Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

THE EIGENVALUE DISTRIBUTION OF SCHUR COMPLEMENTS OF NONSTRICTLY DIAGONALLY DOMINANT MATRICES AND GENERAL $H-$MATRICES*

CHENG-YI ZHANG†, SHUANGHUA LUO‡, FENGMIN XU§, AND CHENGXIAN XU¶

Abstract. The paper studies the eigenvalue distribution of Schur complements of some special matrices, including nonstrictly diagonally dominant matrices and general $H-$matrices. Zhang, Xu, and Li [Theorem 4.1, The eigenvalue distribution on Schur complements of H-matrices. *Linear Algebra Appl.,* 422:250–264, 2007] gave a condition for an $n\times n$ diagonally dominant matrix A to have $|J_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)|$ eigenvalues with negative real part, where $|J_{R_+}(A)|$ ($|J_{R_-}(A)|$) denotes the number of diagonal entries of A with positive (negative) real part. This condition is applied to establish some results about the eigenvalue distribution for the Schur complements of nonstrictly diagonally dominant matrices and general $H-$matrices with complex diagonal entries. Several conditions on the $n\times n$ matrix A and the subset $\alpha\subseteq N=\{1,2,\cdots,n\}$ are presented so that the Schur complement A/α of A has $|J_{R_+}(A)|-|J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)|-|J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part, where $|J^{\alpha}_{R_+}(A)|$ ($|J^{\alpha}_{R_-}(A)|$) denotes the number of diagonal entries of the principal submatrix $A(\alpha)$ of A with positive (negative) real part.

Key words. Eigenvalue distribution, Schur complements, (Generalized) Diagonally dominant matrices, General $H-$matrices.

AMS subject classifications. 15A15, 15A18.

1. Introduction.

The eigenvalue distribution of special matrices has various important applications in many aspects of matrix theory and in applied mathematics; see [9], [13] and [15]. A great deal of work has been devoted on the topic by a number of authors; see e.g., the references within [9, 10, 16, 18, 21, 24, 14]. Recently, con-

*Received by the editors March 5, 2009. Accepted for publication November 25, 2009. Handling Editor: Joao Filipe Queiro.

†Department of Mathematics of School of Science, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, P.R. China (chengyizhang@yahoo.com.cn).

‡Department of Mathematics of School of Science, Lanzhou University of Technology, Lanzhou, 730050, P.R. China (iwantflyluo@163.com).

§School of Electronic and Information Engineering and Institute for Information and System Science, Xi'an Jiaotong University, Xi'an, 710049, P.R. China (fengminxu@mail.xjtu.edu.cn). The work of this author was supported by the National Natural Science Foundations of China (70531030), the National 973 Program of China (2007CB311002) and the open fund of Key Laboratory of Precision Navigation and Timing Technology, Chinese Academy of Sciences.

¶SKLMSE Lab. and Department of Mathematics of School of Science, Xi'an Jiaotong University, Xi'an, 710049, P.R. China (mxxu@mail.xjtu.edu.cn). The work of this author was supported by the National Natural Science Foundation of China (10671152).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

siderable interest in the work on Schur complements for strictly diagonally dominant matrices and nonsingular $H-$matrices has been witnessed and some properties, such as diagonal dominance and the eigenvalue distribution of Schur complements, have been proposed. Readers are referred to [3, 4, 6, 7, 8, 9, 10, 11, 16, 18, 20, 21, 22, 24]. For example, Liu et al. [9, 10, 11], as well as Zhang, Xu, and Li [21] studied the eigenvalue distribution on the Schur complements of strictly diagonally dominant matrices and nonsingular $H-$matrices and proposed some useful results, respectively. But little attention has been paid to work on the eigenvalue distribution of the Schur complements for nonstrictly diagonally dominant matrices and general $H-$matrices.

In this paper, we study the eigenvalue distribution on the Schur complements of some special matrices including nonstrictly diagonally dominant matrices and general H-matrices. Let A be either a nonstrictly diagonally dominant matrix or a general $H-$matrix and $\alpha \subseteq N = \{1, 2, \cdots, n\}$, and denote the Schur complement matrix of the matrix A by A/α. Zhang et al. in [21] give the following result for a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$. *If* $\widehat{A} = (\widehat{a}_{ij})$ *is nonsingular diagonally dominant, where* $\widehat{a}_{ij}$ *is defined in* (2.3)*, then* A *has* $|J_{R_+}(A)|$ *eigenvalues with positive real part and* $|J_{R_-}(A)|$ *eigenvalues with negative real part.* Here $|J_{R_+}(A)|$ (or $|J_{R_-}(A)|$) denotes the number of diagonal entries of A with positive (or negative) real part. Applying this result to the matrix A/α, some properties on diagonal dominance and nonsingularity for the matrix $\widehat{A/\alpha}$ will be presented to establish some results about the eigenvalue distribution for the Schur complements of nonstrictly diagonally dominant matrices and general $H-$matrices with complex diagonal entries. Several conditions on the $n \times n$ matrix A and the subset $\alpha \subseteq N = \{1, 2, \cdots, n\}$ are presented such that the Schur complement matrix A/α of the matrix A has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part, where $|J^{\alpha}_{R_+}(A)|$ (or $|J^{\alpha}_{R_-}(A)|$) denotes the number of diagonal entries of the principal submatrix $A(\alpha)$ of A with positive (or negative) real part.

The paper is organized as follows. Some notation and preliminary results about special matrices are given in Section 2. Some conditions on diagonal dominance and nonsingularity for the matrix $\widehat{A/\alpha}$ are then presented in Section 3. The main results of this paper are given in Section 4, where we give the different conditions for a nonstrictly diagonally dominant matrix or general $H-$matrix A with complex diagonal entries and the subset $\alpha \subseteq N$ so that the Schur complement A/α has $|J_{R_+}(A)|-|J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)|-|J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part. Conclusions are given in Section 5.

2. Preliminaries. In this section we give some notions and preliminary results about special matrices that are used in this paper. $\mathbb{C}^{n\times n}$ ($\mathbb{R}^{n\times n}$) will be used to denote the set of all $n \times n$ complex (real) matrices. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n\times n}$, we write $A \geq B$, if $a_{ij} \geq b_{ij}$ holds for all $i, j = 1, 2, \cdots, n$. A

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a $Z-$*matrix* if $a_{ij} \leq 0$ for all $i \neq j$. We will use Z_n to denote the set of all $n \times n$ $Z-$matrices. A matrix $A = (a_{ij}) \in Z_n$ is called an $M-$*matrix* if A can be expressed in the form $A = sI - B$, where $B \geq 0$, and $s \geq \rho(B)$, the spectral radius of B. If $s > \rho(B)$, A is called a *nonsingular* $M-$*matrix*; if $s = \rho(B)$, A is called a *singular* $M-$*matrix*. M_n, $M_n^{\bullet}$ and M_n^0 will be used to denote the set of all $n \times n$ $M-$matrices, the set of all $n \times n$ nonsingular $M-$matrices and the set of all $n \times n$ singular $M-$matrices, respectively. It is easy to see that

$$M_n = M_n^{\bullet} \cup M_n^0 \quad \textit{and} \quad M_n^{\bullet} \cap M_n^0 = \emptyset. \tag{2.1}$$

The *comparison matrix* of a given matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, denoted by $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} |a_{ii}|, & \text{if } i = j, \\ -|a_{ij}|, & \text{if } i \neq j. \end{cases}$$

It is clear that $\mu(A) \in Z_n$ for a matrix $A \in \mathbb{C}^{n\times n}$. A matrix $A \in \mathbb{C}^{n\times n}$ is called a *general* $H-$*matrix* if $\mu(A) \in M_n$ (see [2]). If $\mu(A) \in M_n^{\bullet}$, A is called a *nonsingular* $H-$*matrix* and if $\mu(A) \in M_n^0$, A is called a *singular* $H-$*matrix*. H_n, $H_n^{\bullet}$ and H_n^0 will denote the set of all $n \times n$ general $H-$matrices, the set of all $n \times n$ nonsingular $H-$matrices and the set of all $n \times n$ singular $H-$matrices, respectively. Similar to equalities (2.1), we have

$$H_n = H_n^{\bullet} \cup H_n^0 \quad \textit{and} \quad H_n^{\bullet} \cap H_n^0 = \emptyset. \tag{2.2}$$

REMARK 2.1. A matrix $A \in H_n^0$ is not necessarily singular.

