

ALGEBRAIC REFLEXIVITY FOR SEMIGROUPS OF OPERATORS*

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Abstract. Algebraic reflexivity of sets and semigroups of linear transformations are studied in this paper. Some new examples of algebraically reflexive sets and semigroups of linear transformations are given. Using known results on algebraically orbit reflexive linear transformations, those linear transformations on a complex Banach space that are determined by their invariant subsets are characterized.

Key words. Semigroup of linear transformations, Algebraic reflexivity.

AMS subject classifications. 47D05, 15A04.

1. Introduction. Let $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V})$ be a set of linear transformations on a vector space \mathcal{V} . Here $\mathcal{L}(\mathcal{V})$ means the algebra of all linear transformations on \mathcal{V} . If $S \in \mathcal{L}(\mathcal{V})$ is such that $Sx = T_x x$ holds for every $x \in \mathcal{V}$ and a linear transformation $T_x \in \mathcal{T}$, which depends on x , then S is said to be locally in \mathcal{T} . Let $\text{Ref } \mathcal{T}$ denote the set of all linear transformations that are locally in \mathcal{T} , i.e. $\text{Ref } \mathcal{T} = \{S \in \mathcal{L}(\mathcal{V}); Sx \in \mathcal{T}x \text{ for all } x \in \mathcal{V}\}$. It is obvious that $\mathcal{T} \subseteq \text{Ref } \mathcal{T}$, so one can pose the following very natural question: For which sets $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V})$ one has the equality $\text{Ref } \mathcal{T} = \mathcal{T}$? Sets which satisfy this equality are said to be algebraically reflexive.

Since the concept of algebraic reflexivity is so simple and natural it was studied during the last few decades by many authors [3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 17] (we have given here just a few references and we did not mention papers that are concerned with algebraic reflexivity of linear spaces of operators). It seems that the basic idea of algebraic reflexivity for sets of linear transformations was given by [11, remark (5) in §6]. However, the idea of reflexive spaces of operators on a Banach space is more than ten years older (see Shulman's definition in §5 of [18]) and itself relies on the concept of a reflexive algebra of operators introduced by Halmos in 1960's.

Recall that a closed algebra \mathcal{A} of bounded linear operators on a complex Banach space \mathcal{X} is *reflexive* if it is determined by $\text{Lat } \mathcal{A}$, the lattice of closed \mathcal{A} -invariant subspaces of \mathcal{X} . More precisely, denote by $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded linear operators on \mathcal{X} and let $\text{Alg Lat } \mathcal{A} \subseteq \mathcal{B}(\mathcal{X})$ be the weakly closed subalgebra

*Received by the editors March 28, 2009. Accepted for publication October 19, 2009. Handling Editor: Harm Bart.

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of all operators T that satisfy $\text{Lat } \mathcal{A} \subseteq \text{Lat } T$. Then \mathcal{A} is reflexive if and only if $\text{Alg Lat } \mathcal{A} = \mathcal{A}$. If \mathcal{A} contains the identity operator Id , then $\text{Alg Lat } \mathcal{A} = \text{Ref}_t \mathcal{A}$, where $\text{Ref}_t \mathcal{A} = \{T \in \mathcal{B}(\mathcal{X}); Tx \in [\mathcal{A}x] \text{ for every } x \in \mathcal{X}\}$ is the topological reflexive cover of \mathcal{A} . Here $[\mathcal{A}x]$ denotes the closure of the orbit $\mathcal{A}x$.

The aim of this paper is to use the ideas and concepts mentioned above in the study of algebraic reflexivity for semigroups of linear transformations. Of course, ideas from the theory of reflexive algebras have already been used in the study of algebraic reflexivity of sets, however we wish to point out some analogies more explicitly. This is done in Section 2 where we prove some general statements. In Section 3, we give a few examples. For instance, we show that every finite set of linear transformations is algebraically reflexive provided that the underlying field is infinite. We show that the group of all invertible linear transformations on a vector space \mathcal{V} is algebraically reflexive if and only if $\dim(\mathcal{V}) < \infty$. In the last example of the section we prove that every continuous one-parameter semigroup on a finite-dimensional complex Banach space is algebraically reflexive. The simplest semigroups are those with one generator. A linear transformation T is said to be algebraically orbit reflexive (see [9]) if the semigroup $\mathcal{O}(T) = \{T^n; n \geq 0\}$ is algebraically reflexive. It seems that the problem which linear transformations are algebraically orbit reflexive is not solved in general. Probably [9, Theorems 9 and 10] gives the most general result in this direction. However, on a complex Banach space every linear transformation is algebraically orbit reflexive. In Section 4 we use results from [9] and consider the question whether a linear transformation is uniquely determined by their own lattice of invariant sets.

2. Algebraic reflexivity. Let \mathcal{V} be a vector space over a field \mathbb{F} . A non-empty subset $\mathcal{M} \subseteq \mathcal{V}$ is said to be invariant for a linear transformation $T \in \mathcal{L}(\mathcal{V})$ if $T\mathcal{M} \subseteq \mathcal{M}$. More generally, \mathcal{M} is *invariant* for a non-empty set of linear transformations $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V})$ if it is invariant for every transformation in \mathcal{T} . In this case, we write briefly $\mathcal{T}\mathcal{M} \subseteq \mathcal{M}$ and say that \mathcal{M} is \mathcal{T} -invariant.

Let $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V})$ be a non-empty set. It is obvious that the intersection and union of any collection of \mathcal{T} -invariant subsets are \mathcal{T} -invariant as well. Thus, the family $\text{Lst } \mathcal{T}$ of all \mathcal{T} -invariant subsets of \mathcal{V} is a lattice with respect to these two operations.

For a non-empty family \mathfrak{M} of non-empty subsets of \mathcal{V} , let $\text{Sgr } \mathfrak{M}$ be the set of all linear transformations $T \in \mathcal{L}(\mathcal{V})$ such that $\mathfrak{M} \subseteq \text{Lst } T$. Since Id leaves invariant every subset of \mathcal{V} we have $Id \in \text{Sgr } \mathfrak{M}$. If $T_1, T_2 \in \text{Sgr } \mathfrak{M}$, then $T_1 T_2 \mathcal{M} = T_1 (T_2 \mathcal{M}) \subseteq T_1 \mathcal{M} \subseteq \mathcal{M}$. Thus, $\text{Sgr } \mathfrak{M}$ is a semigroup of linear transformations. The following relations are easy to check:

- (a) $\mathcal{T} \subseteq \text{Sgr Lst } \mathcal{T}$; (d) $\text{Sgr } \mathfrak{M}_1 \supseteq \text{Sgr } \mathfrak{M}_2$ if $\mathfrak{M}_1 \subseteq \mathfrak{M}_2$;
- (b) $\mathfrak{M} \subseteq \text{Lst Sgr } \mathfrak{M}$; (e) $\text{Lst } \mathcal{T}_1 \supseteq \text{Lst } \mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$;
- (c) $\text{Sgr Lst Sgr } \mathfrak{M} = \text{Sgr } \mathfrak{M}$; (f) $\text{Lst Sgr Lst } \mathcal{T} = \text{Lst } \mathcal{T}$.

