular.

In the literature [1, 2], it has been considered whether it is possible to compute the polar factors of a complex matrix using only arithmetic operations and extraction of radicals of integer degree. In [1], the authors note that this can be done if the largest and smallest singular values are known, e.g., the class of singular real symmetric stochastic matrices and the class of companion matrices. In [4, 6], explicit formulae are given for polar factors of companion and block companion matrices. In this note, we generalize the explicit formulae in [5] for the polar factors of nonsingular real square matrices of order 2, and obtain explicit formulae for polar factors of $n-\text{by}-n$ complex matrices with rank at least $n - 1$.

[†]Mathematics Department, University of Utah, 155 South 1400 East, Salt Lake City, Utah 84112-0090, USA (rhorn@math.utah.edu).
[‡]Dipartimento di Matematica, Politecnico di Bari, Via Orabona 4, I–70125 Bari, Italy. This paper is dedicated to the memory of Professor Giovanni Di Lena.
[§]Dipartimento di Matematica, Politecnico di Bari, Via Orabona 4, I–70125 Bari, Italy (politi@poliba.it).

ELA

R.A. Horn, G. Piazza, and T. Politi

A singular value decomposition of $A \in \mathbb{C}^{n \times n}$ is a factorization

$$A = V\Sigma W^*,$$

in which $V, W \in \mathbb{C}^{n \times n}$ are unitary and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ is a real diagonal matrix whose diagonal entries are $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, the non-increasingly ordered singular values of A. The matrices $H = W\Sigma W^*$ and $U = VW^*$ are corresponding polar factors of A: H is positive semidefinite and $H = (A^*A)^{1/2}$ (always unique), U is unitary (unique only if A is nonsingular), and $A = UH$.

We let $\mathrm{adj}A$ denote the adjugate of A, that is, the transposed matrix of cofactors of A (see [3], p. 20). If A is nonsingular, then

$$\mathrm{adj}A = \det A \cdot A^{-1}.$$

In general,

$$\mathrm{adj}A^* = (\mathrm{adj}A)^*,$$

$$\mathrm{adj}AB = \mathrm{adj}B \, \mathrm{adj}A,$$

and

$$\mathrm{adj}\Sigma = \mathrm{diag}\left(\prod_{j \neq 1} \sigma_j, \prod_{j \neq 2} \sigma_j, \ldots, \prod_{j \neq n} \sigma_j\right).$$

Moreover,

$$\mathrm{adj}A^* = \mathrm{adj}(W\Sigma V^*) = (\mathrm{adj}V^*)(\mathrm{adj}\Sigma)(\mathrm{adj}W) = \det(V^*W)V(\mathrm{adj}\Sigma)W^*.$$

If A is nonsingular, then $e^{i\theta} := (\det A)/|\det A|$ is a well-defined complex number with unit modulus and

$$\mathrm{adj}A^* = e^{-i\theta}V(\mathrm{adj}\Sigma)W^*.$$

2. Orders 2 **and** 3. In this section, we exhibit an explicit formula for the polar factors of a nonzero order 2 complex matrix. Our formula reduces to the representation in [5] when A is real and nonsingular. For order 3 complex matrices with rank at least two, we give an explicit formula for a unitary factor in the polar decomposition, given the positive semidefinite factor. This formula suggests a representation for a unitary factor in the polar decomposition for order n complex matrices with rank at least $n-1$, which we discuss in the next section.

ELA

THEOREM 2.1. *Let $A \in \mathbb{C}^{2 \times 2}$ be nonzero and let θ be any real number such that* $\det A = e^{i\theta} |\det A|$. *Then $A = UH$, in which*

$$U = \left| \det(A + e^{i\theta} \mathrm{adj} A^*) \right|^{-1/2} (A + e^{i\theta} \mathrm{adj} A^*)$$

is unitary and $H = U^* A$ is positive semidefinite. If A is real, then H is real and U may be chosen to be real.

Proof. Suppose that A is nonsingular, and compute

$$
\begin{aligned}
Z_\theta \quad &:= A + e^{i\theta} \mathrm{adj} A^* \\[2mm]
&= V \Sigma W^* + e^{i\theta} e^{-i\theta} V (\mathrm{adj} \Sigma) W^* \\[2mm]
&= V(\Sigma + \mathrm{adj} \Sigma) W^* \\[2mm]
&= V \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} + \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \end{bmatrix} \right) W^* \\[2mm]
&= V((\sigma_1 + \sigma_2)I)W^* = (\sigma_1 + \sigma_2)VW^*,
\end{aligned}
$$

which is a positive scalar multiple of the unitary factor of A corresponding to the positive definite factor $H = (A^* A)^{1/2}$. Moreover,

$$\left| \det(A + e^{i\theta} \mathrm{adj} A^*) \right| = (\sigma_1 + \sigma_2)^2.$$

