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THE STRUCTURE OF LINEAR PRESERVERS OF LEFT MATRIX MAJORIZATION ON $\mathbb{R}^P$ *

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Abstract. For vectors $X, Y \in \mathbb{R}^n$, Y is said to be left matrix majorized by X ($Y \prec_\ell X$) if for some row stochastic matrix R, $Y = RX$. A linear operator $T\colon \mathbb{R}^p \to \mathbb{R}^n$ is said to be a linear preserver of $\prec_\ell$ if $Y \prec_\ell X$ on $\mathbb{R}^p$ implies that $TY \prec_\ell TX$ on $\mathbb{R}^n$. The linear operators $T\colon \mathbb{R}^p \to \mathbb{R}^n$ ($n < p(p-1)$) which preserve $\prec_\ell$ have been characterized. In this paper, linear operators $T\colon \mathbb{R}^p \to \mathbb{R}^n$ which preserve $\prec_\ell$ are characterized without any condition on n and p.

Key words. Row stochastic matrix, Doubly stochastic matrix, Matrix majorization, Weak matrix majorization, Left (right) multivariate majorization, Linear preserver.

AMS subject classifications. 15A04, 15A21, 15A51.

1. Introduction. Let M_{nm} be the algebra of all $n \times m$ real matrices. A matrix $R = [r_{ij}] \in M_{nm}$ is called a *row stochastic* (resp., *row substochastic*) matrix if $r_{ij} \geq 0$ and $\Sigma_{k=1}^{m} r_{ik} = 1$ (resp., ≤ 1) for all i, j. For A, B in M_{nm}, A is said to be *left matrix majorized* by B ($A \prec_\ell B$), if $A = RB$ for some $n \times n$ row stochastic matrix R. These notions were introduced in [11]. If $A \prec_\ell B \prec_\ell A$, we write $A \sim_\ell B$. Let $T\colon \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. T is said to be a linear preserver of $\prec_\ell$ if $Y \prec_\ell X$ on $\mathbb{R}^p$ implies that $TY \prec_\ell TX$ on $\mathbb{R}^n$. For more information about types of majorization see [1], [5] and [10]; for their preservers see [2]-[4], [6] and [9].

We shall use the following conventions throughout the paper: Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a nonzero linear operator and let $[T] = [t_{ij}]$ denote the matrix representation of T with respect to the standard bases $\{e_1, e_2, \ldots, e_p\}$ of $\mathbb{R}^p$ and $\{f_1, f_2, \ldots, f_n\}$ of $\mathbb{R}^n$. If $p = 1$, then all linear operators on $\mathbb{R}^1$ are preservers of $\prec_\ell$. Thus, we assume $p \geq 2$. Let A_i be $m_i \times p$ matrices, $i = 1, \ldots, k$. We use the notation $[A_1/A_2/\ldots/A_k]$ to denote the corresponding $(m_1 + m_2 + \ldots + m_k) \times p$ matrix. We let $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^p$, and denote

$$\begin{aligned} a &:= \max\{\max T(e_1), \ldots, \max T(e_p)\}, \\ b &:= \min\{\min T(e_1), \ldots, \min T(e_p)\}. \end{aligned} \tag{1.1}$$

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Theorem 1.1. ([9, Theorem 2.2]) *Let* $T: \mathbb{R}^p \to \mathbb{R}^n$ *be a nonzero linear preserver of* $\prec_\ell$ *and suppose* $p \geq 2$. *Then* $p \leq n$, $b \leq 0 \leq a$ *and for each* $i \in \{1, \ldots, p\}$, $a = \max T(e_i)$ *and* $b = \min T(e_i)$. *In particular, every column of* $[T]$ *contains at least one entry equal to* a *and at least one entry equal to* b.

Definition 1.2. Let $T: \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. We denote by P_i (resp., N_i) the sum of the nonnegative (resp., non positive) entries in the i^{th} row of $[T]$. If all the entries in the i^{th} row are positive (resp., negative), we define $N_i = 0$ (resp., $P_i = 0$).