Let \mathcal{V} and \mathcal{W} be two vector spaces and let $\mathcal{L}(\mathcal{V}, \mathcal{W})$ be the vector space of all linear transformations from \mathcal{V} to \mathcal{W} . For $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$, let $\mathcal{T}x \subseteq \mathcal{W}$ be the orbit of \mathcal{T} at $x \in \mathcal{V}$. The *algebraic reflexive cover* of \mathcal{T} is $\text{Ref } \mathcal{T} = \{A \in \mathcal{L}(\mathcal{V}, \mathcal{W}) : Ax \in \mathcal{T}x, \text{ for every } x \in \mathcal{V}\}$. It is obvious that $\mathcal{T} \subseteq \text{Ref } \mathcal{T}$. Note that the orbits of \mathcal{T} and $\text{Ref } \mathcal{T}$ coincide at every $x \in \mathcal{V}$. Actually, $\text{Ref } \mathcal{T}$ is the largest subset of $\mathcal{L}(\mathcal{V}, \mathcal{W})$ with this property. Note also that

$$\bigcap_{T \in \text{Ref } \mathcal{T}} \ker T = \bigcap_{T \in \mathcal{T}} \ker T \quad \text{and} \quad \text{Lin} \left(\bigcup_{T \in \text{Ref } \mathcal{T}} \text{im } T \right) = \text{Lin} \left(\bigcup_{T \in \mathcal{T}} \text{im } T \right),$$

where $\ker T \subseteq \mathcal{V}$ is the kernel of $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, $\text{im } T \subseteq \mathcal{W}$ is its image, and Lin denotes the linear span of a set of vectors.

PROPOSITION 2.1. *Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ be a semigroup. Then $\text{Ref } \mathcal{S}$ is a semigroup contained in the semigroup $\text{Sgr Lst } \mathcal{S}$. The equality $\text{Ref } \mathcal{S} = \text{Sgr Lst } \mathcal{S}$ holds if and only if $\text{Id} \in \text{Ref } \mathcal{S}$; in particular, it holds if $\text{Id} \in \mathcal{S}$.*

Proof. Let $T_1, T_2 \in \text{Ref } \mathcal{S}$. Then, for $x \in \mathcal{V}$, there exist $S_1, S_2 \in \mathcal{S}$ such that $T_2x = S_2x$ and $T_1(S_2x) = S_1(S_2x)$. It follows, $T_1T_2x = T_1S_2x = S_1S_2x \in \mathcal{S}x$. Thus, $T_1T_2 \in \text{Ref } \mathcal{S}$. Let $T \in \text{Ref } \mathcal{S}$ and $\mathcal{M} \in \text{Lst } \mathcal{S}$. Then $Tx \in \mathcal{S}x \subseteq \mathcal{M}$, for every $x \in \mathcal{M}$, and therefore $\text{Ref } \mathcal{S} \subseteq \text{Sgr Lst } \mathcal{S}$. Since $\text{Sgr Lst } \mathcal{S}$ always contains the identity operator, one has $\text{Id} \in \text{Ref } \mathcal{S}$ whenever the equality $\text{Ref } \mathcal{S} = \text{Sgr Lst } \mathcal{S}$ holds. To see the opposite implication assume that $\text{Id} \in \text{Ref } \mathcal{S}$. Then $x \in \mathcal{S}x$, for every $x \in \mathcal{V}$. Thus, if $T \in \text{Sgr Lst } \mathcal{S}$, then $T(\mathcal{S}x) \subseteq \mathcal{S}x$ gives $Tx \in \mathcal{S}x$, i.e., $T \in \text{Ref } \mathcal{S}$. \square

It is easy to find a semigroup \mathcal{S} such that $\mathcal{S} = \text{Ref } \mathcal{S} \neq \text{Sgr Lst } \mathcal{S}$. Indeed, let $P \in \mathcal{L}(\mathcal{V})$, $P \neq \text{Id}$, be an idempotent. Then $\mathcal{P} = \{P\}$ is a semigroup. Since $Tx \in \mathcal{P}x$, for every $x \in \mathcal{V}$, if and only if $T = P$ one has $\text{Ref } \mathcal{P} = \mathcal{P}$. Next example shows that there exist semigroups $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ such that $\text{Id} \notin \mathcal{S}$ and $\text{Id} \in \text{Ref } \mathcal{S}$.

EXAMPLE 2.2. Let \mathcal{V}' be the dual of \mathcal{V} . For non-zero elements $e \in \mathcal{V}$ and $\xi \in \mathcal{V}'$, let $e \otimes \xi$ be the rank-one linear transformation on \mathcal{V} that is defined by $(e \otimes \xi)x = \xi(x)e$. We denote by \mathcal{F}_1 the set of all linear transformations in $\mathcal{L}(\mathcal{V})$ whose rank is less or equal to 1. Since $(e_1 \otimes \xi_1)(e_2 \otimes \xi_2) = \xi_1(e_2)e_1 \otimes \xi_2$ the set \mathcal{F}_1 is a multiplicative semigroup.

It is easily seen that $\mathcal{F}_1x = \mathcal{V}$, for any non-zero vector $x \in \mathcal{V}$. Thus, $Tx \in \mathcal{F}_1x$, for every $x \in \mathcal{V}$ and every $T \in \mathcal{L}(\mathcal{V})$. We conclude that $\text{Ref } \mathcal{F}_1 = \mathcal{L}(\mathcal{V}) = \text{Sgr Lst } \mathcal{F}_1$. However, if $\dim(\mathcal{V}) \geq 2$, then $\text{Id} \notin \mathcal{F}_1$.

Even in the pure algebraic case there is a natural topological structure on $\mathcal{L}(\mathcal{V}, \mathcal{W})$. For $T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and a finite set $\mathcal{E} \subseteq \mathcal{V}$, let $\mathcal{U}(T; \mathcal{E}) = \{S \in \mathcal{L}(\mathcal{V}, \mathcal{W}) : Sx = Tx \text{ for every } x \in \mathcal{E}\}$. Then $\{\mathcal{U}(T; \mathcal{E}) : \mathcal{E} \subseteq \mathcal{V} \text{ finite}\}$ is the family of basic neighbour-

hoods of T in the strict topology on $\mathcal{L}(\mathcal{V}, \mathcal{W})$.

PROPOSITION 2.3. *Let $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ be a non-empty set. Reflexive cover $\text{Ref } \mathcal{T}$ is closed in the strict topology.*

Proof. Assume that $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is not in $\text{Ref } \mathcal{T}$. Then there exists $x \in \mathcal{V}$ such that $Ax \notin \mathcal{T}x$. It follows, for an arbitrary $B \in \mathcal{U}(A; x)$, that $Bx \notin \mathcal{T}x$. Thus, $\mathcal{U}(A; x)$ is a subset of the complement of $\text{Ref } \mathcal{T}$, which means that this complement is open in the strict topology. \square

Proposition 2.1 shows that the property being a semigroup is transferred from \mathcal{S} to its algebraic reflexive cover. Some other properties are transferred from $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ to $\text{Ref } \mathcal{T}$ as well.