Now suppose that A is singular and nonzero, so $\mathrm{rank} A = 1$ and $\Sigma = \mathrm{diag}(\sigma_1, 0)$ with $\sigma_1 > 0$. For any $\theta \in [0, 2\pi)$ we must show that Z_θ is a positive scalar multiple of a unitary matrix and that $Z_\theta^* A$ is positive semidefinite. Let $\gamma = e^{i\theta} \det(V^* W)$ and compute

$$
\begin{aligned}
Z_\theta \quad &= V(\Sigma + \mathrm{adj} \Sigma) W^* \\[2mm]
&= V \left(\begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 0 & \sigma_1 \end{bmatrix} \right) W^* \\[2mm]
&= \sigma_1 V \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} W^*,
\end{aligned}
$$

which is a positive scalar multiple of a unitary matrix (a product of three unitary matrices). Moreover,

$$Z_\theta^* A \quad = \sigma_1 W \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma} \end{bmatrix} V^* \cdot V \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} W^* = \sigma_1 W \Sigma W^* = \sigma_1 H,$$

ELA

http://math.technion.ac.il/iic/ela

696   R.A. Horn, G. Piazza, and T. Politi

which is positive semidefinite.

Finally, suppose that A is real. Then so is $\mathrm{adj}A^*$. If A is nonsingular, then $\theta \in \{0, \pi\}$; if A is singular, take $\theta = 0$. □

If A is nonsingular, then $e^{i\theta}$ and U in the preceding theorem are uniquely determined. If A is singular, then $e^{i\theta}$ may be any complex number with modulus one, and U is not uniquely determined.

THEOREM 2.2. *Let $A \in \mathbb{C}^{3\times3}$ and suppose that $\mathrm{rank}A \geq 2$. Let H be the positive semidefinite square root of A^*A, and let θ be any real number such that $\det A = e^{i\theta}|\det A|$. Then $A = UH$, in which*

$$(2.1) \qquad U = \frac{1}{\mathrm{tr}(\mathrm{adj}H)}((\mathrm{tr}H)A - AH + e^{i\theta}\mathrm{adj}A^*)$$

is unitary. If A is real, then H is real and U may be chosen to be real.

Proof. Suppose that A is nonsingular, and compute

$$
\begin{aligned}
Z_\theta \quad &:= (\mathrm{tr}H)A - AH + e^{i\theta}\mathrm{adj}A^* \\
&= (\mathrm{tr}\Sigma)V\Sigma W^* - (V\Sigma W^*)(W\Sigma W^*) + e^{i\theta}e^{-i\theta}V(\mathrm{adj}\Sigma)W^* \\
&= V((\mathrm{tr}\Sigma)\Sigma - \Sigma^2 + \mathrm{adj}\Sigma)W^* \\
&= (\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)VW^* = \mathrm{tr}(\mathrm{adj}\Sigma)VW^*,
\end{aligned}
$$

which is a positive scalar multiple of the unitary polar factor of A corresponding to the positive definite factor H.

Now suppose that $\mathrm{rank}A = 2$, so $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, 0)$ and $\sigma_1 \geq \sigma_2 > 0$. Let

$\gamma = e^{i\theta}\det(V^*W)$ and compute

$$Z_\theta = V\left((\mathrm{tr}\Sigma)\Sigma - \Sigma^2 + \gamma\mathrm{adj}\Sigma\right)W^*$$

$$= V\left(\begin{bmatrix} \sigma_1(\sigma_1+\sigma_2) & 0 & 0 \\ 0 & \sigma_2(\sigma_1+\sigma_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right.$$

$$\left. + \gamma\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix}\right)W^*$$

$$= \sigma_1\sigma_2 V\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{bmatrix}W^*,$$

which is a positive scalar multiple of a unitary matrix. One checks that $Z_\theta^* A = \sigma_1\sigma_2 W\Sigma W^* = \sigma_1\sigma_2 H$. $\square$

3. Order n for $n \geq 3$. Let $p_H(z)$ be the characteristic polynomial of the n-by-n positive semidefinite matrix $H = (A^*A)^{1/2}$:

(3.1)
$$p_H(z) = z^n + \sum_{k=1}^{n} p_k z^{n-k}.$$

The coefficients p_k are elementary symmetric functions of the eigenvalues of H, which are the singular values of A:

$$p_k = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} \sigma_{i_j}, \qquad k = 1, \ldots, n.$$

Alternatively, $p_k = (-1)^k E_k(H)$, in which $E_k(H)$ is the sum of the $\binom{n}{k}$ principal minors of H (see [3], pp. 41-42). In particular, $p_n = p_H(0) = (-1)^n E_n(H) = (-1)^n \det H$, $p_{n-1} = p_H'(0) = (-1)^{n-1}E_{n-1}(H) = (-1)^{n-1}\mathrm{tr}(\mathrm{adj}H)$, and $p_{n-2} = \frac{1}{2}p_H''(0) = (-1)^{n-2}E_{n-2}(H)$.