We know that T is a linear preserver of $\prec_\ell$ if and only if αT is also a linear preserver of $\prec_\ell$ for some nonzero real number α. Without loss of generality we make the following assumption.

Assumption 1.3. *Let* $T: \mathbb{R}^p \to \mathbb{R}^n$ *be a nonzero linear preserver of* $\prec_\ell$. *Let* a *and* b *be as in (1.1). We assume that* $0 \leq -b \leq 1 = a$.

Definition 1.4. Let P be the permutation matrix such that $P(e_i) = e_{i+1}$, $1 \leq i \leq p-1$, $P(e_p) = e_1$. Let I denote the $p \times p$ identity matrix, and let $r, s \in \mathbb{R}$ be such that $rs < 0$. Define the $p(p-1) \times p$ matrix $\mathcal{P}_p(r,s) = [P_1/P_2/\ldots/P_{p-1}]$, where $P_j = rI + sP^j$, for all $j = 1, 2, \ldots, p-1$. It is clear that up to a row permutation, the matrices $\mathcal{P}_p(r,s)$ and $\mathcal{P}_p(s,r)$ are equal. Also define $\mathcal{P}_p(r,0) := rI$, $\mathcal{P}_p(0,s) := sI$ and $\mathcal{P}_p(0,0)$ as a zero row.

The structure of all linear operators $T: M_{nm} \to M_{nm}$ preserving matrix majorizations was considered in [6, 7, 8]. Also the linear operators T from $\mathbb{R}^p$ to $\mathbb{R}^n$ that preserve the left matrix majorization $\prec_\ell$ were characterized in [9] for $n < p(p-1)$. In the present paper, we will characterize all linear preservers of $\prec_\ell$ mapping $\mathbb{R}^p$ to $\mathbb{R}^n$ without any additional conditions.

2. Left matrix majorization.

In this section we obtain a key condition that is necessary for $T : \mathbb{R}^p \to \mathbb{R}^n$ to be a linear preserver of $\prec_\ell$. We first need the following.

Lemma 2.1. *Let* $T : \mathbb{R}^p \to \mathbb{R}^n$ *be a linear operator such that* $\min T(Y) \leq \min T(X)$ *for all* $X \prec_\ell Y$. *Then* T *is a preserver of* $\prec_\ell$.

Proof. Let $X \prec_\ell Y$. It is enough to show that $\max T(X) \leq \max T(Y)$. Since $X \prec_\ell Y$, $-X \prec_\ell -Y$, and hence $\min T(-Y) \leq \min T(-X)$. This means that $\max T(X) \leq \max T(Y)$. Then T is a preserver of $\prec_\ell$. □

Remark 2.2. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and let a and b be as in Assumption 1.3. By Theorem 1.1 we know that in each column of $[T] = [t_{ij}]$ there is at least one entry equal to $a(= 1)$ and at least one entry equal to b. For $1 \leq k \leq p$,

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we define

$$I_k = \{i : 1 \le i \le n, t_{ik} = 1\}, \qquad J_k = \{j : 1 \le j \le n, t_{jk} = b\}.$$

Next we state the key theorem of this paper.

THEOREM 2.3. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and let a and b be as in Assumption 1.3. Then there exist $0 \le \alpha \le 1$ and $b \le \beta \le 0$ such that $\mathcal{P}_p(1,\beta)$ and $\mathcal{P}_p(\alpha, b)$ are submatrices of $[T]$, where $\mathcal{P}_p(r,s)$ is as in Definition 1.4.*