For example, given a matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Obviously,

$$\mu(A) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = I - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I - B,$$

where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \geq 0$, and the spectral radius $\rho(B) = 1$, which implies that $\mu(A) \in M_n^0$ and thus $A \in H_n^0$. But, $detA = 2 \neq 0$ shows that the singular $H-$matrix A is nonsingular.

Another example is seen in [2, Example 1].

For $n \geq 2$, an $n \times n$ complex matrix A is *reducible* if there exists an $n \times n$ permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

where A_{11} is an $r \times r$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then A is called *irreducible*. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and a set $\alpha \subseteq N = \{1, 2, \cdots, n\}$, we define the matrix $\widehat{A} = (\widehat{a}_{ij}) \in \mathbb{C}^{n\times n}$ and the matrix $A_\alpha = (\breve{a}_{ij}) \in \mathbb{C}^{n\times n}$ by

$$\widehat{a}_{ij} = \begin{cases} Re(a_{ii}), & i = j, \\ a_{ij}, & otherwise \end{cases} \quad \text{and } \breve{a}_{ij} = \begin{cases} Re(a_{ii}), & i = j \in N - \alpha, \\ a_{ij}, & otherwise, \end{cases} \tag{2.3}$$

respectively. Here, the matrix A_α is introduced for further study on the eigenvalue distribution for the Schur complement of nonstrictly diagonally dominant matrices and general $H-$matrices (see [21] and section 4 in this paper). Obviously, $A_\alpha = A$ if $\alpha = N$ and $A_\alpha = \widehat{A}$ if $\alpha = \emptyset$.

Let $|\alpha|$ denote the cardinality of the set $\alpha \subseteq N = \{1, 2, \cdots, n\}$. For nonempty index sets $\alpha, \beta \subseteq N$, $A(\alpha, \beta)$ is the submatrix of $A \in \mathbb{C}^{n\times n}$ with row indices in α and column indices in β. The submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Let $\alpha \subset N$, $\alpha' = N - \alpha$, and $A(\alpha)$ be nonsingular, the matrix

$$A/\alpha = A/A(\alpha) = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') \tag{2.4}$$

is called the *Schur complement with respect to* $A(\alpha)$, where indices in both α and α' are arranged with increasing order. We shall confine ourselves to the nonsingular $A(\alpha)$ as far as A/α is concerned.

DEFINITION 2.2. A matrix $A \in \mathbb{C}^{n\times n}$ is called *diagonally dominant by rows* if

$$|a_{ii}| \geq \sum_{j=1, j\neq i}^{n} |a_{ij}| \tag{2.5}$$

holds for all $i \in N$. If inequality in (2.5) holds strictly for all $i \in N$, A is called *strictly diagonally dominant by rows*. If A is irreducible and the inequality in (2.5) holds strictly for at least one $i \in N$, A is called *irreducibly diagonally dominant by rows*. If (2.5) holds with equality for all $i \in N$, A is called *diagonally equipotent by rows*.

D_n (SD_n, ID_n) and DE_n will be used to denote the sets of all $n \times n$ (strictly, irreducibly) diagonally dominant matrices, and the set of all $n \times n$ diagonally equipotent matrices, respectively.

DEFINITION 2.3. A matrix $A \in \mathbb{C}^{n\times n}$ is called *generalized diagonally dominant by rows* if there exist positive constants α_i, $i = 1, 2, \cdots, n$, such that

$$\alpha_i |a_{ii}| \geq \sum_{j=1, j\neq i}^{n} \alpha_j |a_{ij}| \tag{2.6}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

holds for all $i \in N$. If inequality in (2.6) holds strictly for all $i \in N$, A is called *generalized strictly diagonally dominant by rows.* If (2.6) holds with equality for all $i \in N$, A is called *generalized diagonally equipotent by rows.*

We will denote the sets of all $n \times n$ generalized (strictly) diagonally dominant matrices and the set of all $n \times n$ generalized diagonally equipotent matrices by GD_n (GSD_n) and GDE_n, respectively.

DEFINITION 2.4. A matrix A is called nonstrictly diagonally dominant, if either (2.5) or (2.6) holds with equality for at least one $i \in N$.

REMARK 2.5. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be nonstrictly diagonally dominant and $\alpha = N - \alpha' \subset N$. If $A(\alpha)$ is a (generalized) diagonally equipotent principal submatrix of A, then the following hold:

- $A(\alpha, \alpha') = 0$;
- $A(i_1) = (a_{i_1 i_1})$ being (generalized) diagonally equipotent implies $a_{i_1 i_1} = 0$.

REMARK 2.6. Definition 2.2 and Definition 2.3 show that

$$D_n \subset GD_n \quad and \quad GSD_n \subset GD_n.$$

The following properties of diagonally dominant matrices and $H-$matrices will be used in the rest of the paper.

LEMMA 2.7. (see [17]) *If $A \in SD_n \cup ID_n$, then A is nonsingular.*

LEMMA 2.8. (see [21]) *A matrix $A \in D_n$ is singular if and only if the matrix A has at least either one zero principal submatrix or one irreducible and diagonally equipotent principal submatrix $A_k = A(i_1, i_2, \cdots, i_k), 1 < k \leq n$, which satisfies condition that there exists a $k \times k$ unitary diagonal matrix U_k such that*

$$U_k^{-1} D_{A_k}^{-1} A_k U_k = \mu(D_{A_k}^{-1} A_k),$$

where $D_{A_k} = diag(a_{i_1 i_1}, a_{i_2 i_2}, \cdots, a_{i_k i_k})$.

LEMMA 2.9. (see [22]) *Let $A \in D_n$. Then A is singular if and only if A has at least one singular principal submatrix.*

LEMMA 2.10. (see [19, 22]) *Let $A \in D_n$. Then $A \in H_n^{\bullet}$ if and only if A has no diagonally equipotent principal submatrices. Furthermore, if $A \in D_n \cap Z_n$, then $A \in M_n^{\bullet}$ if and only if A has no diagonally equipotent principal submatrices.*

LEMMA 2.11. (see [1, 24]) *A matrix $A \in \mathbb{C}^{n\times n}$ is generalized (strictly) diagonally dominant if and only if there exists a positive diagonal matrix D such that $D^{-1}AD$ is (strictly) diagonally dominant.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

LEMMA 2.12. (see [1]) $H_n^{\bullet} = GSD_n$

LEMMA 2.13. *A matrix $A \in H_n^{\bullet}$ if and only if $A \in GD_n$ and has no generalized diagonally equipotent principal submatrices.*

Proof. Lemma 2.11, Lemma 2.12 and Lemma 2.10 give the conclusion of this lemma. □

LEMMA 2.14. (see [2]) $H_n = GD_n$.