PROPOSITION 2.4. *Let $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ be a non-empty set.*

- (i) *Ref \mathcal{T} is closed under addition whenever \mathcal{T} is closed.*
- (ii) *If $\mathcal{C} \subseteq \mathcal{L}(\mathcal{W})$ is a non-empty set such that $\mathcal{CT} := \{CT : C \in \mathcal{C}, T \in \mathcal{T}\}$ is contained in \mathcal{T} , then $(\text{Ref } \mathcal{C})(\text{Ref } \mathcal{T}) \subseteq \text{Ref } \mathcal{T}$. Similarly, if $\mathcal{TC} \subseteq \mathcal{T}$, then $(\text{Ref } \mathcal{T})(\text{Ref } \mathcal{C}) \subseteq \text{Ref } \mathcal{T}$.*

Proof. (i) If $A_1, A_2 \in \text{Ref } \mathcal{T}$ and $x \in \mathcal{V}$, then there exist $T_1, T_2 \in \mathcal{T}$ such that $A_1x = T_1x$ and $A_2x = T_2x$. It follows $(A_1 + A_2)x = (T_1 + T_2)x \in \mathcal{T}x$.

(ii) Let $C \in \mathcal{C}$ and $A \in \text{Ref } \mathcal{T}$. Then, for $x \in \mathcal{V}$, there exists $T_x \in \mathcal{T}$ such that $Ax = T_x x$. It follows $CAx = CT_x x \in \mathcal{T}x$ since $\mathcal{CT} \subseteq \mathcal{T}$. This proves that $\mathcal{C}\text{Ref } \mathcal{T} \subseteq \text{Ref } \mathcal{T}$. Now, let $D \in \text{Ref } \mathcal{C}$ and $A \in \text{Ref } \mathcal{T}$ be arbitrary. Then, for $x \in \mathcal{V}$, there is $C_{Ax} \in \mathcal{C}$ such that $D(Ax) = C_{Ax}Ax$. By the first part of the proof, $C_{Ax}Ax \in \mathcal{T}x$ and therefore $DA \in \text{Ref } \mathcal{T}$.

Assume that $\mathcal{TC} \subseteq \mathcal{T}$ and let $A \in \text{Ref } \mathcal{T}$, $C \in \mathcal{C}$ be arbitrary. For $x \in \mathcal{V}$, there exists $T_{Cx} \in \mathcal{T}$ such that $A(Cx) = T_{Cx}Cx \in \mathcal{T}x$. Thus, $(\text{Ref } \mathcal{T})\mathcal{C} \subseteq \text{Ref } \mathcal{T}$. Let $A \in \text{Ref } \mathcal{T}$ and $D \in \text{Ref } \mathcal{C}$ be arbitrary. For $x \in \mathcal{V}$, there exists $C_x \in \mathcal{C}$ such that $Dx = C_x x$, which gives $ADx = AC_x x \in \mathcal{T}x$. This proves that $(\text{Ref } \mathcal{T})(\text{Ref } \mathcal{C}) \subseteq \text{Ref } \mathcal{T}$. \square

It follows immediately from Proposition 2.4 that $\text{Ref } \mathcal{T}$ is closed for multiplication with scalars if \mathcal{T} is closed. Moreover, an algebraic reflexive cover of a linear space (an algebra) is a linear space (an algebra). Note also that the converse of (i) does not hold, in general. Namely, let $\dim(\mathcal{V}) \geq 2$. Then $\mathcal{L}(\mathcal{V}) \setminus \{0\}$ is not closed under addition however $\text{Ref } (\mathcal{L}(\mathcal{V}) \setminus \{0\}) = \mathcal{L}(\mathcal{V})$.

A non-empty subset \mathcal{I} of a semigroup $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ is called a left (respectively, right) semigroup ideal of \mathcal{S} (briefly, a left/right ideal) if $\mathcal{SI} \subseteq \mathcal{I}$ (respectively, $\mathcal{IS} \subseteq \mathcal{I}$).

COROLLARY 2.5. *Let $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ be a semigroup. If $\mathcal{I} \subseteq \mathcal{S}$ is a left (right) ideal, then $\text{Ref } \mathcal{I}$ is a left (right) ideal of $\text{Ref } \mathcal{S}$.*

Let $\mathcal{V}, \mathcal{V}', \mathcal{W}$, and \mathcal{W}' be vector spaces. For a non-empty set $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ and operators $C \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$, $D \in \mathcal{L}(\mathcal{V}', \mathcal{V})$, let $CTD = \{CTD : T \in \mathcal{T}\}$.

PROPOSITION 2.6. *If \mathcal{T} , C , and D are as above, then*

$$C(\text{Ref } \mathcal{T})D \subseteq \text{Ref } (CTD).$$

Moreover, for C and D invertible, the equality holds.

Proof. Let $A \in \text{Ref } \mathcal{T}$. For $x' \in \mathcal{V}'$, there exists $T_{Dx'} \in \mathcal{T}$ such that $ADx' = T_{Dx'}Dx'$. It follows $CADx' = CT_{Dx'}Dx' \in (CTD)x'$, which gives $C(\text{Ref } \mathcal{T})D \subseteq \text{Ref } (CTD)$.

Assume that C and D are invertible and denote $\mathcal{T}' = CTD$. Then $\mathcal{T} = C^{-1}\mathcal{T}'D^{-1}$. By the first part of the proof, $C^{-1}(\text{Ref } \mathcal{T}')D^{-1} \subseteq \text{Ref } (C^{-1}\mathcal{T}'D^{-1}) = \text{Ref } \mathcal{T}$. Thus, $\text{Ref } (CTD) \subseteq C(\text{Ref } \mathcal{T})D$. \square

According to Proposition 2.1 the following is a natural extension of the definition of algebraically reflexive semigroups to arbitrary sets of operators.

DEFINITION 2.7. A non-empty subset $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ is *algebraically reflexive* if $\mathcal{T} = \text{Ref } \mathcal{T}$.

So, we have the following corollary of Proposition 2.6.

COROLLARY 2.8. *Let $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ be a non-empty set and let $C \in \mathcal{L}(\mathcal{W}, \mathcal{W}')$, $D \in \mathcal{L}(\mathcal{V}', \mathcal{V})$ be invertible operators. Then \mathcal{T} is algebraically reflexive if and only if CTD is algebraically reflexive. In particular, a semigroup $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ is algebraically reflexive if and only if there exists an invertible $A \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ such that the semigroup $ASA^{-1} \subseteq \mathcal{L}(\mathcal{W})$ is algebraically reflexive.*

3. Examples. In this section, we list some examples of algebraically reflexive and algebraically non-reflexive sets and semigroups.

3.1. Finite sets. Denote by $|\mathbb{F}|$ the cardinality of \mathbb{F} . Let $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ be a finite set, say $\mathcal{T} = \{T_1, \dots, T_n\}$. If $n \leq |\mathbb{F}|$, then \mathcal{T} is algebraically reflexive. Indeed, if $A \in \text{Ref } \mathcal{T}$, then $Ax \in Tx$, for every $x \in \mathcal{V}$, which gives $\mathcal{V} = \cup_{i=1}^n \ker(A - T_i)$. By [2, Lemma 2], \mathcal{V} cannot be a union of $n \leq |\mathbb{F}|$ proper subspaces. Thus, one among the kernels is the whole space. We have $A = T_i$, for some $i \in \{1, \dots, n\}$. We wish to point out two particular cases.

