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ON λ_1-EXTREMAL NON-REGULAR GRAPHS*

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Abstract. Let G be a connected non-regular graph with n vertices, maximum degree Δ and minimum degree δ, and let λ_1 be the greatest eigenvalue of the adjacency matrix of G. In this paper, by studying the Perron vector of G, it is shown that type-I-a graphs and type-I-b (resp. type-II-a) graphs with some specified properties are not λ_1-extremal graphs. Moreover, for each connected non-regular graph some lower bounds on the difference between Δ and λ_1 are obtained.

Key words. Spectral radius, Non-regular graph, λ_1-extremal graph, Perron vector, Degree.

AMS subject classifications. 05C50, 15A48.

1. Introduction.

Throughout this paper, let $G = (V, E)$ be a connected, simple and undirected graph with vertex set V and edge set E, where $|V| = n$. Let uv denote the edge joining vertices u and v. For a vertex u, let $N(u)$ be the set of all neighbors of u, and let $d(u) = |N(u)|$ be the degree of u. The maximum and minimum degree of G are denoted by Δ and δ respectively. The sequence $\pi = (d_1,\ d_2,\ \dots\ ,\ d_n)$ (always with $d_1 \geq d_2 \geq \cdots \geq d_n$) is called the degree sequence of G, where $d_1,\ d_2,\ \dots\ ,\ d_n$ are the degrees of the vertices of G. Let $G + uv$ be the graph obtained from G by inserting an edge $uv \notin E(G)$, where $u, v \in V(G)$. Similarly, $G - uv$ is the graph obtained from G by deleting the edge $uv \in E(G)$, and $G - v$ is the graph obtained from G by deleting the vertex v and all edges $uv \in E(G)$ for some $u \in V(G)$.

Let $A(G)$ be the adjacency matrix of G. The spectral radius of G, denoted by $\lambda_1(G)$, is the largest eigenvalue of $A(G)$. Thus, by the Perron-Frobenius Theorem (see [8]), when G is connected, $\lambda_1(G)$ is simple and there is a corresponding unique positive unit eigenvector. We refer to such eigenvector f as the *Perron vector* of G.

Given a degree sequence π, let C_π denote the set of connected graphs with degree sequence π. We say that the graph G has the *greatest maximum eigenvalue* in class C_π provided $\lambda_1(G) \geq \lambda_1(G^*)$ for every G^* in C_π.

Let G be a connected non-regular graph. In [11], G is called λ_1-*extremal* if $\lambda_1(G) > \lambda_1(G^*)$ for every other connected non-regular graph G^* with the same num-

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ber of vertices and maximum degree as G. Let $g(n,\Delta)$ denote the set of all connected non-regular graphs with n vertices and maximum degree Δ.

Let G be a λ_1-extremal graph of $g(n,\Delta)$. As we know that if $\Delta = 2$, then G is necessarily a path and $\lambda_1(G) = 2cos(\frac{\pi}{n+1})$, while G is isomorphic to $K_n - e$ and $\lambda_1(G) = \frac{n-3+\sqrt{(n+1)^2-8}}{2}$ if $\Delta = n-1$ $(n \geq 4)$ (see [11]). In the following, we can suppose that $2 < \Delta < n-1$.

Let $V_\Delta = \{u \mid d(u) = \Delta\}$ and $V_{<\Delta} = \{u \mid d(u) < \Delta\}$. It has been shown that a λ_1-extremal graph of $g(n,\Delta)$ has the following special properties.

Lemma 1.1. ([9]) *Suppose* $2 < \Delta < n-1$. *If* G *is a* λ_1*-extremal graph of* $g(n,\Delta)$*, then* G *must have one of the following properties:*

(1) $|V_{<\Delta}| \geq 2$, *and* $V_{<\Delta}$ *induces (i.e.,* $G[V_{<\Delta}]$*) a complete graph.*

(2) $|V_{<\Delta}| = 1$.

(3) $V_{<\Delta} = \{u, v\}$, $uv \notin E(G)$ *and* $d(u) = d(v) = \Delta - 1$.