LEMMA 2.15. *A matrix $A \in H_n^0$ if and only if $A \in GD_n$ and has at least one generalized diagonally equipotent principal submatrix.*

Proof. It follows from (2.2), Lemma 2.13, and Lemma 2.14 that the conclusion of this lemma is obtained immediately. □

3. Further results on the Schur complements of diagonally dominant matrices. As is shown in [3, 7], the Schur complement of a diagonally dominant matrix is diagonally dominant. This section will present some further results on the Schur complements of diagonally dominant matrices. Applying these results to the Schur complement matrix A/α of nonstrictly diagonally dominant matrices and singular $H-$matrices will in Section 4 establish some theorems on the eigenvalue distribution for the Schur complements of nonstrictly diagonally dominant matrices and singular $H-$matrices. The following lemmas will be used in this section.

LEMMA 3.1. (see [21]) *Given a matrix $A \in \mathbb{C}^{n\times n}$, if $\widehat{A} \in D_n$ and is nonsingular, then A is nonsingular.*

LEMMA 3.2. (see [5, 9, 20]) *If $A \in H_n^{\bullet}$, then*

$$[\mu(A)]^{-1} \geq |A^{-1}| \geq 0.$$

LEMMA 3.3. (see [12]) *Let $A \in \mathbb{C}^{n\times n}$ and be partitioned as*

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{21} = (a_{21}, a_{31}, \cdots, a_{n1})^T$, $A_{12} = (a_{12}, a_{13}, \cdots, a_{1n})$. If A_{22} is nonsingular, then

$$\frac{det\ A}{det\ A_{22}} = a_{11} - (a_{12}, a_{13}, \cdots, a_{1n})[A_{22}]^{-1} \begin{pmatrix} a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

LEMMA 3.4. *Given a matrix $A \in D_n$ and a set $\alpha = N - \alpha' \subseteq N$, if $A(\gamma)$ is the largest diagonally equipotent principal submatrix of $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$, then $A/\alpha = A(\alpha' \cup \gamma')/\gamma'$, where*

$$A(\alpha' \cup \gamma') = \begin{bmatrix} A(\gamma') & A(\gamma', \alpha') \\ A(\alpha', \gamma') & A(\alpha') \end{bmatrix}. \tag{3.1}$$

Proof. If $\gamma = \alpha$, i.e., $A(\alpha)$ is diagonally equipotent and nonsingular, then it follows from Remark 2.5 that $A(\alpha, \alpha') = 0$. As a result,

$$A/\alpha = A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') = A(\alpha') = A(\alpha' \cup \emptyset)/\emptyset.$$

Now we consider the case when $\gamma \subset \alpha$. Since $A \in D_n$ and $A(\gamma)$ is diagonally equipotent, we have with Remark 2.5 that $A(\gamma, \gamma') = 0$ and $A(\gamma, \alpha') = 0$. Thus, there exists an $|\alpha| \times |\alpha|$ permutation matrix P_α such that

$$P_\alpha A(\alpha) P_\alpha^T = \begin{bmatrix} A(\gamma) & 0 \\ A(\gamma', \gamma) & A(\gamma') \end{bmatrix},$$

correspondingly

$$\begin{aligned} A(\alpha', \alpha)P_\alpha^T &= (A(\alpha', \gamma), A(\alpha', \gamma')) \qquad \textit{and} \\ P_\alpha A(\alpha, \alpha') &= \begin{pmatrix} A(\gamma, \alpha') \\ A(\gamma', \alpha') \end{pmatrix} = \begin{pmatrix} 0 \\ A(\gamma', \alpha') \end{pmatrix}. \end{aligned} \tag{3.2}$$

Since $A(\alpha)$ is nonsingular, $A(\gamma)$ and $A(\gamma')$ are both nonsingular. Therefore, we have

$$[A(\alpha)]^{-1} = P_\alpha^T \begin{bmatrix} [A(\gamma)]^{-1} & 0 \\ -[A(\gamma')]^{-1}A(\gamma', \gamma)[A(\gamma)]^{-1} & [A(\gamma')]^{-1} \end{bmatrix} P_\alpha. \tag{3.3}$$

Let $\mathscr{B} = \begin{bmatrix} [A(\gamma)]^{-1} & 0 \\ -[A(\gamma')]^{-1}A(\gamma', \gamma)[A(\gamma)]^{-1} & [A(\gamma')]^{-1} \end{bmatrix}$. Then $[A(\alpha)]^{-1} = P_\alpha^T \mathscr{B} P_\alpha$. As a consequence, it follows from (3.2) and (3.3) that

$$\begin{aligned} A/\alpha &= A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') \\ &= A(\alpha') - A(\alpha', \alpha)P_\alpha^T \mathscr{B} P_\alpha A(\alpha, \alpha') \\ &= A(\alpha') - (A(\alpha', \gamma), A(\alpha', \gamma'))\mathscr{B} \begin{bmatrix} 0 \\ A(\gamma', \alpha') \end{bmatrix} \\ &= A(\alpha') - A(\alpha', \gamma')[A(\gamma')]^{-1}A(\gamma', \alpha') \\ &= A(\alpha' \cup \gamma')/\gamma'. \end{aligned}$$

This completes the proof. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

LEMMA 3.5. (see [1, 5]) *Let $A \in \mathbb{R}^{n\times n}$. If $A \in M_n$, then det $A \geq 0$. Furthermore, if $A \in M_n^{\bullet}$, then det $A > 0$.*

LEMMA 3.6. (see [22]) *Let $A \in \mathbb{C}^{n\times n}$ and $\alpha \subset N$. If both A and $A(\alpha)$ are nonsingular, then A/α is also nonsingular.*

THEOREM 3.7. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, if $\widehat{A} \in D_n$ and is nonsingular, where $\widehat{A}$ is given in* (2.3)*, then $\widehat{A/\alpha} \in D_{n-|\alpha|}$ and is nonsingular for any given $\alpha \subset N$.*

Proof. We firstly prove the conclusion that $\widehat{A/\alpha} \in D_{n-|\alpha|}$ for any given $\alpha \subset N$ by proving the following two cases: (i) If $A(\alpha) \in H^{\bullet}_{|\alpha|}$, then $\widehat{A/\alpha} \in D_{n-|\alpha|}$; (ii) if $A(\alpha) \notin H^{\bullet}_{|\alpha|}$, then $\widehat{A/\alpha} \in D_{n-|\alpha|}$.