Proof. Let $1 \le k \le p$ be a fixed number and let I_k and J_k be as in Remark 2.2. Since T is a linear preserver of $\prec_\ell$, it follows that I_k and J_k are nonempty sets. Also $e_k + e_l \prec_\ell e_k, l \ne k$. Thus, the other entries in the i^{th} row, $i \in I_k$ (resp., j^{th} row, $j \in J_k$) are non positive (resp., nonnegative). Hence, $t_{il} \le 0,\ t_{jl} \ge 0,\ l \ne k,\ i \in I_k$, and $j \in J_k$. Let $\beta_k^i = \sum_{l \ne k} t_{il} \le 0,\ i \in I_k$ and $\alpha_k^j = \sum_{l \ne k} t_{jl} \ge 0,\ j \in J_k$. Set

$$\beta_k := \min\{\beta_k^i,\ i \in I_k\}, \quad \alpha_k := \max\{\alpha_k^j,\ j \in J_k\}. \tag{2.1}$$

Define $X_k = -(N+1)e_k + e$. Choose N_0 large enough such that for all $N \ge N_0$ and $1 \le i \le n$,

$$\min T(X_k) = -N + \beta_k \le -N t_{ik} + \sum_{l \ne k} t_{il} \le -Nb + \alpha_k = \max T(X_k). \tag{2.2}$$

We know that $X_k \sim_\ell X_r = -(N+1)e_r + e,\ 1 \le r \le p$ and T is a linear preserver of $\prec_\ell$. Hence by (2.2), $\alpha := \alpha_k = \alpha_r$ and $\beta := \beta_k = \beta_r, 1 \le r \le p$. Also, $X_k \sim_\ell -Ne_i + e_j,\ i \ne j$. For each $N \ge N_0$, there exists $1 \le h \le n$ such that $-Nt_{hi} + t_{hj} = \min T(-Ne_i + e_j) = \min T(X_k) = -N + \beta$ and for each $1 \le i \le p,\ 1 \le j \le p$ and $N \ge N_0$, there exists $1 \le h \le n$ such that $-N(1 - t_{hi}) = t_{hj} - \beta$. It follows that $t_{hi} = 1,\ t_{hj} = \beta$. Hence $\mathcal{P}_p(1,\beta)$ is a submatrix of $[T]$. Similarly, there exists N_1, such that for each $N \ge N_1$ there exists $1 \le h \le n$ so that $-Nt_{hi} + t_{hj} = \max T(-Ne_i + e_j) = \max T(X_k) = -Nb + \alpha$ and $-N(b - t_{hi}) = t_{hj} - \alpha$. Thus, $t_{hi} = b$ and $t_{hj} = \alpha$. Since $1 \le i \ne j \le p$ was arbitrary, $\mathcal{P}_p(b,\alpha)$ is a submatrix of $[T]$. Therefore, $\mathcal{P}_p(1,\beta)$ and $\mathcal{P}_p(b,\alpha)$ are submatrices of $[T]$. ∎

REMARK 2.4. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ and $\widehat{T} : \mathbb{R}^p \to \mathbb{R}^m$ be two linear operators such that $[T] = [T_1/T_2/\ldots/T_n]$ and let $[\widehat{T}] = [\widehat{T}_1/\widehat{T}_2/\ldots/\widehat{T}_m]$ be the matrix representation of these operators with respect to the standard basis. Let $\mathcal{R}(T) = \{T_1, T_2, \ldots, T_n\}$ be the set of all rows of $[T]$. If $\mathcal{R}(T) = \mathcal{R}(\widehat{T})$, then T preserves $\prec_\ell$ if and only if $\widehat{T}$ preserves $\prec_\ell$.

LEMMA 2.5. *Let T be a linear operator on $\mathbb{R}^p$. If $[T] = \mathcal{P}_p(\alpha,\beta)$, $\alpha\beta \le 0$, then T is a preserver of $\prec_\ell$.*

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Proof. Without loss of generality, let $\beta \leq 0 \leq \alpha$ and let $X = (x_1, \dots, x_p)^t, Y = (y_1, \dots, y_p)^t \in \mathbb{R}^p$ such that $X \prec_\ell Y$. Then $y_m = \min Y \leq x_i \leq \max Y = y_M$, for all $1 \leq i \leq p$. It is easy to check that $\alpha y_m + \beta y_M \leq \alpha x_i + \beta x_j$, for all $i \neq j \in \{1, \dots, p\}$, which implies $\min TY \leq \min TX$. Hence by Lemma 2.1, $TX \prec_\ell TY$. □

3. Left matrix majorization on $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator and let a, b, be as in Assumption 1.3. We consider the square $S = [b, 1] \times [b, 1]$ in $\mathbb{R}^2$.