Moreover, $G \in g(n,\Delta)$ *is called a type-I (resp. type-II or type-III) graph if* G *has the property* (1) *(resp.* (2) *or* (3)*).*

By studying the properties of λ_1-extremal graphs, B. Liu et al. proved that

Lemma 1.2. ([10]) *Suppose* $2 < \Delta < n-1$ *and* G *is a* λ_1*-extremal graph of* $g(n,\Delta)$*, then* G *must be a type-I or type-II graph.*

Now we divide type-I (resp. type-II) graphs into two classes as follows.

Definition 1.3. (1) Let $G \in g(n,\Delta)$ be a type-I graph. If there exist $u_1, v_1 \in V_\Delta$ and $u_2, v_2 \in V_{<\Delta}$ such that $u_1u_2, v_1v_2 \in E$ and $u_1v_2, v_1u_2 \notin E$, then G is called a type-I-a graph. Otherwise, G is a type-I-b graph.

(2) Let $G \in g(n,\Delta)$ be a type-II graph. Then G is called a type-II-a graph if $\delta < \Delta - 2$. Otherwise, G is called a type-II-b graph.

In this paper, by investigating the Perron vector of G, we show that type-I-a graphs are not λ_1-extremal graphs, and type-I-b (resp. type-II-a) graphs with some specified properties are also not λ_1-extremal graphs, which provide more evidence to confirm the following conjecture in [9].

Conjecture 1.4. ([9]) *Let* $G \in g(n,\Delta)$ *with* $2 < \Delta < n-1$. *Then* G *is a* λ_1*-extremal graph if and only if* G *is a graph with greatest maximum eigenvalue in the class* C_π, *where* $\pi = (\Delta, \Delta, \cdots, \Delta, \delta)$, *and* $\delta = \begin{cases} \Delta - 1, & \text{when } n\Delta \text{ is odd,} \\ \Delta - 2, & \text{when } n\Delta \text{ is even.} \end{cases}$

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Let G be a connected non-regular graph of order n. Stevanović first derived a lower bound of $\Delta - \lambda_1$ for G in [13]. Later this bound was improved in [3, 4, 11]. Let D (resp. $\bar{d}$) denote the diameter (resp. the average degree) of G. In [3, 11], the authors showed that

$$\Delta - \lambda_1 > \frac{1}{nD} \quad ([3]) \tag{1.1}$$

and

$$D \leq \frac{3n + \Delta - 5}{\Delta + 1} \quad ([11]). \tag{1.2}$$

Thus combining (1.1) and (1.2), we have

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 5)} \quad ([9,\ 11]). \tag{1.3}$$

Applying Lemma 1.2, the authors in [9] made further improvement on Inequality (1.2) and obtained the following inequality which improves (1.3).

$$D \leq \frac{3n + \Delta - 8}{\Delta + 1} \ \textit{and}\ \Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)} \quad ([9]). \tag{1.4}$$

Recently, L. Shi [12] established another strong inequality as follows.

$$\Delta - \lambda_1 > [(n - \delta)D + \frac{1}{\Delta - \bar{d}} - \binom{D}{2}]^{-1} \quad ([12]). \tag{1.5}$$

Remark 1.5. For most almost regular graphs of constant degree and large order (graphs where $n\Delta - 2m$ is a constant and $D = o(\sqrt{n})$), Inequality (1.1) is better than (1.5). However, for many non-regular graphs, i.e. graphs with $(\Delta - \bar{d})[\binom{D}{2} + D\delta] \geq 1$, L. Shi's inequality is better. For example, in graphs where the diameter is a constant fraction of the number of vertices (1.5) is better.

In Section 3, we obtain the following inequalities which improve (1.4).

$$D \leq \frac{3n + \Delta - 3\delta - 5}{\Delta + 1},$$

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 3\delta - 5)}$$

and

$$\Delta - \lambda_1 > [-\frac{(3n + \Delta - 3\delta - 5)^2}{2(\Delta + 1)^2} + \frac{(0.5 + n - \delta)(3n + \Delta - 3\delta - 5)}{\Delta + 1} + \frac{1}{\Delta - \bar{d}}]^{-1}.$$

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2. On λ_1-extremal graphs. As is known to all, the Rayleigh quotient of the adjacency matrix $A(G)$ on vectors f on V is the fraction

$$R_G(f) = \frac{\langle Af, f\rangle}{\langle f, f\rangle} = \frac{2\sum_{uv\in E} f(u)f(v)}{\sum_{v\in V} f(v)^2}. \quad ([8])$$

By the Rayleigh-Ritz Theorem we have the following well known property for the spectral radius of G.