First, we prove case (i). Since $\widehat{A} \in D_n$ and is nonsingular, it follows from Lemma 3.1 that A is nonsingular. Thus, Lemma 2.9 indicates that $A(\alpha)$ is nonsingular. As a result, A/α exists. Assume $\alpha = \{i_1, i_2, \cdots, i_k\}$, $\alpha' = N-\alpha = \{j_1, j_2, \cdots, j_m\}$, $k+m = n$. Let $A/\alpha = (\widetilde{a}_{j_l,j_t})_{m\times m}$. According to definition (2.4) of the matrix A/α, we have the entries of the Schur complement A/α,

$$\widetilde{a}_{j_l,j_t} = a_{j_l,j_t} - \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_t} \\ a_{i_2,j_t} \\ \vdots \\ a_{i_k,j_t} \end{pmatrix}\right], \tag{3.4}$$

$$l, t = 1, 2, \cdots, m,$$

and the diagonal entries,

$$\widetilde{a}_{j_l,j_l} = a_{j_l,j_l} - \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix}\right], \tag{3.5}$$

$$l = 1, 2, \cdots, m.$$

Since $A(\alpha) \in H^{\bullet}_{|\alpha|}$, Lemma 3.2 gives

$$\{\mu[A(\alpha)]\}^{-1} \geq |[A(\alpha)]^{-1}| \geq 0. \tag{3.6}$$

Then from (3.4), (3.5), (3.6), and Lemma 3.3, we have

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

$$
\begin{aligned}
|Re(\widetilde{a}_{j_l,j_l})| \quad - \quad & \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| = \left| Re \left[a_{j_l,j_l} - (a_{j_l,i_1},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \right| \\
& - \sum_{i=1,i\neq l}^{m} \left| \left[a_{j_l,j_i} - (a_{j_l,i_1},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_i} \\ a_{i_2,j_i} \\ \vdots \\ a_{i_k,j_i} \end{pmatrix} \right] \right| \\
\geq \quad & |Re(a_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| \\
& - \sum_{i=1}^{m} \left| \left[(a_{j_l,i_1},a_{j_l,i_2},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_i} \\ a_{i_2,j_i} \\ \vdots \\ a_{i_k,j_i} \end{pmatrix} \right] \right| \\
\geq \quad & |Re(a_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| \\
& - \sum_{i=1}^{m} \left[(|a_{j_l,i_1}|,|a_{j_l,i_2}|,\cdots,|a_{j_l,i_k}|)|[A(\alpha)]^{-1}| \begin{pmatrix} |a_{i_1,j_i}| \\ |a_{i_2,j_i}| \\ \vdots \\ |a_{i_k,j_i}| \end{pmatrix} \right] \\
\geq \quad & |Re(a_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| \\
& - \sum_{i=1}^{m} \left[(|a_{j_l,i_1}|,|a_{j_l,i_2}|,\cdots,|a_{j_l,i_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1,j_i}| \\ |a_{i_2,j_i}| \\ \vdots \\ |a_{i_k,j_i}| \end{pmatrix} \right] \\
= \quad & \frac{detB_l}{det\mu[A(\alpha)]}, \quad l = 1,2,\cdots,m,
\end{aligned} \tag{3.7}
$$

where

$$
B_l = \begin{pmatrix} |Re(a_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| & h^T \\ g & \mu[A(\alpha)]) \end{pmatrix}_{(k+1)\times(k+1)},
$$

$$
g = (-\sum_{i=1}^{m} |a_{i_1,j_i}|,\cdots,-\sum_{i=1}^{m} |a_{i_k,j_i}|)^T,
$$

$$
h = (-|a_{j_l,i_1}|,\cdots,-|a_{j_l,i_k}|)^T.
$$

It is clear that $B_l \in Z_{k+1}$. Since $\widehat{A} \in D_n$, we have

$$
|Re(a_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |a_{j_l,j_i}| \geq \sum_{t=1}^{k} |a_{j_l,i_t}|. \tag{3.8}
$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

Since $A \in D_n$ for $\widehat{A} \in D_n$, we obtain

$$|a_{i_r,i_r}| \geq \sum_{t=1,t\neq r}^{k} |a_{i_r,i_t}| + \sum_{i=1}^{m} |a_{i_r,j_i}|,\ r = 1, 2, \cdots, k. \tag{3.9}$$

Inequalities (3.8) and (3.9) indicate that $B_l \in D_{k+1}$. Therefore, it follows from Remark 2.6 and Lemma 2.14 that $B_l \in H_{k+1} \cap Z_{k+1}$ and consequently $B_l \in M_{k+1}$. Lemma 3.5 gives $detB_l = det\mu(B_l) \geq 0$. Again, since $A(\alpha) \in H^{\bullet}_{|\alpha|}$, $\mu[A(\alpha)] \in M^{\bullet}_{|\alpha|}$. Using Lemma 3.5, we have $det\mu[A(\alpha)] > 0$. Thus, by (3.7), we have

$$|Re(\widetilde{a}_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| \geq \frac{detB_l}{det\mu[A(\alpha)]} \geq 0,\ l = 1, 2, \cdots, m, \tag{3.10}$$

and thus

$$|Re(\widetilde{a}_{j_l,j_l})| \geq \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}|,\ l = 1, 2, \cdots, m, \tag{3.11}$$

which shows $\widehat{A/\alpha} \in D_m$. This completes the proof of case (i).

Next, we prove case (ii): Assume $A(\alpha) \notin H^{\bullet}_{|\alpha|}$, it then follows from Lemma 2.10 that $A(\alpha)$ has at least one diagonally equipotent principal submatrix. Let $A(\gamma)$ be the largest diagonally equipotent principal submatrix of the matrix $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. Then $A(\gamma')$ has no diagonally equipotent principal submatrix and hence $A(\gamma') \in H^{\bullet}_{|\gamma'|}$ from Lemma 2.10. Since $A \in D_n$ for $\widehat{A} \in D_n$ and $A(\gamma)$ is the largest diagonally equipotent principal submatrix of the matrix $A(\alpha)$, it follows from Lemma 3.4 that $A/\alpha = A(\alpha' \cup \gamma')/\gamma'$, where $\alpha' = N - \alpha \subseteq N$ and $A(\alpha' \cup \gamma')$ is given in (3.1). Let $B = A(\alpha' \cup \gamma')$. Since $\widehat{B} = \widehat{A}(\alpha' \cup \gamma') \in D_{|\alpha' \cup \gamma'|}$ for $\widehat{A} \in D_n$ and $A(\gamma') \in H^{\bullet}_{|\gamma'|}$, it follows from the proof of case (i) that $\widehat{B/\gamma'} \in D_{|\alpha'|}$. Since $A/\alpha = A(\alpha' \cup \gamma')/\gamma' = B/\gamma'$, $\widehat{A/\alpha} = \widehat{B/\gamma'} \in D_{|\alpha'|}$, which shows that the proof of case (ii) is completed.

In what follows, the conclusion that $\widehat{A/\alpha}$ is nonsingular for any given $\alpha \subset N$ will be proved by proving the following two cases: (1) $\widehat{A/\alpha}$ is irreducible; otherwise (2), $\widehat{A/\alpha}$ is reducible.