1. If \mathbb{F} is infinite, then every finite set $\mathcal{T} \subseteq \mathcal{L}(\mathcal{V}, \mathcal{W})$ is algebraically reflexive.

2. If $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ is a semigroup that contains Id and has less than $|\mathbb{F}|$ elements, then $\text{Sgr Lst } \mathcal{S} = \mathcal{S}$.

Let \mathcal{X} and \mathcal{Y} be real or complex Banach spaces. The Banach space of all bounded linear transformations from \mathcal{X} to \mathcal{Y} is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. If $\mathcal{T} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is any countable set of operators, then $\mathcal{T} = \text{Ref } \mathcal{T} \cap \mathcal{B}(\mathcal{X}, \mathcal{Y})$, see [9, Lemma 1].

3.2. Semigroups of invertible/non-invertible linear transformations.

Let \mathcal{V} be a vector space over \mathbb{F} . Denote by $\text{Inj}(\mathcal{V})$ and $\text{Sur}(\mathcal{V})$ the sets of all injective, respectively surjective, linear transformations on \mathcal{V} . It is easily seen that these sets are semigroups; their intersection is $\text{Inv}(\mathcal{V})$, the group of all invertible linear transformations.

PROPOSITION 3.1. *Semigroup $\text{Inj}(\mathcal{V})$ is algebraically reflexive; moreover one has $\text{Ref}(\text{Inv}(\mathcal{V})) = \text{Inj}(\mathcal{V})$.*

Proof. Let $T \in \text{Ref}(\text{Inj}(\mathcal{V}))$ and x be a non-zero vector. Then $Tx = S_x x \neq 0$, for some $S_x \in \text{Inj}(\mathcal{V})$. Thus, the inclusion $\text{Inv}(\mathcal{V}) \subseteq \text{Inj}(\mathcal{V})$ gives $\text{Ref}(\text{Inv}(\mathcal{V})) \subseteq \text{Inj}(\mathcal{V})$. For the opposite inclusion, assume that $T \in \mathcal{L}(\mathcal{V})$ is injective. Let $x \in \mathcal{V}$ be arbitrary. Of course, if $x = 0$, then $Tx = 0 \in \{0\} = \text{Inj}(\mathcal{V})x$. Assume therefore that $x \neq 0$. Then $Tx \neq 0$, which means that there is an invertible linear transformation A on $\mathcal{Y} := \text{Lin}\{x, Tx\}$ such that $Ax = Tx$. Let \mathcal{Z} be a complement of \mathcal{Y} , i.e. a subspace of \mathcal{V} such that $\mathcal{V} = \mathcal{Y} \oplus \mathcal{Z}$. Define linear transformations S and S' on \mathcal{V} in the following way. If $v = y + z$ is the unique decomposition with $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, then let $Sv = Ay + z$ and $S'v = A^{-1}y + z$. It is obvious that $S'S = SS' = Id$. Since $Sx = Ax = Tx$ we have $T \in \text{Ref}(\text{Inv}(\mathcal{V}))$. Now, since $\text{Inv}(\mathcal{V}) \subseteq \text{Inj}(\mathcal{V})$, we may conclude that $\text{Ref}(\text{Inv}(\mathcal{V})) = \text{Inj}(\mathcal{V})$. \square

If \mathcal{V} is not finite dimensional, then the group of invertible linear transformations is a proper subsemigroup in $\text{Inj}(\mathcal{V})$. On the other hand, if \mathcal{V} is finite dimensional, then a linear transformation is injective if and only if it is surjective (see [16, Corollary 2.9]), i.e. $\text{Inv}(\mathcal{V}) = \text{Inj}(\mathcal{V}) = \text{Sur}(\mathcal{V})$. Thus, we have the following result.

COROLLARY 3.2. *Semigroup $\text{Inv}(\mathcal{V})$ is algebraically reflexive if and only if \mathcal{V} is finite dimensional.*

Corollary 3.2 shows that the algebraic reflexive cover of a group is not necessary a group, in general. However, the following holds.

PROPOSITION 3.3. *If $\mathcal{S} \subseteq \mathcal{L}(\mathcal{V})$ is a semigroup of invertible linear transformations and $T \in \text{Ref } \mathcal{S}$ is invertible, then $T^{-1} \in \text{Ref } \mathcal{S}$.*

Proof. Let $x \in \mathcal{V}$. Then there exists $S_{T^{-1}x} \in \mathcal{S}$ such that $x = T(T^{-1}x) = S_{T^{-1}x}(T^{-1}x)$. It follows $T^{-1}x = S_{T^{-1}x}^{-1}x \in \mathcal{S}x$. Thus, $T^{-1} \in \text{Ref } \mathcal{S}$. \square

It is an immediate consequence of Corollary 3.2 and Proposition 3.3 that the algebraic reflexive cover of a group of linear transformations is a group whenever the underlying space is finite dimensional.

PROPOSITION 3.4. *If \mathcal{V} is infinite-dimensional, then $\text{Ref}(\text{Sur}(\mathcal{V})) = \mathcal{L}(\mathcal{V})$.*

Proof. Let $T \in \mathcal{L}(\mathcal{V})$ be arbitrary. If $x \in \mathcal{V}$ is not in the kernel of T , then there exists an invertible S such that $Tx = Sx$ (see the proof of Proposition 3.1). Assume therefore that $Tx = 0$, for a non-zero $x \in \mathcal{V}$. Let \mathcal{X} be a subspace of \mathcal{V} spanned by x and let \mathcal{Y} be a complement of \mathcal{X} , i.e. $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$. Since $\dim(\mathcal{V}) = \dim(\mathcal{Y})$ there exists a bijective linear transformation $A : \mathcal{Y} \rightarrow \mathcal{V}$. Let $v = \alpha x + y$ be the unique decomposition of $v \in \mathcal{V}$ with respect to $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$. Let $S \in \mathcal{L}(\mathcal{V})$ be given by $Sv = S(\alpha x + y) = Ay$. It is obvious that S is surjective and that $Sx = 0 = Tx$. Thus, $T \in \text{Ref}(\text{Sur}(\mathcal{V}))$. \square

3.3. Continuous one-parameter semigroups in finite dimensional case.

Let \mathcal{X} be a complex Banach space. Recall from [6] that a family $(T(t))_{t \geq 0} \subseteq \mathcal{B}(\mathcal{X})$ is a strongly continuous semigroup if it satisfies the functional equation $T(t+s) = T(t)T(s)$ with $T(0) = Id$ and, for every $x \in \mathcal{X}$, the map $t \mapsto T(t)x$ is continuous from $[0, \infty)$ to \mathcal{X} . If \mathcal{X} is finite-dimensional, then every strongly continuous semigroup with values in $\mathcal{B}(\mathcal{X})$ is uniformly continuous, i.e. the map $t \mapsto T(t)$ is continuous from $[0, \infty)$ to $\mathcal{B}(\mathcal{X})$. This follows from the fact that in the finite-dimensional case all Hausdorff topologies on $\mathcal{B}(\mathcal{X})$ coincide. During this subsection, it is assumed that \mathcal{X} is a complex Banach space of dimension $m \in \mathbb{N}$ and that $(T(t))_{t \geq 0} \subseteq \mathcal{B}(\mathcal{X})$ is a continuous semigroup.