PROPOSITION 2.1. ([8]) *Let S denote the set of unit vectors on V. Then*

$$\lambda_1(G) = max_{f\in S} R_G(f) = 2max_{f\in S} \sum_{uv\in E} f(u)f(v).$$

If $R_G(f) = \lambda_1(G)$ for a (positive) function $f \in S$, then f is a Perron vector.

The following technical lemma is useful in this paper.

LEMMA 2.2. (Shifting [1, 2]) *Let $G(V, E)$ be a connected graph with $uv_1 \in E$ and $uv_2 \notin E$. Let $G^* = G + uv_2 - uv_1$. Suppose f is a Perron vector of G. If $f(v_2) \geq f(v_1)$, then $\lambda_1(G^*) > \lambda_1(G)$.*

Analogously, we introduce another technique called *Splitting*.

LEMMA 2.3. (Splitting) *Let $G(V, E)$ be a connected graph with $u_1u_2 \in E$ and $u_1w_1, u_2w_2 \notin E$ (maybe $w_1 = w_2$). Let $G^* = G + u_1w_1 + u_2w_2 - u_1u_2$. Suppose f is a Perron vector of G. If $f(w_1) + f(w_2) \geq max\{f(u_1), f(u_2)\}$, then $\lambda_1(G^*) > \lambda_1(G)$.*

Proof. Without loss of generality, suppose $f(u_1) \geq f(u_2)$. Then

$$\begin{aligned} R_{G^*}(f) - R_G(f) &= \langle A(G^*)f, f\rangle - \langle A(G)f, f\rangle \\ &= 2(\sum_{xy\in E^*-E} f(x)f(y) - \sum_{uv\in E-E^*} f(u)f(v)) \\ &= 2[f(u_1)f(w_1) + f(u_2)f(w_2) - f(u_1)f(u_2)] \\ &\geq 2\{f(u_2)[f(w_1) + f(w_2)] - f(u_1)f(u_2)\} \\ &= 2f(u_2)[f(w_1) + f(w_2) - f(u_1)] \geq 0. \end{aligned}$$

Hence $\lambda_1(G^*) \geq R_{G^*}(f) \geq R_G(f) = \lambda_1(G)$ by Proposition 2.1. Assume that $\lambda_1(G^*) = \lambda_1(G)$, which implies f would also be a Perron vector of G^*. Then

$$\lambda_1(G^*)f(w_1) = \sum_{xw_1\in E} f(x) + \sum_{yw_1\in E^*-E} f(y) > \sum_{xw_1\in E} f(x) = \lambda_1(G)f(w_1).$$

This is a contradiction. Consequently, $\lambda_1(G^*) > \lambda_1(G)$. ∎

Now let's turn to the study of λ_1-extremal graphs.

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THEOREM 2.4. *Let $G = (V, E) \in g(n, \Delta)$ be a type-I-a graph with $2 < \Delta < n-1$. Then G is not a λ_1-extremal graph of $g(n, \Delta)$.*

Proof. By contradiction, suppose G is a λ_1-extremal graph.

Since $G \in g(n, \Delta)$ is a type-I-a graph, there exist $u_1, v_1 \in V_\Delta$ and $u_2, v_2 \in V_{<\Delta}$ such that $u_1u_2, v_1v_2 \in E$ and $u_1v_2, v_1u_2 \notin E$. Let f be the Perron vector of G. We consider the next two cases:

Case 1. $f(u_2) \geq f(v_2)$. Let $G^* = G - v_1v_2 + v_1u_2$. Note that $G[V_{<\Delta}]$ is a complete graph and $d(u_2) < \Delta$, thus $G^* \in g(n, \Delta)$. By Lemma 2.2, we have $\lambda_1(G^*) > \lambda_1(G)$, which is a contradiction.