We prove case (1) first. If $\widehat{A/\alpha}$ is irreducible and the inequality in (3.11) holds strictly for at least one l, $l = 1, 2, \cdots, m$, Lemma 2.7 yields that $\widehat{A/\alpha}$ is nonsingular. Otherwise, if $\widehat{A/\alpha}$ is irreducible and the equality in (3.11) holds for all l, $l = 1, 2, \cdots, m$, i.e.,

$$|Re(\widetilde{a}_{j_l,j_l})| - \sum_{i=1,i\neq l}^{m} |\widetilde{a}_{j_l,j_i}| = 0,\ l = 1, 2, \cdots, m, \tag{3.12}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

(3.12), (3.7), and (3.10) imply that

$$\begin{aligned} |Re(\widetilde{a}_{j_l,j_l})| &= \left| Re\left[a_{j_l,j_l} - (a_{j_l,i_1},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \right| \\ &= |Re(a_{j_l,j_l})| - \left| \left[(a_{j_l,i_1}, a_{j_l,i_2},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \right| \end{aligned}$$

and thus

$$\begin{aligned} Re(\widetilde{a}_{j_l,j_l}) &= Re\left[a_{j_l,j_l} - (a_{j_l,i_1},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \\ &= Re(a_{j_l,j_l}) - \left[(a_{j_l,i_1}, a_{j_l,i_2},\cdots,a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \end{aligned} \tag{3.13}$$

for all $l,\ l = 1, 2, \cdots, m$. Let $\breve{A} = A - E$, where $E = i \cdot diag(e_1, \cdots, e_n)$ with $i = \sqrt{-1}$, $e_s = Im(a_{ss})$ for $s \in \alpha' = N - \alpha$ and $e_s = 0$ for $s \in \alpha$. Therefore, (3.13) yields

$$\widehat{A/\alpha} = \breve{A}/\alpha.$$

Since $\widehat{A} \in D_n$ is nonsingular and $\widehat{\breve{A}} = \widehat{A}$, it follows from Lemma 3.1 that $\breve{A}$ is also nonsingular. Again, since $A(\alpha)$ is nonsingular, Lemma 3.6 shows that $\widehat{A/\alpha} = \breve{A}/\alpha$ is nonsingular. This completes the proof of case (i).

The following will prove case (2). If $\widehat{A/\alpha}$ is reducible, so is A/α. Then there exists an $m \times m$ permutation matrix P such that

$$\begin{aligned} P[A/\alpha]P^T &= PA(\alpha')P^T - PA(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha')P^T \\ &= \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ 0 & B_{22} & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{ss} \end{bmatrix}, \end{aligned} \tag{3.14}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

and correspondingly,

$$P[A(\alpha')]P^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix},$$

$$P[A(\alpha', \alpha)] = \begin{bmatrix} A_{10} & A_{20} & \cdots & A_{s0} \end{bmatrix}^T,$$

and

$$[A(\alpha, \alpha')]P^T = \begin{bmatrix} A_{01} & A_{02} & \cdots & A_{0s} \end{bmatrix},$$

where B_{ii} is irreducible for $i = 1, 2, \cdots, s$ $(s \geq 2)$, $B_{ij} = B(\alpha_i, \alpha_j)$ and $A_{ij} = A(\alpha_i, \alpha_j)$ are the submatrices of the matrix A and $A/\alpha = B$, respectively, with row indices in α_i and column indices in α_j, $\bigcup_{j=1}^{s} \alpha_j = \alpha'$ and $\alpha_i \cap \alpha_j = \emptyset$ for $i \neq j$, $i, j = 0, 1, 2, \cdots, s$ $(s \geq 2)$, $\alpha_0 = \alpha$. Using (3.14), direct calculation gives

$$B_{ii} = A_{ii} - A_{i0}[A(\alpha)]^{-1}A_{0i}, \quad i = 1, 2, \cdots, s \ (s \geq 2), \quad i = 1, 2, \cdots, s$$

which is the Schur complement of the matrix

$$A(\alpha \cup \alpha_i) = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha_i) \\ A(\alpha_i, \alpha) & A(\alpha_i) \end{bmatrix} = \begin{bmatrix} A(\alpha) & A_{0i} \\ A_{i0} & A_{ii} \end{bmatrix}$$

with respect to $A(\alpha_i) = A_{ii}$. Since $\widehat{A} \in D_n$ and is nonsingular, $\widehat{A}(\alpha \cup \alpha_i) \in D_{|\alpha \cup \alpha_i|}$ and is nonsingular. Again, since B_{ii} is irreducible, so is $\widehat{B}_{ii}$. It follows from the proof of case (1) that $\widehat{B}_{ii}$ is nonsingular for $i = 1, 2, \cdots, s$. By (3.14) we have

$$P[\widehat{A/\alpha}]P^T = \begin{bmatrix} \widehat{B}_{11} & B_{12} & \cdots & B_{1s} \\ 0 & \widehat{B}_{22} & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{B}_{ss} \end{bmatrix}. \tag{3.15}$$

The nonsingularity of $\widehat{B}_{ii}$ for all $i = 1, 2, \cdots, s$ and (3.15) show that $P[\widehat{A/\alpha}]P^T$ is nonsingular, so is $\widehat{A/\alpha}$, which shows that the proof of case (2) is completed. This completes the proof. □

THEOREM 3.8. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and a set $\alpha \subset N$, if $A_\alpha \in D_n$ and $A(\alpha)$ is nonsingular, where A_α is given in* (2.3)*, then $\widehat{A/\alpha} \in D_{n-|\alpha|}$ and is nonsingular.*

Proof. The proof on diagonal dominance of $\widehat{A/\alpha}$ follows the proof of Theorem 3.7. It follows from the definition of A_α that $A(\alpha) = A_\alpha(\alpha) \in D_{|\alpha|}$ for $A_\alpha \in D_n$. Then, if

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

$A(\alpha) \in H^{\bullet}_{|\alpha|}$, (3.7) holds. Furthermore, $A_\alpha \in D_n$ indicates that both (3.8) and (3.9) hold. As a result, (3.11) holds, which shows that $\widehat{A/\alpha} \in D_{n-|\alpha|}$. If $A(\alpha) \notin H^{\bullet}_{|\alpha|}$, the proof on diagonal dominance of $\widehat{A/\alpha}$ is similar to the proof of case (ii) in the proof Theorem 3.7.

In what follows, we will prove the nonsingularity of the matrix $\widehat{A/\alpha}$. Since A_α is nonsingular, Lemma 2.9 indicates $a_{ii} \neq 0$ for all $i \in \alpha$ and $Re(a_{ii}) \neq 0$ for all $i \in \alpha'$. Furthermore, since $A_\alpha \in D_n$, $A \in D_n$. Let $\alpha = \{i_1, i_2, \cdots, i_k\}$ and $\alpha' = \{j_1, j_2, \cdots, j_m\}$ with $k + m = n$. Definite a unitary diagonal matrix $U = \begin{bmatrix} U_\alpha & 0 \\ 0 & I_{\alpha'} \end{bmatrix}$, where $U_\alpha = diag(u_{i_1}, \cdots, u_{i_k})$, $u_i = \dfrac{\bar{a}_{ii}}{|a_{ii}|}$ for all $i \in \alpha$, $\bar{a}_{ii}$ is the conjugate complex number of the complex number a_{ii} and $I_{\alpha'}$ is the $|\alpha'| \times |\alpha'|$ identity matrix, such that $B = UA = \begin{bmatrix} U_\alpha A(\alpha) & U_\alpha A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha', \alpha') \end{bmatrix} \in D_n$. Therefore, $\widehat{B} = UA_\alpha = \begin{bmatrix} U_\alpha A(\alpha) & U_\alpha A(\alpha, \alpha') \\ A(\alpha', \alpha) & \widehat{A}(\alpha', \alpha') \end{bmatrix} \in D_n$ and is nonsingular for $A_\alpha \in D_n$ and is nonsingular. It follows from Theorem 3.7 that $\widehat{B/\alpha}$ is nonsingular. Since

$$\begin{aligned} A/\alpha &= A(\alpha') - A(\alpha', \alpha)[A(\alpha)]^{-1}A(\alpha, \alpha') \\ &= A(\alpha') - A(\alpha', \alpha)[U_\alpha A(\alpha)]^{-1}[U_\alpha A(\alpha, \alpha')] \\ &= [UA]/[U_\alpha A(\alpha)] \\ &= B/\alpha, \end{aligned}$$

$\widehat{A/\alpha}$ is also nonsingular. This completes the proof. □

Applying Theorem 3.7 and Theorem 3.8, we will in the next section establish some results on the eigenvalue distribution for the Schur complements of nonstrictly diagonally dominant matrices and general $H-$matrices.