Definition 3.1. Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/ \dots /T_n]$, where $T_i = (t_{i1}, t_{i2})$, $1 \leq i \leq n$. Define

$$\Delta := \text{Conv}\left(\{(t_{i1}, t_{i2}), (t_{i2}, t_{i1}), 1 \leq i \leq n\}\right) \subseteq \mathbb{R}^2.$$

Also, let $C(T)$ denote the set of all corners of Δ.

Lemma 3.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and $[T] = [T_1/ \dots /T_n]$, where $T_j = (t_{j1}, t_{j2})$, $1 \leq j \leq n$. If for some $1 \leq i \leq n$, $t_{i1}t_{i2} > 0$, then $T_i \notin C(T)$, where $C(T)$ is as in Definition 3.1.*

Proof. Assume that, if possible, there exists $1 \leq i \leq n$ such that $T_i \in C(T)$ and $t_{i1}t_{i2} > 0$. By Remark 2.4 we can assume that $[T]$ has no identical rows. Without loss of generality, we assume that there exist $1 \leq i \leq n$ and real numbers $m \leq M$ such that $t_{i1} > 0, t_{i2} > 0$ and $mt_{i1} + Mt_{i2} < mt_{j1} + Mt_{j2}, j \neq i$. Choose $\varepsilon > 0$ small enough so that $mt_{i1} + (M + \varepsilon)t_{i2} < mt_{j1} + (M + \varepsilon)t_{j2}, j \neq i$. Since $(m, M)^t \prec_\ell (m, M + \varepsilon)^t$, $T(m, M)^t \prec_\ell T(m, M + \varepsilon)^t$. But $\min(T(m, M + \varepsilon)^t) = mt_{i1} + (M + \varepsilon)t_{i2} > mt_{i1} + Mt_{i2} = \min(T(m, M)^t)$, a contradiction. □

Next we shall characterize all linear operators $T : \mathbb{R}^2 \to \mathbb{R}^n$ which preserve $\prec_\ell$.

Theorem 3.3. *Let $T : \mathbb{R}^2 \to \mathbb{R}^n$ be a linear operator. Then T is a linear preserver of $\prec_\ell$ if and only if $\mathcal{P}_2(x, y)$ is a submatrix of $[T]$ and $xy \leq 0$ for all $(x, y) \in C(T)$.*

Proof. Let T be a linear preserver of $\prec_\ell$ with $0 \leq -b \leq 1 = a$. Let $(x, y) \in C(T)$, then by Lemma 3.2, $xy \leq 0$. Without loss of generality, let $T_i = (t_{i1}, t_{i2}) \in C(T)$ and $t_{i1}t_{i2} \leq 0$. By Remark 2.4, we assume that $[T]$ has no identical rows. Then there exist real numbers $m, M \in \mathbb{R}$ such that $mt_{i1} + Mt_{i2} < mt_{j1} + Mt_{j2}, j \neq i$. Choose $\varepsilon_0 > 0$ small enough so that $(m-\varepsilon)t_{i1} + (M+\varepsilon)t_{i2} < (m-\varepsilon)t_{j1} + (M+\varepsilon)t_{j2}, j \neq i, 0 < \varepsilon \leq \varepsilon_0$. Since $(M + \varepsilon, m - \varepsilon)^t \sim_\ell (m - \varepsilon, M + \varepsilon)^t$, $T(M + \varepsilon, m - \varepsilon)^t \sim_\ell T(m - \varepsilon, M + \varepsilon)^t$. Hence, for all $0 < \varepsilon \leq \varepsilon_0$, there exist $1 \leq k \leq n$ such that $T_k = (t_{k1}, t_{k2}) \in C(T)$ and $(m - \varepsilon)t_{i1} + (M + \varepsilon)t_{i2} = \min T(m - \varepsilon, M + \varepsilon)^t = \min T(M + \varepsilon, m - \varepsilon)^t = (M + \varepsilon)t_{k1} + (m - \varepsilon)t_{k2}$. Since $k \in \{1, 2, \dots, n\}$ is a finite set, there exists k such that $t_{k1} = t_{i2}$ and $t_{k2} = t_{i1}$. Therefore, $\mathcal{P}_2(t_{i1}, t_{i2})$ is a submatrix of $[T]$.