Case 2. $f(u_2) \leq f(v_2)$. Let $G^* = G - u_1u_2 + u_1v_2$. Similarly, since $G[V_{<\Delta}]$ is a complete graph and $d(v_2) < \Delta$, we have $G^* \in g(n, \Delta)$. It follows from Lemma 2.2 that $\lambda_1(G^*) > \lambda_1(G)$, also a contradiction. ◻

EXAMPLE 2.5. Let G_0 be a graph with vertex set $V = \{v_1,\ v_2,\ \ldots\ ,\ v_8\}$ and edge set $E = \{v_iv_j \mid i,\ j = 1,\ 2,\ \ldots\ ,\ 5 \textit{ and } i \neq j\} \cup \{v_6v_i \mid i = 4,\ 7,\ 8\} \cup \{v_7v_i \mid i = 5,\ 8\} - \{v_4v_5\}$. Note that $G_0 \in g(8,\ 4)$ is a type-I-a graph. By Theorem 2.4, G_0 is not a λ_1-extremal graph.

PROPOSITION 2.6. *Let $G = (V, E) \in g(n, \Delta)$ be a type-I-b graph with $2 < \Delta < n-1$ and let f be a Perron vector of G. Assume that $\delta \neq \Delta - 1$ when $|V_{<\Delta}| = 2$. If there exist $u_1, u_2 \in V_\Delta$, $w_1, w_2 \in V_{<\Delta}$ such that $u_1u_2 \in E$, $u_1w_1, u_2w_2 \notin E$ and $f(w_1) + f(w_2) \geq max\{f(u_1), f(u_2)\}$, then G is not a λ_1-extremal graph of $g(n, \Delta)$.*

Proof. Let $G^* = G + u_1w_1 + u_2w_2 - u_1u_2$. Note that either $|V_{<\Delta}| \geq 3$ or $|V_{<\Delta}| = 2$ ($\delta \neq \Delta - 1$). Because $w_1, w_2 \in V_{<\Delta}$ and $G[V_{<\Delta}]$ is a complete graph, we have $G^* \in g(n, \Delta)$. Since $f(w_1) + f(w_2) \geq max\{f(u_1), f(u_2)\}$, by Lemma 2.3, $\lambda_1(G^*) > \lambda_1(G)$, which implies that G is not a λ_1-extremal graph of $g(n, \Delta)$. ◻

EXAMPLE 2.7. Let G_1 be a graph with vertex set $V = \{v_1,\ v_2,\ \ldots\ ,\ v_{11}\}$ and edge set $E = \{v_iv_j \mid i,\ j = 1,\ \ldots\ ,\ 6 \textit{ and } i \neq j\} \cup \{v_7v_i \mid i = 4,\ 5,\ 9,\ 10,\ 11\} \cup \{v_8v_i \mid i = 3,\ 6,\ 9,\ 10,\ 11\} \cup \{v_9v_{10}, v_9v_{11}, v_{10}v_{11}\} - \{v_3v_6, v_4v_5\}$. It is not difficult to see that $G_1 \in g(11,\ 5)$ is a type-I-b graph with $\lambda_1 \approx 4.82843$ and degree sequence $(5, 5, 5, 5, 5, 5, 5, 5, 4, 4, 4)$.

Directly calculating, $f(v_i) \approx 0.36725$ ($i = 1,\ 2$), $f(v_j) \approx 0.35150$ ($3 \leq j \leq 6$), $f(v_k) \approx 0.25969$ ($k = 7,\ 8$), and $f(v_l) \approx 0.18363$ ($l = 9,\ 10,\ 11$). Note that $v_5v_6 \in E$, $v_5v_9, v_6v_{10} \notin E$ and $f(v_9) + f(v_{10}) \geq max\{f(v_5), f(v_6)\}$, it follows from Proposition 2.6 that G_1 is not a λ_1-extremal graph.

PROPOSITION 2.8. *Let $G = (V, E) \in g(n, \Delta)$ be a type-II-a graph with $2 < \Delta < n-1$ and let f be a Perron vector of G. Suppose $V_{<\Delta} = \{w\}$ and $d(w) = \delta$. If there*

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exist $u_1, u_2 \in V_\Delta$ *such that* $u_1u_2 \in E$, $u_1w, u_2w \notin E$ *and* $2f(w) \geq max\{f(u_1), f(u_2)\}$, *then* G *is not a* λ_1*-extremal graph of* $g(n, \Delta)$.