4. The eigenvalue distribution on the Schur complement of nonstrictly diagonally dominant matrices and general $H-$matrices. This section follows [9], [10], [11], and [21], and continues to study the eigenvalue distribution on the Schur complements of some special matrices including nonstrictly diagonally dominant matrices and singular $H-$matrices. The results in [9] and [21] will be extended to the Schur complements for nonstrictly diagonally dominant matrices and general $H-$matrices with complex diagonal entries. The following lemma will be used in this section.

LEMMA 4.1. (see [21]) *Given a matrix $A \in \mathbb{C}^{n\times n}$, if $\widehat{A} \in D_n$ and is nonsingular, then A has $|J_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)|$ eigenvalues with negative real part, where $J_{R_+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in N\}$, $J_{R_-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in N\}$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

THEOREM 4.2. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and a set $\alpha \subset N$, if $\widehat{A} \in D_n$ and is nonsingular, where $\widehat{A}$ is given in* (2.3)*, then A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part, where $J_{R_+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in N\}$, $J_{R_-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in N\}$, $J^{\alpha}_{R_+}(A) = \{i \mid Re(a_{ii}) > 0,\ i \in \alpha\}$, $J^{\alpha}_{R_-}(A) = \{i \mid Re(a_{ii}) < 0,\ i \in \alpha\}$.*

Proof. The conclusion of this theorem will be proved by showing the results: (i) $\widehat{A/\alpha} \in D_m$ and is nonsingular; (ii) $Sign\ Re(\widetilde{a}_{j_l,j_l}) = Sign\ Re(a_{j_l,j_l})$, $l = 1, 2, \cdots, m$, where $m = n - |\alpha|$, $\widetilde{a}_{j_l,j_l}$ and a_{j_l,j_l} are the diagonal entries of the matrices A/α and A, respectively. With these two results, the conclusion of the theorem comes from the results of Lemma 4.1.

Since $\widehat{A} \in D_n$ and is nonsingular, it follows from Theorem 3.7 that $\widehat{A/\alpha} \in D_m$ and is nonsingular. This completes the proof of result (i). Next, we prove result (ii):

$$Sign\ Re(\widetilde{a}_{j_l,j_l}) = Sign\ Re(a_{j_l,j_l}),\ l = 1, 2, \cdots, m. \tag{4.1}$$

Assume $\alpha = \{i_1, i_2, \cdots, i_k\}$, $\alpha' = N - \alpha = \{j_1, j_2, \cdots, j_m\}$, $k + m = n$. Let $A/\alpha = (\widetilde{a}_{j_l,j_t})_{m\times m}$, then it follows from (2.4) that the diagonal entries of A/α,

$$\widetilde{a}_{j_l,j_l} = a_{j_l,j_l} - \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right], \quad l = 1, 2, \cdots, m.$$

Since $\widehat{A/\alpha} \in D_m$ and is nonsingular, it then from Lemma 2.9 that

$$\begin{aligned} Re(\widetilde{a}_{j_l,j_l}) &= Re(a_{j_l,j_l}) - \\ & Re\left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \neq 0 \end{aligned} \tag{4.2}$$

for $l = 1, 2, \cdots, m$. Therefore, equation (4.2) implies that (4.1) is true if

$$|Re(a_{j_l,j_l})| \geq \left| Re\left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \right|, \quad l = 1, 2, \cdots, m \tag{4.3}$$

holds. The inequality (4.3) will be proved by the following two cases.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

Case (I.) If $A(\alpha) \in H^{\bullet}_{|\alpha|}$, then Lemma 3.2 gives

$$\{\mu[A(\alpha)]\}^{-1} \geq |[A(\alpha)]^{-1}| \geq 0. \tag{4.4}$$

Using inequality (4.4) and Lemma 3.3, we have

$$\begin{aligned}
& |Re(a_{j_l,j_l})| - \left| Re \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \right| \\
\geq\ & |Re(a_{j_l,j_l})| - \left| (a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right| \\
\geq\ & |Re(a_{j_l,j_l})| - \left[(|a_{j_l,i_1}|, |a_{j_l,i_2}|, \cdots, |a_{j_l,i_k}|)|[A(\alpha)]^{-1}| \begin{pmatrix} |a_{i_1,j_l}| \\ |a_{i_2,j_l}| \\ \vdots \\ |a_{i_k,j_l}| \end{pmatrix} \right] \\
\geq\ & |Re(a_{j_l,j_l})| - \left[(|a_{j_l,i_1}|, |a_{j_l,i_2}|, \cdots, |a_{j_l,i_k}|)\{\mu[A(\alpha)]\}^{-1} \begin{pmatrix} |a_{i_1,j_l}| \\ |a_{i_2,j_l}| \\ \vdots \\ |a_{i_k,j_l}| \end{pmatrix} \right] \\
=\ & \frac{detC_l}{det\mu[A(\alpha)]}, \quad l = 1, 2, \cdots, m,
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
C_l &= \begin{pmatrix} |Re(a_{j_l,j_l})| & h_l^T \\ g_l & \mu[A(\alpha)] \end{pmatrix}_{(k+1)\times(k+1)}, \\
g_l &= (-|a_{i_1,j_l}|, \cdots, -|a_{i_k,j_l}|)^T, \\
h_l &= (-|a_{j_l,i_1}|, \cdots, -|a_{j_l,i_k}|)^T.
\end{aligned}$$

It is clear that the matrix $C_l \in Z_{k+1}$. Since $\widehat{A} \in D_n$ and hence $A \in D_n$, we have

$$|Re(a_{j_l,j_l})| \geq \sum_{t=1}^{k} |a_{j_l,i_t}|, \tag{4.6}$$

$$|a_{i_r,i_r}| \geq \sum_{t=1,t\neq r}^{k} |a_{i_r,i_t}| + |a_{i_r,j_l}|,\ r = 1, 2, \cdots, k. \tag{4.7}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

Inequalities (4.6) and (4.7) indicate that the matrix $C_l \in D_{k+1} \cap Z_{k+1}$. It then follows from Remark 2.6 and Lemma 2.14 that C_l is an $H-$matrix and thus an $M-$matrix. Hence $det\ C_l \geq 0$ coming from Lemma 3.5. Since $A(\alpha) \in H^{\bullet}_{|\alpha|}$, $\mu[A(\alpha)] \in M^{\bullet}_{|\alpha|}$ and it follows from Lemma 3.5 that $det\mu[A(\alpha)] > 0$, it follows from (4.5) that we have

$$|Re(a_{j_l,j_l})| - \left| Re \left[(a_{j_l,i_1}, a_{j_l,i_2}, \cdots, a_{j_l,i_k})[A(\alpha)]^{-1} \begin{pmatrix} a_{i_1,j_l} \\ a_{i_2,j_l} \\ \vdots \\ a_{i_k,j_l} \end{pmatrix} \right] \right| \geq 0,\ l = 1, 2, \cdots, m,$$

which proves (4.3), and hence (4.1) holds for $l = 1, 2, \cdots, m$.