THEOREM 3.1. *Let $n \geq 3$, let $A \in \mathbb{C}^{n\times n}$, and suppose that $\mathrm{rank}A \geq n-1$. Let H be the positive semidefinite square root of A^*A, let θ be any real number such that $\det A = e^{i\theta}|\det A|$, let $p_H(z)$ be the characteristic polynomial (3.1) of H, and let*

$$q_H(z) = (p_H(z) - p_{n-2}z^2 - p_{n-1}z - p_n)/z^2.$$

Then $A = UH$, in which

(3.2) $$U = \frac{1}{\mathrm{tr}(\mathrm{adj}H)}\left((-1)^n p_{n-2}A + (-1)^n Aq_H(H) + e^{i\theta}\mathrm{adj}A^*\right)$$

ELA

http://math.technion.ac.il/iic/ela

R.A. Horn, G. Piazza, and T. Politi

is unitary. If A is real, then H is real and U may be chosen to be real.

Proof. Notice that $q_H(z)$ is a monic polynomial of degree $n - 2$ and $q_H(0) = 0$. Suppose that A is nonsingular, and compute

$$
\begin{aligned}
Z_\theta \quad &:= (-1)^n p_{n-2} A + (-1)^n A q_H(H) + e^{i\theta} \mathrm{adj} A^* \\[2mm]
&= V \left((-1)^n p_{n-2} \Sigma + (-1)^n \Sigma q_H(\Sigma) + \mathrm{adj}\Sigma \right) W^*.
\end{aligned}
$$

It suffices to show that

$$
(-1)^n p_{n-2} \Sigma + (-1)^n \Sigma q_H(\Sigma) + \mathrm{adj}\Sigma = \mathrm{tr}(\mathrm{adj}\Sigma) I.
$$

That is, we must show that

$$
\sigma_k q_H(\sigma_k) = -p_{n-2} \sigma_k - p_{n-1} - (-1)^n \prod_{i \neq k} \sigma_i, \qquad k = 1, 2, \ldots, n.
$$

Since each $\sigma_k > 0$ and $p_H(\sigma_k) = 0$, we have

$$
\begin{aligned}
\sigma_k q_H(\sigma_k) \quad &= (p_H(\sigma_k) - p_{n-2}\sigma_k^2 - p_{n-1}\sigma_k - p_n)/\sigma_k \\[2mm]
&= -p_{n-2}\sigma_k - p_{n-1} - p_n/\sigma_k \\[2mm]
&= -p_{n-2}\sigma_k - p_{n-1} - (-1)^n \prod_{i \neq k} \sigma_i.
\end{aligned}
$$

Now suppose that $\mathrm{rank} A = n - 1$, so $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n-1}, 0)$ and $\sigma_1 \geq \cdots \geq \sigma_{n-1} > 0$. Let $\gamma = e^{i\theta} \det(V^* W)$ and compute

$$
Z_\theta = V \left((-1)^n p_{n-2} \Sigma + (-1)^n \Sigma q_H(\Sigma) + \gamma \mathrm{adj}\Sigma \right) W^*.
$$

For $k = 1, \ldots, n - 1$, the kth main diagonal entry of $V^* Z_\theta W$ is

$$
(-1)^n p_{n-2} \sigma_k + (-1)^n (-p_{n-2}\sigma_k - p_{n-1}) = (-1)^{n-1} p_{n-1} = \mathrm{tr}(\mathrm{adj} H)
$$

because $\sigma_n = 0$. The nth main diagonal entry of $V^* Z_\theta W$ is $\gamma \sigma_1 \cdots \sigma_{n-1} = \gamma \mathrm{tr}(\mathrm{adj} H)$. Thus, $Z_\theta = \mathrm{tr}(\mathrm{adj} H) V \mathrm{diag}(1, \ldots, 1, \gamma) W^*$, which is a positive scalar multiple of a unitary matrix. One checks that $Z_\theta^* A = \mathrm{tr}(\mathrm{adj} H) W^* \Sigma W = \mathrm{tr}(\mathrm{adj} H) H$. $\square$

For $n = 3$, (3.2) becomes

(3.3) $$ U = \frac{1}{\mathrm{tr}(\mathrm{adjH})} \left((-(-\mathrm{tr} H)) A - A H + e^{i\theta} \mathrm{adj} A^* \right), $$

which is (2.1).

ELA

http://math.technion.ac.il/iic/ela

REFERENCES

[1] A. George and Kh. Ikramov. Is the polar decomposition finitely computable? *SIAM J. Matrix Anal. Appl.*, 17:348–354, 1996.

[2] A. George and Kh. Ikramov. Addendum: Is the polar decomposition finitely computable? *SIAM J. Matrix Anal. Appl.*, 18:264, 1997.

[3] R. Horn and C. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

[4] G. Kalogeropoulos and P. Psarrakos. The polar decomposition of block companion matrices. *Comput. Math. Appl.*, 50:529–537, 2005.

[5] F. Uhlig. Explicit polar decomposition and a near-characteristic polynomial: The 2×2 case. *Linear Algebra Appl.*, 38:239–249, 1981.

[6] P. van den Driessche and H.K. Wimmer. Explicit polar decomposition of companion matrices. *Electron. J. Linear Algebra*, 1:64–69, 1996.