Proof. Let $G^* = G + u_1w + u_2w - u_1u_2$. Since G is a type-II-a graph, we have $d(w) = \delta < \Delta - 2$, and then $G^* \in g(n, \Delta)$. Note that $2f(w) \geq max\{f(u_1), f(u_2)\}$, by Lemma 2.3, $\lambda_1(G^*) > \lambda_1(G)$. Therefore, G is not a λ_1-extremal graph. □

Example 2.9. Let G_2 be a graph with vertex set $V = \{v_1,\ v_2,\ \dots,\ v_9\}$ and edge set $E = \{v_iv_j \mid i,\ j = 1,\ \dots,\ 8 \text{ and } i \neq j\} \cup \{v_9v_i \mid i = 5,\ \dots,\ 8\} - \{v_5v_7, v_6v_8\}$.

It is easy to see that $G_2 \in g(9,\ 7)$ is a type-II-a graph with $\lambda_1 \approx 6.79944$ and degree sequence $(7,7,7,7,7,7,7,7,4)$. Directly computing, $f(v_i) \approx 0.35531$ $(1 \leq i \leq 4)$, $f(v_j) \approx 0.33749$ $(5 \leq j \leq 8)$, and $f(v_9) \approx 0.19854$. Note that $v_1v_2 \in E$, $v_1v_9, v_2v_9 \notin E$ and $2f(v_9) \geq max\{f(v_1), f(v_2)\}$, by Proposition 2.8, we conclude that G_2 is not a λ_1-extremal graph.

Remark 2.10. By Theorem 2.4, Propositions 2.6 and 2.8, type-I-a graphs, and type-I-b graphs (resp. type-II-a graphs) with some specified properties are not λ_1-extremal graphs. In other words, if G is a λ_1-extremal graph of $g(n, \Delta)$, then G is most likely to be a type-II-b graph ($|V_{<\Delta}| = 1$ and $\delta = \Delta - 2$ *or* $\Delta - 1$). Hence this provides more evidence to confirm Conjecture 1.4.

Remark 2.11. Conjecture 1.4 is true for small n, where $n \leq 7$. Since $2 < \Delta < n - 1$, it need to be verified for $n = 5,\ 6,\ 7$ as follows.

The λ_1-extremal graph of $g(5,\ 3)$ is the Graph 1.17 ([7], pp. 273) with the degree sequence $\pi = (3,3,3,3,2)$, and $\lambda_1 \approx 2.8558$.

The λ_1-extremal graph of $g(6,\ 3)$ is the Graph 65 ([5]) with the degree sequence $\pi = (3,3,3,3,3,1)$, and $\lambda_1 \approx 2.895$.

The λ_1-extremal graph of $g(6,\ 4)$ is the Graph 14 ([5]) with the degree sequence $\pi = (4,4,4,4,4,2)$, and $\lambda_1 \approx 3.820$.

The λ_1-extremal graph of $g(7,\ 3)$ is the Graph 10-261 ([6], pp. 193) with the degree sequence $\pi = (3,3,3,3,3,3,2)$, and $\lambda_1 \approx 2.9107$.

The λ_1-extremal graph of $g(7,\ 4)$ is the Graph 13-643 ([6], pp. 218) with the degree sequence $\pi = (4,4,4,4,4,4,2)$, and $\lambda_1 \approx 3.8558$.

The λ_1-extremal graph of $g(7,\ 5)$ is the Graph 17-835 ([6], pp. 231) with the degree sequence $\pi = (5,5,5,5,5,5,4)$, and $\lambda_1 \approx 4.8809$.

3. The largest eigenvalue of non-regular graphs.

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THEOREM 3.1. *Let $G \in g(n, \Delta)$ $(2 < \Delta < n-1)$ be a type-I-b graph (or type-II graph) with diameter D and minimum degree δ. Then*

$$D \leq \frac{3n + \Delta - 3\delta - 5}{\Delta + 1}.$$

Proof. Since $G \in g(n, \Delta)$ with $2 < \Delta < n - 1$ is non-regular, we have $D \geq 2$. Let u, v be vertices at distance D and let $P : u = u_0 \leftrightarrow u_1 \leftrightarrow \cdots \leftrightarrow u_D = v$ be a shortest path connecting u and v. We observe $|V_{<\Delta} \cap V(P)| \leq 2$. Otherwise G is a type-I-b graph. Assume $\{u_p, u_q, u_r\} \subseteq V_{<\Delta} \cap V(P)$ with $p < q < r$. Since $G[V_{<\Delta}]$ is a complete graph, $u_p u_r \in E(G)$, contradicting the choice of P.