Case (Π.) If $A(\alpha) \notin H^{\bullet}_{|\alpha|}$, it then follows from Lemma 2.10 that $A(\alpha)$ has at least one diagonally equipotent principal submatrix. Let $A(\gamma)$ be the largest diagonally equipotent principal submatrix of the matrix $A(\alpha)$ for $\gamma = \alpha - \gamma' \subseteq \alpha$. Then $A(\gamma')$ has no diagonally equipotent principal submatrix and hence $A(\gamma') \in H^{\bullet}_{|\gamma'|}$ from Lemma 2.10. Since $A \in D_n$ for $\widehat{A} \in D_n$ and $A(\gamma)$ is the largest diagonally equipotent principal submatrix of the matrix $A(\alpha)$, it follows from Lemma 3.4 that $A/\alpha = A(\alpha' \cup \gamma')/\gamma'$, where $\alpha' = N - \alpha \subseteq N$ and $A(\alpha' \cup \gamma')$ is given in (3.1). Assume $\gamma' = \{r_1, r_2, \cdots, r_s\} \subseteq \alpha$ $(s \leq k)$. Then the diagonal entries of $A/\alpha = A(\alpha' \cup \gamma')/\gamma'$ are

$$\widetilde{a}_{j_l,j_l} = a_{j_l,j_l} - \left[(a_{j_l,r_1}, a_{j_l,r_2}, \cdots, a_{j_l,r_s})[A(\gamma')]^{-1} \begin{pmatrix} a_{r_1,j_l} \\ a_{r_2,j_l} \\ \vdots \\ a_{r_s,j_l} \end{pmatrix} \right],$$
$$l = 1, 2, \cdots, m.$$

Since $\widehat{A/\alpha} \in D_m$ and is nonsingular, Lemma 2.9 indicates that

$$Re(\widetilde{a}_{j_l,j_l}) = Re(a_{j_l,j_l}) -$$

$$Re \left[(a_{j_l,r_1}, a_{j_l,r_2}, \cdots, a_{j_l,r_s})[A(\gamma')]^{-1} \begin{pmatrix} a_{r_1,j_l} \\ a_{r_2,j_l} \\ \vdots \\ a_{r_s,j_l} \end{pmatrix} \right] \neq 0 \tag{4.8}$$

for $l = 1, 2, \cdots, m$. Therefore, equation (4.8) implies that (4.1) is true if

$$|Re(a_{j_l,j_l})| \geq \left| Re \left[(a_{j_l,r_1}, a_{j_l,r_2}, \cdots, a_{j_l,r_s})[A(\gamma')]^{-1} \begin{pmatrix} a_{r_1,j_l} \\ a_{r_2,j_l} \\ \vdots \\ a_{r_s,j_l} \end{pmatrix} \right] \right|, \tag{4.9}$$
$$l = 1, 2, \cdots, m$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

holds. Similar to the proof of Case (I), we can prove that (4.9) holds and hence (4.1) also holds for $l = 1, 2, \cdots, m$.

Result (i) shows that the matrix A/α satisfies the conditions of Lemma 4.1. Applying Lemma 4.1 and the result (i) and (ii) give the conclusion that the matrix A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part. The proof is completed. □

LEMMA 4.3. (see [22, 23]) *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and a positive diagonal matrix $D = diag(d_1, \cdots, d_n)$. Let $B = D^{-1}AD$ and $\alpha \subset N$ given. If $A(\alpha)$ is nonsingular, then $B/\alpha = D^{-1}_{\alpha'}(A/\alpha)D_{\alpha'}$, where $\alpha' = N - \alpha = \{j_1, \cdots, j_m\}$ and $D_{\alpha'} = diag(d_{j_1}, \cdots, d_{j_m})$.*

THEOREM 4.4. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and a set $\alpha \subset N$, if $\widehat{A} \in H_n$ and is nonsingular, where $\widehat{A}$ is given in* (2.3)*, then A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part.*

Proof. Since $\widehat{A} \in H_n$, Lemma 2.14 and Lemma 2.11 indicate that there exists a positive diagonal matrix $D = diag(d_1, d_2, \cdots, d_n)$ such that $D^{-1}\widehat{A}D \in D_n$. Then, it follows from the definition of $\widehat{A}$ that $D^{-1}AD \in D_n$. Let $B = D^{-1}AD = (b_{ij})$, then $b_{ij} = d_i^{-1}a_{ij}d_j$, $i \neq j$ and $b_{ii} = a_{ii}$ for all $i, j \in N$. Thus, $|J_{R_+}(B)| - |J^{\alpha}_{R_+}(B)| = |J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ and $|J_{R_-}(B)| - |J^{\alpha}_{R_-}(B)| = |J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$. Since the matrix B satisfies the condition of Theorem 4.2, it follows that the matrix B/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part. Again, Lemma 4.3 gives

$$B/\alpha = D^{-1}_m(A/\alpha)D_m, \tag{4.10}$$

where $D_m = diag(d_{j_1}, d_{j_2}, \cdots, d_{j_m})$. Equation (4.10) implies that both the matrices B/α and A/α have the same number of eigenvalues with positive real part and the same number of eigenvalues with negative real part. This completes the proof. □

EXAMPLE 4.5. Given a matrix

$$A = \begin{bmatrix} 5-i & -1 & -1 & -1 & -1 & -1 \\ 1 & -6+i & -2 & -i & -1 & i \\ 2i & 1 & -10 & -2 & -3 & -2 \\ 0 & 0 & 0 & 6+3i & -2 & -4i \\ 0 & 0 & 0 & 2 & 8-5i & -6 \\ 0 & 0 & 0 & 3 & 1 & -4+3i \end{bmatrix}$$

and $\alpha = \{1, 3, 4\}$, we consider the eigenvalue distribution on the Schur complement matrix A/α.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

Since $\widehat{A} \in D_n \subset H_n$ and is nonsingular, it follows from Theorem 4.4 that A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)| = 3 - 2 = 1$ eigenvalue with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)| = 3 - 1 = 2$ eigenvalues with negative real part. In fact, by direct computations, it is easy to verify that A/α has an eigenvalue with positive real part and 2 eigenvalues with negative real part. This illustrates that the conclusion of Theorem 4.4 is true.

The fist equality in (2.2) implies that the following corollary holds.

COROLLARY 4.6. *Given a matrix $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ and a set $\alpha \subset N$, if $\widehat{A} \in H^0_n$ and is nonsingular, where $\widehat{A}$ is given in* (2.3)*, then A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part.*

In fact, the condition that "$\widehat{A} \in D_n$ and is nonsingular" in Theorem 4.2 and the condition that "$\widehat{A} \in H_n$ and is nonsingular" in Theorem 4.4 seem strict and can be weakened while the conclusions of the theorems still hold. The following theorems give the weakened conditions on the matrix A.

THEOREM 4.7. *Given a matrix $A \in \mathbb{C}^{n\times n}$ and a set $\alpha \subset N$, if $A_\alpha \in D_n$ and is nonsingular, where A_α is given in* (2.3)*, then A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part.*

Proof. The proof is similar to the proof of Theorem 4.2. Theorem 3.8 gives that $\widehat{A/\alpha} \in D_m$ and is nonsingular, which shows that the result (i) in the proof of Theorem 4.2 holds.

The definition of the matrix A_α and $A_\alpha \in D_n$ shows $A_\alpha(\alpha) = A(\alpha) \in D_{|\alpha|}$. If $A(\alpha) \in H^{\bullet}_{|\alpha|}$ and if $A(\alpha) \notin H^{\bullet}_{|\alpha|}$, we can prove that the result (ii) in the proof of Theorem 4.2 also holds by using the methods of the proof for Case (I) and Case (II) in the proof of Theorem 4.2, respectively.