Since the space \mathcal{X} is isomorphic to \mathbb{C}^m the semigroup $(T(t))_{t \geq 0}$ is similar to a continuous semigroup included in $M_m(\mathbb{C})$, the algebra of all m -by- m complex matrices. By Corollary 2.8, similar semigroups are simultaneously reflexive. Thus, it is enough to consider the reflexivity of continuous semigroups in $M_m(\mathbb{C})$.

By [6, Theorem I.2.8], every continuous semigroup in $M_m(\mathbb{C})$ is of the form $t \mapsto e^{tA}$ ($t \geq 0$), for some $A \in M_m(\mathbb{C})$. Since each $A \in M_m(\mathbb{C})$ can be written as $A = S^{-1}DS$, where D has the Jordan canonical form and $S \in M_m(\mathbb{C})$ is invertible one has, by [6, Lemma I.2.4], $e^{tA} = S^{-1}e^{tD}S$, which means that we may reduce our consideration to the semigroups that are of the form $T(t) = e^{tD}$ ($t \geq 0$).

Let $\alpha \in \mathbb{C}$ and let $J_n \in M_n(\mathbb{C})$ be the $n \times n$ Jordan block. We consider first the semigroup \mathcal{T} which is given by

$$T(t) = e^{t(\alpha Id + J_n)} = e^{t\alpha} \left(Id + tJ_n + \frac{t^2}{2}J_n^2 + \cdots + \frac{t^{n-1}}{(n-1)!}J_n^{n-1} \right).$$

LEMMA 3.5. *Semigroup \mathcal{T} is algebraically reflexive.*

Proof. Let e_1, \dots, e_n be the standard basis in \mathbb{C}^n . Assume that $S = [s_{ij}] \in \text{Ref}(\mathcal{T})$. Then, for every $j \in \{1, \dots, n\}$, there exists t_j such that $Se_j = T(t_j)e_j$. It follows that

$$S = \begin{pmatrix} e^{t_1\alpha} & t_2e^{t_2\alpha} & \frac{t_3^2}{2}e^{t_3\alpha} & \dots & \frac{t_n^{n-1}}{(n-1)!}e^{t_n\alpha} \\ 0 & e^{t_2\alpha} & t_3e^{t_3\alpha} & \dots & \frac{t_n^{n-2}}{(n-2)!}e^{t_n\alpha} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t_n e^{t_n\alpha} \\ 0 & \dots & \dots & 0 & e^{t_n\alpha} \end{pmatrix}.$$

We have to show that $t_1 = t_2 = \dots = t_n$. For $j \in \{1, \dots, n-1\}$ and $r > 0$, let $f_{r,j} = r e_j + e_{j+1}$. Then there exists $\tau_j(r) \geq 0$ such that $Sf_{r,j} = T(\tau_j(r))f_{r,j}$. Comparing the rows j and $j+1$ we get the equalities

$$(3.1) \quad re^{t_j\alpha} + t_{j+1}e^{\alpha t_{j+1}} = re^{\alpha\tau_j(r)} + \tau_j(r)e^{\alpha\tau_j(r)}$$

and

$$(3.2) \quad e^{\alpha t_{j+1}} = e^{\alpha\tau_j(r)}.$$

It follows from (3.2) that there exists an integer k_j such that

$$(3.3) \quad \tau_j(r) = t_{j+1} - \frac{2k_j\pi i}{\alpha}.$$

Use (3.2) and (3.3) in (3.1):

$$re^{t_j\alpha} + t_{j+1}e^{\alpha t_{j+1}} = re^{\alpha t_{j+1}} + \left(t_{j+1} - \frac{2k_j\pi i}{\alpha}\right)e^{\alpha t_{j+1}}.$$

So, one has

$$e^{\alpha(t_j - t_{j+1})} = 1 - \frac{2k_j\pi i}{r\alpha}.$$

Since r is an arbitrary positive number the last equality holds only if $k_j = 0$. We conclude that

$$e^{\alpha t_j} = e^{\alpha t_{j+1}}.$$

Thus,

$$S = e^{t_1\alpha} \begin{pmatrix} 1 & t_2 & \frac{t_3^2}{2} & \dots & \frac{t_n^{n-1}}{(n-1)!} \\ 0 & 1 & t_3 & \dots & \frac{t_n^{n-2}}{(n-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & t_n \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Now, let $g = e_1 + \dots + e_n$. Then there exists $\sigma \geq 0$ such that $Sg = T(\sigma)g$. It is easily to see that this equality gives $t_1 = \dots = t_n = \sigma$. \square

Assume now that $\mathcal{T} = (e^{tD})_{t \geq 0} \subset M_m(\mathbb{C})$, where

$$D = (\alpha_1 Id_{n_1} + J_{n_1}) \oplus \dots \oplus (\alpha_k Id_{n_k} + J_{n_k}), \quad n_1 + \dots + n_k = m.$$

It is obvious that

$$(3.4) \quad e^{tD} = e^{t\alpha_1} e^{tJ_{n_1}} \oplus \dots \oplus e^{t\alpha_k} e^{tJ_{n_k}}$$

with respect to the decomposition

$$(3.5) \quad \mathbb{C}^m = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_k}.$$

PROPOSITION 3.6. *Every continuous semigroup \mathcal{T} on a finite dimensional complex Banach space is algebraically reflexive.*

Proof. As mentioned above, without loss of generality, one can assume that \mathcal{T} is a continuous semigroup of m -by- m complex matrices and that it is of the form (3.4). Let $S \in \text{Ref}(\mathcal{T})$ have the block-matrix $[S_{ij}]$ with respect to the decomposition (3.5). For $1 \leq j \leq k$, let $x_j \in \mathbb{C}^{n_j}$ be an arbitrary vector. Then there exists $t = t_{x_j} \geq 0$ such that

$$S(0 \oplus \dots \oplus x_j \oplus \dots \oplus 0) = (e^{t\alpha_1} e^{tJ_{n_1}} \oplus \dots \oplus e^{t\alpha_k} e^{tJ_{n_k}})(0 \oplus \dots \oplus x_j \oplus \dots \oplus 0).$$

It is easily seen that this gives $S_{ij} = 0$ if $i \neq j$ and $S_{jj}x_j \in \{e^{t\alpha_j} e^{tJ_{n_j}} x_j : t \geq 0\}$. Thus, $S_{jj} \in \text{Ref}((e^{t\alpha_j} e^{tJ_{n_j}})_{t \geq 0})$. Since, by Lemma 3.5, the semigroup $(e^{t\alpha_j} e^{tJ_{n_j}})_{t \geq 0}$ is reflexive there exists $t_j \geq 0$ such that $S_{jj} = e^{t_j \alpha_j} e^{t_j J_{n_j}}$, which gives

$$S = e^{t_1 \alpha_1} e^{t_1 J_{n_1}} \oplus \dots \oplus e^{t_k \alpha_k} e^{t_k J_{n_k}}.$$