Case 1. $V_{<\Delta} \cap V(P) = \{u_p, u_{p+1}\}$.

Subcase 1.1. $D \equiv 2 \ (mod\ 3)$. Define $T = \{i \mid i \equiv 0 \ (mod\ 3) \ and\ i < p\} \cup \{i \mid i \equiv D \ (mod\ 3) \ and\ p + 1 < i \leq D\}$. Then $|T| = \frac{D+1}{3}$.

Let $d(u_i, u_j)$ denote the distance between u_i and u_j. Since P is a shortest path connecting u and v, we have $d(u_i, u_j) \geq 3$ and thus $N(u_i) \cap N(u_j) = \emptyset$ for any distinct $i, j \in T$. Note that $|\{p, p + 1\} \cap \{0, D\}| \leq 1$ since $D \geq 2$. Thus

$$n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \geq (D + 1) + [(\Delta - 1) + (|T| - 1)(\Delta - 2)] \quad (3.1)$$

$$= (D + 1) + [(\Delta - 1) + (\frac{D + 1}{3} - 1)(\Delta - 2)] = \frac{D(\Delta + 1) + \Delta + 4}{3}. \quad (3.2)$$

Subcase 1.2. $D \not\equiv 2 \ (mod\ 3)$. Define $T = \{i \mid i \equiv 0 \ (mod\ 3) \ and\ 0 \leq i \leq D - 3\} \cup \{D\}$. Then $|T| = \lceil \frac{D+2}{3} \rceil$. Note that there is at most one $j \in T$ such that $\delta \leq d(u_j) < \Delta$. Similarly as Subcase 1.1, we have

$$n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \quad (3.3)$$

$$\geq (D + 1) + [(\Delta - 1) + (\delta - 1) + (|T| - 2)(\Delta - 2)] \quad (3.4)$$

$$\geq (D + 1) + [(\Delta - 1) + (\delta - 1) + (\frac{D + 2}{3} - 2)(\Delta - 2)] \quad (3.5)$$

$$= \frac{D(\Delta + 1) - \Delta + 3\delta + 5}{3}. \quad (3.6)$$

Case 2. $V_{<\Delta} \cap V(P) = \{u_p\}$.

Subcase 2.1. $D \not\equiv 0 \ (mod\ 3)$. Define $T = \{i \mid i \equiv 0 \ (mod\ 3) \ and\ i < p\} \cup \{i \mid i \equiv D \ (mod\ 3) \ and\ p < i \leq D\}$. Then $|T| = \lceil \frac{D+1}{3} \rceil$. Similarly as Subcase 1.1,

$$n \geq |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| \geq \frac{D(\Delta + 1) + \Delta + 4}{3}. \quad (3.7)$$

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Subcase 2.2. $D \equiv 0$ $(mod\ 3)$. Define $T = \{i \mid i \equiv 0\ (mod\ 3)\ and\ 0 \le i \le D\}$. Then $|T| = \frac{D+3}{3}$ and $0, D \in T$. Note that there is at most one $j \in T$ such that $\delta \le d(u_j) < \Delta$. Similarly as Subcase 1.2, we have

$$\begin{aligned} n &\ge |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| && (3.8)\\ &\ge (D+1) + [(\Delta-1) + (\delta-1) + (|T|-2)(\Delta-2)] && (3.9)\\ &= \frac{D(\Delta+1)+3\delta+3}{3}. && (3.10) \end{aligned}$$