Result (i) in the proof of Theorem 4.2 shows that the matrix A/α satisfies the conditions of Lemma 4.1. Applying Lemma 4.1 and the result (i) and (ii) in the proof of Theorem 4.2 give the conclusion that the matrix A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part. The proof is completed. □

THEOREM 4.8. *Given a matrix $A \in \mathbb{C}^{n\times n}$ and a set $\alpha \subset N$, if $A_\alpha \in H_n$ and is nonsingular, where A_α is given in* (2.3)*, then A/α has $|J_{R_+}(A)| - |J^{\alpha}_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^{\alpha}_{R_-}(A)|$ eigenvalues with negative real part.*

Proof. Similar to the proof of Theorem 4.4, the conclusion of this theorem can be obtained immediately from Lemma 2.14, Lemma 2.11, Lemma 4.3, and Theorem 4.7. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

EXAMPLE 4.9. Given a matrix

$$A = \begin{bmatrix} 3i & -1 & i & -1 \\ 1 & 2+i & 0 & -i \\ 2 & i & -5 & -2 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

and $\alpha = \{1\}$, we consider the eigenvalue distribution on the Schur complement matrix A/α.

Since $\widehat{A} \notin H_n$, we fail to get the eigenvalue distribution of the Schur complement matrix B/α by Theorem 4.4. However, $A_\alpha \in D_n \subset H_n$ and is nonsingular. It follows from Theorem 4.8 that A/α has $|J_{R_+}(A)| - |J^\alpha_{R_+}(A)| = 2 - 0 = 2$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^\alpha_{R_-}(A)| = 1 - 0 = 1$ eigenvalue with negative real part. In fact, by direct computations, it is easy to verify that A/α has 2 eigenvalues with positive real part and an eigenvalue with negative real part, which demonstrates that the conclusion of Theorem 4.8 is true.

COROLLARY 4.10. *Given a matrix $A \in \mathbb{C}^{n\times n}$ and a set $\alpha \subset N$, if $A_\alpha \in H^0_n$ and is nonsingular, where A_α is given in* (2.3)*, then A/α has $|J_{R_+}(A)| - |J^\alpha_{R_+}(A)|$ eigenvalues with positive real part and $|J_{R_-}(A)| - |J^\alpha_{R_-}(A)|$ eigenvalues with negative real part.*

REMARK 4.11. It follows from the former equality of (2.2) that Theorem 5.7 in [21] and Corollary 4.10 in this paper are special cases of Theorem 4.8. Besides, Example 4.9 and the relationship between the matrix $\widehat{A}$ and A_α show that Theorems 4.4 and Corollary 4.6 are also special cases of Theorem 4.8.

5. Conclusions. This paper studies the eigenvalue distribution for the Schur complements of nonstrictly diagonally dominant matrices and general $H-$matrices. Above all, different conditions on the matrix A and the set $\alpha \subseteq N$ are presented such that the matrix $\widehat{A/\alpha}$ is diagonally dominant and nonsingular. Then, the result of Zhang et al. in [21] is applied to establish some results about the eigenvalue distribution for the Schur complements of nonstrictly diagonally dominant matrices and general $H-$matrices with complex diagonal entries.

Acknowledgment. The first author would like to thank Professor Michele Benzi, Emory University, for his sincere help during the preparation and revision of this paper. The authors would like to thank the anonymous referees for their valuable comments and suggestions, which actually stimulated this work.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 801-820, December 2009

ELA
http://math.technion.ac.il/iic/ela

REFERENCES

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Academic Press, New York, 1979.
[2] R. Bru., C. Corral, I. Gimenez and J. Mas. Classes of general H-matrices. *Linear Algebra Appl.*, 429:2358–2366, 2008.
[3] D. Carlson and T. Markham. Schur complements of diagonally dominant matrices. *Czechoslovak Math. J.*, 29(104):246–251, 1979.
[4] Ljiljana Cvetković, Vladimir Kostić, Maja Kovačević, and Tomasz Szulc. Further results on $H-$matrices and their Schur complements. *Appl. Math. Comput.*, 198:506–510, 2008.
[5] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.
[6] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.
[7] T.-G. Lei, C.-W. Woo, J.-Z. Liu, and F. Zhang. On the Schur Complements of Diagonally Dominant Matrices. *Proceedings of the SIAM Conference on Applied Linear Algebra*, (http://www.siam.org/meetings/la03/proceedings/LeiTGWLZ.pdf), 2003.
[8] Jianzhou Liu and Yinghong Xu. The note on a method for judging nonsingular $M-$matrix and its parallel algorithm. *Numer. Math. J. Chinese Univ.*, 25:317–320, 2003.
[9] Jianzhou Liu and Yungqing Huang. Some properties on Schur complements of H-matrices and diagonally dominant matrices. *Linear Algebra Appl.*, 389:365–380, 2004.
[10] Jianzhou Liu and Fuzhen Zhang. Disc separation of the Schur complement of diagonally dominant matrices and determinantal bounds. *SIAM J. Matrix Anal. Appl.*, 27:665–674, 2005.
[11] Jianzhou Liu, Jicheng Li, Zhuohong Huang, and Xu Kong. Some properties of Schur complements and diagonal-Schur complements of diagonally dominant matrices. *Linear Algebra Appl.*, 428:1009–1030, 2008.
[12] A.W. Marshall and I. Olkin. *Inequalities: theory of majorization and its applications.* Academic Press, New York, 1979.
[13] A.N. Michel and R.K. Miller. *Qualitative analysis of large scale dynamical systems.*. Academic Press, New York, 1977.
[14] K.J. Palmer. A sufficient condition that a matrix have eigenvalues with nonzero real parts. *Linear Multilinear Algebra*, 7:43–45, 1979.
[15] D.D. Siljak. *Large-scale dynamic systems. Stability and structure.*. North-Holland Publishing Co., New York-Amsterdam, 1979.
[16] Wen-ting Tong. On the distribution of eigenvalues of some matrices. *Acta Math. Sinica*, 20:273–275, 1977.
[17] R.S. Varga. *Matrix Iterative Analysis, second edition.* Springer-Verlag, Berlin, 2000.
[18] Zhaoyong You and Lei Li. Distribution of the eigenvalues for conjugate generalized diagonally dominant matrices. *J. Math. Res. Exposition*, 9:309–310, 1989).
[19] Cheng-yi Zhang and Yao-tang Li. Diagonal Dominant Matrices and the Determining of $H-$matrices and $M-$matrices. *Guangxi Sci.*, 12:1161–164, 2005.
[20] Cheng-yi Zhang, Yao-tang Li, and Feng Chen. On Schur complements of block diagonally dominant matrices. *Linear Algebra Appl.*, 414:533–546, 2006.
[21] Cheng-yi Zhang, Chengxian Xu and Yao-tang Li. The eigenvalue distribution on Schur complements of H-matrices. *Linear Algebra Appl.*, 422:250-264, 2007.
[22] Cheng-yi Zhang, Shuanghua Luo, Chengxian Xu, and Hongying Jiang. Schur complements of generally diagonally dominant matrices and criterion for irreducibility of matrices. *Electron. J. Linear Algebra*, 18:69–87, 2009.
[23] Fuzhen Zhang, ed. *The Schur complement and its applications*. Springer-Verlag, New York, 2005.
[24] Jia Ju Zhang. Distributions of the eigenvalues of conjugate diagonal dominant matrices. *Acta Math. Sinica*, 23:544–546, 1980.