In order to show that $t_1 = \dots = t_k$, let u_1, \dots, u_m be the standard basis in \mathbb{C}^m and let

$$v = u_{n_1-1} + u_{n_1} + u_{n_1+n_2-1} + u_{n_1+n_2} \dots + u_{n_1+n_2+\dots+n_k-1} + u_{n_1+n_2+\dots+n_k}.$$

There exists $\tau \geq 0$ such that

$$(e^{t_1 \alpha_1} e^{t_1 J_{n_1}} \oplus \dots \oplus e^{t_k \alpha_k} e^{t_k J_{n_k}})v = (e^{\tau \alpha_1} e^{\tau J_{n_1}} \oplus \dots \oplus e^{\tau \alpha_k} e^{\tau J_{n_k}})v.$$

From this equation we derive systems of equations

$$e^{t_j \alpha_j} (1 + t_j) = e^{\tau \alpha_j} (1 + \tau) \quad \text{and} \quad e^{t_j \alpha_j} = e^{\tau \alpha_j} \quad (1 \leq j \leq k).$$

It is obvious that $t_j = \tau$ for every $1 \leq j \leq k$. \square

4. Algebraically orbit reflexive transformations. Of course, the most natural examples of semigroups of linear transformations are those generated by a single element. Let $A \in \mathcal{L}(\mathcal{V})$ be a non-zero linear transformation and let $\mathcal{O}(A)$ be the semigroup generated by A . Thus, $\mathcal{O}(A) = \{A^n : n = 0, 1, 2, \dots\}$. In general, $\mathcal{O}(A)$ is an infinite semigroup (even in the case when \mathcal{V} is finite dimensional); it is finite in very special cases.

PROPOSITION 4.1. *For $A \in \mathcal{L}(\mathcal{V})$, the semigroup $\mathcal{O}(A)$ is finite if and only if there exist positive integers m and n such that the minimal polynomial of A divides $p(z) = z^m(z^n - 1)$.*

Proof. Let $m_A(z)$ be the minimal polynomial of A . If $m_A(z) | z^m(z^n - 1)$, where m, n are positive integers, then $A^m(A^n - Id) = 0$ and therefore $A^{m+n} = A^m$, which means that $\mathcal{O}(A) \subseteq \{Id, A, \dots, A^{m+n-1}\}$. On the other hand, if $\mathcal{O}(A)$ is a finite semigroup, then there exist positive integers m and n such that $A^{m+n} = A^m$. It follows that $A^m(A^n - Id) = 0$ and therefore $m_A(z) | z^m(z^n - 1)$. \square

A linear transformation $A \in \mathcal{L}(\mathcal{V})$ is said to be *algebraically orbit-reflexive* if the semigroup $\mathcal{O}(A)$, which is generated by A , is algebraically reflexive (in the sense of Definition 2.7). If the field \mathbb{F} is large enough, then many linear transformations are algebraically orbit-reflexive.

Recall that a linear transformation $A \in \mathcal{L}(\mathcal{V})$ is locally algebraic if, for each $x \in \mathcal{V}$, there exists a polynomial $p_x \neq 0$ such that $p_x(A)x = 0$. If one has a polynomial $p \neq 0$ such that $p(A)x = 0$ for all $x \in \mathcal{V}$, which means $p(A) = 0$, then A is said to be algebraic. Of course, if \mathcal{V} is a finite dimensional vector space, then every linear transformation is algebraic. By the celebrated Kaplansky Theorem, every locally algebraic bounded linear transformation on a Banach space is algebraic. However, there exist vector spaces and locally algebraic linear transformations on them that are not algebraic. For instance, on $\mathbb{F}[z]$ the linear transformation that maps a polynomial p to its derivative p' is locally algebraic but not algebraic.

THEOREM 4.2 ([9], Theorems 9 and 10). *Let \mathcal{V} be a vector space over \mathbb{F} .*

1. *If \mathbb{F} is uncountable, then*
 - (i) *every algebraic linear transformation is algebraically orbit-reflexive; in particular, on a finite dimensional space every linear transformation is algebraically orbit-reflexive;*
 - (ii) *for every locally algebraic $A \in \mathcal{L}(\mathcal{V})$, the reflexive cover of $\mathcal{O}(A)$ is just the closure of $\mathcal{O}(A)$ in the strict topology.*
2. *If \mathbb{F} is infinite and $A \in \mathcal{L}(\mathcal{V})$ is not locally algebraic, then A is algebraically orbit-reflexive.*

In this section, we will employ Theorem 4.2 in the consideration of the question

if a linear transformation A is determined by the lattice $\text{Lst } A$. More precisely, have $A, B \in \mathcal{L}(\mathcal{V})$ to be equal if $\text{Lst } A = \text{Lst } B$? As the following simple example shows, A is not determined by $\text{Lst } A$, in general. Namely, let \mathcal{V} be a complex vector space and let $\zeta = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$. Denote $A = \zeta Id$ and $B = \zeta^2 Id$. Of course, $A \neq B$. However, $\text{Lst } A = \text{Lst } B$. Indeed, for every $x \in \mathcal{V}$, one has $\mathcal{O}(A)x = \{x, \zeta x, \zeta^2 x\} = \mathcal{O}(B)x$. We will see that examples similar to this one are essentially the only exceptions when \mathbb{F} is an uncountable algebraically closed field and A is an algebraic linear transformation. First we state the following corollary of Theorem 4.2.

COROLLARY 4.3. *Let \mathcal{V} be a vector space over an infinite field \mathbb{F} . If $A \in \mathcal{L}(\mathcal{V})$ is not locally algebraic, then A is uniquely determined with $\text{Lst } A$.*

Proof. Assume that $\text{Lst } A = \text{Lst } B$, for some $B \in \mathcal{L}(\mathcal{V})$. Then $\text{Sgr } \text{Lst } B = \text{Sgr } \text{Lst } A$ and therefore $\text{Sgr } \text{Lst } B = \mathcal{O}(A)$, by Theorem 4.2 (2). Hence there exists a non-negative integer n such that $B = A^n$. Linear transformation B cannot be locally algebraic. Namely, if for every $x \in \mathcal{V}$ there exists a non-zero polynomial p_x such that $p_x(B)x = 0$, then one has $q_x(A)x = 0$ for a non-zero polynomial $q_x(z) := p_x(z^n)$, which means A is locally algebraic. Since B is not locally algebraic we have, by Theorem 4.2 (2), $\text{Sgr } \text{Lst } B = \mathcal{O}(B)$. Thus, there is an integer $m \geq 0$ such that $A = B^m$. It follows that $A = A^{mn}$. This equality is reasonable only if $mn = 1$ since A is not algebraic. We conclude $A = B$. \square

It remains to consider locally algebraic transformations.