Case 3. $V_{<\Delta} \cap V(P) = \emptyset$. Define $T = \{i \mid i \equiv 0\ (mod\ 3)\ and\ i \le D-3\} \cup \{D\}$. Then $|T| = \lceil \frac{D+1}{3} \rceil$. Analogously, we obtain that

$$\begin{aligned} n &\ge |V(P)| + \sum_{i \in T} |N(u_i) - V(P)| && (3.11)\\ &\ge (D+1) + [2(\Delta-1) + (|T|-2)(\Delta-2)] && (3.12)\\ &= (D+1) + [2(\Delta-1) + (\frac{D+1}{3} - 2)(\Delta-2)] = \frac{D(\Delta+1)+\Delta+7}{3}. && (3.13) \end{aligned}$$

By combining the above inequalities (3.1)-(3.13), Theorem 3.1 holds. ☐

THEOREM 3.2. *Let $G \in g(n, \Delta)$ with $2 < \Delta < n-1$, and minimum degree δ. Then*

$$\Delta - \lambda_1 > \frac{\Delta+1}{n(3n+\Delta-3\delta-5)}.$$

Proof. We may assume $G \in g(n, \Delta)$ with $2 < \Delta < n-1$ is a λ_1-extremal graph. Applying Theorem 3.1 on Inequality (1.1), we obtain the desired result. ☐

REMARK 3.3. Note that

$$\lambda_1 < \Delta - \frac{\Delta+1}{n(3n+\Delta-5-3\delta)} \le \Delta - \frac{\Delta+1}{n(3n+\Delta-8)}$$

since $\delta \ge 1$, the bound we obtain improves Inequality (1.4) (also see [9]).

The following corollary is a direct consequence of Theorem 3.2.

COROLLARY 3.4. *Let $G \in g(n, \Delta)$ $(2 < \Delta < n-1)$ with no pendant vertices. Then*

$$\Delta - \lambda_1 > \frac{\Delta+1}{n(3n+\Delta-11)}.$$

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THEOREM 3.5. *Let $G \in g(n, \Delta)$ with $2 < \Delta < n-1$, minimum degree δ, and average degree $\bar{d}$. Then*

$$\Delta - \lambda_1 > [-\frac{(3n+\Delta-3\delta-5)^2}{2(\Delta+1)^2} + \frac{(0.5+n-\delta)(3n+\Delta-3\delta-5)}{\Delta+1} + \frac{1}{\Delta-\bar{d}}]^{-1}.$$

Proof. Let

$$f(x) = (n-\delta)x + \frac{1}{\Delta-\bar{d}} - \binom{x}{2}.$$

It is easy to see that

$$\begin{aligned} f(x) &= -\frac{x^2}{2} + (\frac{1}{2}+n-\delta)x + \frac{1}{\Delta-\bar{d}} \\ &= -\frac{1}{2}[x-(\frac{1}{2}+n-\delta)]^2 + \frac{1}{2}\cdot(\frac{1}{2}+n-\delta)^2 + \frac{1}{\Delta-\bar{d}}. \end{aligned}$$

Then for $x \leq \frac{1}{2}+n-\delta$, the function $f(x)$ is monotonically increasing in x.

On the other hand, we have

$$\frac{1}{2}+n-\delta-\frac{3n+\Delta-3\delta-5}{\Delta+1} = \frac{(n-\delta-0.5)(\Delta-2)+4.5}{\Delta+1} \geq 0.$$

Combining this with Theorem 3.1, it follows that

$$D \leq \frac{3n+\Delta-3\delta-5}{\Delta+1} \leq \frac{1}{2}+n-\delta.$$

Hence

$$f(D) \leq -\frac{(3n+\Delta-3\delta-5)^2}{2(\Delta+1)^2} + \frac{(0.5+n-\delta)(3n+\Delta-3\delta-5)}{\Delta+1} + \frac{1}{\Delta-\bar{d}}.$$

By Inequality (1.5), $\Delta - \lambda_1 > f(D)^{-1}$, and this completes the proof. ∎

REMARK 3.6. For the non-regular graphs with

$$(\Delta-\bar{d})[\binom{\frac{3n+\Delta-3\delta-5}{\Delta+1}}{2} + \frac{\delta(3n+\Delta-3\delta-5)}{\Delta+1}] \geq 1,$$

the bound in Theorem 3.5 is better than that in Theorem 3.2. On the other hand, for most almost regular graphs of constant degree and large order, the bound in Theorem 3.2 is better. We conclude that these two bounds are incomparable.

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