PROPOSITION 4.4. *Let \mathcal{V} be a vector space over an arbitrary field \mathbb{F} and let $A, B \in \mathcal{L}(\mathcal{V})$ be commuting linear transformations.*

- (i) *If A and B are algebraic, then $A + B$ and AB are algebraic as well.*
- (ii) *If A and B are locally algebraic, then $A + B$ and AB are locally algebraic.*

Proof. (i) Let A and B be algebraic. Denote by $\mathcal{R} \subseteq \mathcal{L}(\mathcal{V})$ the ring generated by A, B , and Id . Of course, \mathcal{R} is a commutative ring with the identity and $\mathbb{F}Id$ is a subring that contains the identity. In the terminology of [1], elements $A, B \in \mathcal{R}$ are integral over $\mathbb{F}Id$. By [1, Corollary 5.3], the sum $A + B$ and the product AB are integral over $\mathbb{F}Id$ as well, which, in our terminology, means that they are algebraic linear transformations.

(ii) Assume that A and B are locally algebraic. For an arbitrary vector $x \in \mathcal{V}$, there exist polynomials $p_x(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$ and $q_x(z) = z^m + b_{m-1}z^{m-1} + \dots + b_1z + b_0$ such that $p_x(A)x = 0$ and $q_x(B)x = 0$. Denote by \mathcal{W}_x the subspace of \mathcal{V} that is spanned by vectors $A^i B^j x$, where $0 \leq i < n$ and $0 \leq j < m$. Since A and B commute it is easily seen that \mathcal{W}_x is invariant for A and B . Let A_x be the restriction of A to \mathcal{W}_x and, similarly, let B_x be the restriction of B to \mathcal{W}_x . It is straightforward to check that $p_x(A_x) = 0$ and $q_x(B_x) = 0$, which means that

A_x and B_x are algebraic linear transformations in $\mathcal{L}(\mathcal{W}_x)$. By the first part of this proof, $A_x + B_x$ and $A_x B_x$ are algebraic as well. Thus, there are polynomials $u_x(z)$ and $v_x(z)$ such that $u_x(A_x + B_x) = 0$ and $v_x(A_x B_x) = 0$, which gives $u_x(A + B)x = 0$ and $v_x(AB)x = 0$. \square

If \mathbb{F} is infinite and $A \in \mathcal{L}(\mathcal{V})$ is locally algebraic, then $\text{Lst } A = \text{Lst } B$ can hold only for locally algebraic $B \in \mathcal{L}(\mathcal{V})$. This is just a reformulation of Corollary 4.3. In the case of an uncountable field, we can say more.

PROPOSITION 4.5. *Let \mathcal{V} be a vector space over an uncountable field \mathbb{F} and let $A \in \mathcal{L}(\mathcal{V})$.*

- (i) *If A is an algebraic linear transformation, then every linear transformation in $\text{Sgr Lst } A$ is algebraic.*
- (ii) *If A is a locally algebraic linear transformation, then every linear transformation in $\text{Sgr Lst } A$ is locally algebraic.*

Proof. (i) Assume that A is algebraic and let $B \in \text{Sgr Lst } A$. By Theorem 4.2 (1)(i), A is algebraically orbit-reflexive, which means that $\text{Sgr Lst } A = \mathcal{O}(A)$. In particular, there is a non-negative integer m such that $B = A^m$. It follows, by Proposition 4.4 (i), that B is algebraic.

(ii) Let A be locally algebraic and $B \in \text{Sgr Lst } A$. By Theorem 4.2 (1)(ii), B is in the strict closure of $\mathcal{O}(A)$. Note that A and B commute. Indeed, let $x \in \mathcal{V}$ be arbitrary and let $\mathcal{E} = \{x, Ax\}$. Recall that $\mathcal{U}(B; \mathcal{E}) = \{S \in \mathcal{L}(\mathcal{V}) : Bx = Sx \text{ and } BAx = SAx\}$ is a basic neighbourhood of B . Since B is in the strict closure of $\mathcal{O}(A)$, the intersection $\mathcal{U}(B; \mathcal{E}) \cap \mathcal{O}(A)$ cannot be empty. Thus, there is $A^k \in \mathcal{O}(A)$ such that $Bx = A^k x$ and $B(Ax) = A^k(Ax)$. It follows that $BAx = ABx$ and consequently $BA = AB$.

Let $x \in \mathcal{V}$ and let $p_x(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ be a polynomial such that $p_x(A)x = 0$. Denote with \mathcal{W}_x the subspace of \mathcal{V} that is spanned by vectors $x, Ax, \dots, A^{n-1}x$. Since $\mathcal{U}(B; x) \cap \mathcal{O}(A) \neq \emptyset$ one has $Bx = A^m x$, for a non-negative integer m . Hence it is easily seen that \mathcal{W}_x is invariant for B . Let A_x and B_x be restrictions of A and B to \mathcal{W}_x . Then $Bx = A^m x$ gives $B_x = A_x^m$. Since A_x is algebraic one has $B_x \in \mathcal{O}(A_x) = \text{Sgr Lst } A_x$ and therefore, by the first part of this proposition, B_x is algebraic. Thus, there is a polynomial $q_x(z)$ such that $q_x(B_x) = 0$ and we may conclude that $q_x(B)x = 0$. \square

Now we consider an algebraic linear transformation $A \in \mathcal{L}(\mathcal{V})$. It is obvious that the identity transformation Id is the only linear transformation that leaves invariant every subset of \mathcal{V} . Also, the zero linear transformation leaves invariant every subset of \mathcal{V} that contains the zero vector, and it is the unique transformation with this property. So, in sequel, we assume that $A \neq Id$ and $A \neq 0$.

PROPOSITION 4.6. *Let \mathcal{V} be a vector space over an uncountable field \mathbb{F} and let $A \in \mathcal{L}(\mathcal{V})$, $Id \neq A \neq 0$, be an algebraic linear transformation. If $\text{Lst } A = \text{Lst } B$, for some $B \in \mathcal{L}(\mathcal{V})$ such that $A \neq B$, then B is algebraic and there exist integers $m \geq 2$ and $n \geq 2$ such that $A = B^m$ and $B = A^n$, which means that the minimal polynomials of A and B divide the polynomial $q(z) = z^{mn} - z$.*

Proof. The equality $\text{Lst } A = \text{Lst } B$ gives $\text{Sgr Lst } A = \text{Sgr Lst } B$. Since A is algebraic B is algebraic as well, by Proposition 4.5. Thus, by Theorem 4.2, $\mathcal{O}(A) = \mathcal{O}(B)$. It follows that there are non-negative integers m and n such that $A = B^m$ and $B = A^n$. It is easily seen that $A = B$ if $m \in \{0, 1\}$ or $n \in \{0, 1\}$. So, since we have assumed that $A \neq B$, one has $m \geq 2$ and $n \geq 2$. Of course, the equalities $A = B^m$ and $B = A^n$ give $A^{mn} - A = 0$ and $B^{mn} - B = 0$, which means that the minimal polynomials of A and B divide the polynomial $q(z) = z^{mn} - z$. \square

The following corollaries are immediate consequences of Proposition 4.6.

COROLLARY 4.7. *Let \mathcal{V} be a vector space over an uncountable field \mathbb{F} . If $A \in \mathcal{L}(\mathcal{V})$ is an algebraic linear transformation whose minimal polynomial does not divide $q(z) = z^{mn} - z$, for any integers $m, n \geq 2$, then A is uniquely determined with $\text{Lst } A$.*

COROLLARY 4.8. *Let \mathcal{V} be a vector space over an uncountable field \mathbb{F} and let $A \in \mathcal{L}(\mathcal{V})$ be an algebraic linear transformation such that $\text{Lst } A = \text{Lst } B$ for some $B \in \mathcal{L}(\mathcal{V})$ that is distinct from A . Then the minimal polynomial $m_A(z)$ has only simple zeros in the algebraic closure $\overline{\mathbb{F}}$. Thus, if $m_A(z) = r_1(z) \cdots r_k(z)$ is the decomposition of the minimal polynomial of A into polynomials $r_1(z), \dots, r_k(z)$ which are irreducible over \mathbb{F} , then $r_i(z) \neq r_j(z)$ whenever $i \neq j$.*

Proof. By Proposition 4.6, $m_A(z)$ divides $q(z) = z^{mn} - z$, where $m, n \geq 2$. Since $q(z)$ has only simple zeros in $\overline{\mathbb{F}}$, the same holds for $m_A(z)$. \square

COROLLARY 4.9. *Let \mathcal{V} be a vector space over an uncountable field \mathbb{F} and let $A \in \mathcal{L}(\mathcal{V})$ be an algebraic linear transformation. If $\text{Lst } A = \text{Lst } B$, for some $B \in \mathcal{L}(\mathcal{V})$, then $\ker A = \ker B$.*

Proof. If $x \in \ker A$, then, by Proposition 4.6, $Bx = A^n x = 0$. Similarly, if $x \in \ker B$, then $Ax = B^m x = 0$. \square

Now we assume that

- (i) \mathbb{F} is an uncountable algebraically closed field,
- (ii) $A \in \mathcal{L}(\mathcal{V})$ is an algebraic linear transformation with the minimal polynomial

$$m_A(z) = (z - \lambda_1) \cdots (z - \lambda_k),$$

- (iii) $\lambda_j \in \mathbb{F}$ are distinct and are either zero or a root of 1, and
- (iv') $r \geq 1$ is the smallest integer such that $\lambda_j^r = \lambda_j$ holds for all $j = 1, \dots, k$.

For every $j = 1, \dots, k$, let $\mathcal{X}_j = \ker(A - \lambda_j)$. It is well known that

$$\mathcal{V} = \mathcal{X}_1 \oplus \dots \oplus \mathcal{X}_k$$

and one has the corresponding decomposition $A = \lambda_1 Id_1 \oplus \dots \oplus \lambda_k Id_k$ of the linear transformation. Here Id_j is the identical linear transformation on \mathcal{X}_j .

It is easily seen that A is uniquely determined with $\text{Lst } A$ if $m_A(z) = z(z - 1)$, which means $A = 0 \oplus Id$. So, we may assume that at least one eigenvalue of A is not 0 or 1, which means that the above condition (iv') can be replaced with

(iv) $r \geq 2$ is the smallest integer such that $\lambda_j^r = \lambda_j$ holds for all $j = 1, \dots, k$.

PROPOSITION 4.10. *Assume that the conditions (i)-(iv) are satisfied. Then, for $B \in \mathcal{L}(\mathcal{V})$, equality $\text{Lst } A = \text{Lst } B$ holds if and only if $B = A^n$, where $n < r$ is a positive integer satisfying $mn - 1 = p(r - 1)$ for some positive integers m and p . Moreover, let $n_1 = 1 < \dots < n_{\varphi(r-1)}$ be the positive integers which are smaller than r and coprime with $r - 1$ (φ is Euler's totient function). Then $\text{Lst } A = \text{Lst } B$ if and only if $B \in \{A = A^{n_1}, \dots, A^{n_{\varphi(r-1)}}\}$.*

Proof. Assume that $\text{Lst } A = \text{Lst } B$, for some $B \in \mathcal{L}(\mathcal{V})$. If $A = B$, then $n = p = 1$ and $m = r$. Assume therefore that $A \neq B$. By Proposition 4.6, B is algebraic and there exist positive integers m and n such that $B = A^n$ and $A = B^m$. Since $A^r = A$ one can assume that $n < r$. It follows that $\lambda_j^{mn} = \lambda_j$, for all $j = 1, \dots, k$, and therefore $\lambda_j^{mn-1} = 1$, for all $\lambda_j \neq 0$. Since $r \geq 2$ is the smallest integer such that $\lambda_j^{r-1} = 1$, for all $\lambda_j \neq 0$, one has $mn - 1 \geq r - 1$. Let $p \geq 1$ and $0 \leq q < r - 1$ be integers such that $mn - 1 = p(r - 1) + q$. Then $1 = \lambda_j^{mn-1} = (\lambda_j^{r-1})^p \lambda_j^q = \lambda_j^q$, for all $\lambda_j \neq 0$, which gives $q = 0$.

For the opposite implication suppose that $B = A^n$, where $n < r$ is a positive integers satisfying $mn - 1 = p(r - 1)$ for some positive integers m and p . Then $B^m = A^{mn} = A$. It follows that $\text{Lst } A \subseteq \text{Lst } A^n = \text{Lst } B \subseteq \text{Lst } B^m = \text{Lst } A$.

It is obvious that, for $r \geq 2$ and $1 \leq n \leq r - 1$, one can find positive integers m and p satisfying $mn - 1 = p(r - 1)$ if and only if n and $r - 1$ are coprime. If $n_1 = 1 < \dots < n_{\varphi(r-1)}$ are the positive integers which are smaller than r and coprime with $r - 1$, then, because of condition (iv), $A = A^{n_1}, \dots, A^{n_{\varphi(r-1)}}$ are distinct linear transformations and $\text{Lst } A = \text{Lst } A^{n_k}$, for every $k = 1, \dots, \varphi(r - 1)$. \square

COROLLARY 4.11. *Let \mathcal{X} be a complex Banach space. Then $A \in \mathcal{B}(\mathcal{X})$ is uniquely determined by the lattice $\text{Lst } A$ except in the case when*

- A is an algebraic operator with the minimal polynomial

$$m_A(z) = (z - \lambda_1) \cdots (z - \lambda_k),$$

- λ_i ($i = 1, \dots, k$) are distinct numbers, each is either zero or a root of 1
- if r is the minimal positive integer such that $\lambda_i^r = \lambda_i$ for all $i = 1, \dots, k$, then $\varphi(r - 1) > 1$.

In the exceptional case, let $\mathcal{N} = \{n_1 = 1 < \dots < n_{\varphi(r-1)} = r - 2\}$ be the set of all $\varphi(r - 1)$ positive integers that are smaller than $r - 1$ and coprime with $r - 1$. Then $\text{Lst } A = \text{Lst } B$ if and only if $B = A^{n_i}$ for some $n_i \in \mathcal{N}$.

Proof. If \mathcal{X} is a complex Banach space then each locally algebraic bounded operator on it is algebraic. Thus, if A is not algebraic, then it is uniquely determined with $\text{Lst } A$, by Corollary 4.3. On the other hand, if A is algebraic and there exists $B \neq A$ such that $\text{Lst } B = \text{Lst } A$, then, by Corollary 4.7, the minimal polynomial has only simple zeros and each of them is either zero or a root of 1. The rest of the proof follows from Proposition 4.10. \square

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