Conversely, let $\mathcal{P}_2(x, y)$ be a submatrix of $[T]$ and suppose for all $(x, y) \in C(T)$,

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$xy \leq 0$. Define the linear operator $\widehat{T}$ on $\mathbb{R}^2$ such that $[\widehat{T}] = [\mathcal{P}_2(x_1, y_1)/\cdots/\mathcal{P}_2(x_r, y_r)]$, where $(x_i, y_i) \in C(T), 1 \leq i \leq r$. By elementary convex analysis, we know that $\max T(X) = \max \widehat{T}(X)$ and $\min T(X) = \min \widehat{T}(X)$ for all $X \in \mathbb{R}^2$. Hence it is enough to show that $\widehat{T}$ is a linear preserver of $\prec_\ell$. By Lemma 2.5, each $\mathcal{P}_2(x_i, y_i)$ is a linear preserver of $\prec_\ell$. Thus, $\widehat{T}$ is a linear preserver of $\prec_\ell$. □

4. Left matrix majorization on $\mathbb{R}^p$.

In this section we shall characterize all linear operators $T : \mathbb{R}^p \to \mathbb{R}^n$ which preserve $\prec_\ell$. We shall prove several lemmas and prove the main theorem of this paper.

Definition 4.1. Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator and let $[T] = [T_1/\ldots/T_n]$. Define

$$\Omega := \mathrm{Conv}(\{T_i = (t_{i1}, \ldots, t_{ip}), 1 \leq i \leq n\}) \subseteq \mathbb{R}^p.$$

Also, let $C(T)$ be the set of all corners of Ω.

Lemma 4.2. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and $[T] = [T_1/\ldots/T_n]$, where $T_i = (t_{i1}, t_{i2}, \ldots, t_{ip})$, $1 \leq i \leq n$. Suppose there exists $1 \leq i \leq n$ such that $t_{ij} > 0, \forall 1 \leq j \leq p$, or $t_{ij} < 0, \forall 1 \leq j \leq p$. Then $T_i \notin C(T)$, where $C(T)$ is as in Definition 4.1.*

Proof. Assume that, if possible, there exists $1 \leq i \leq n$ such that $T_i \in C(T)$ and $t_{ij} > 0$, for all $1 \leq j \leq p$, or $t_{ij} < 0$, for all $1 \leq j \leq p$. By Remark 2.4, without loss of generality, we can assume that $[T]$ has no identical rows and there exists $1 \leq i \leq n$ such that $t_{ij} > 0$, for all $1 \leq j \leq p$. Since $T_i \in C(T)$, there exists $X = (x_1, \ldots, x_p)^t$ such that $x_1 t_{i1} + x_2 t_{i2} + \cdots + x_p t_{ip} < x_1 t_{j1} + x_2 t_{j2} + \cdots + x_p t_{jp}, j \neq i$. Let $x_k = \max\{x_i, 1 \leq i \leq p\}$. Choose $\varepsilon > 0$ small enough so that $x_1 t_{i1} + \cdots + (x_k + \varepsilon) t_{ik} + \cdots + x_p t_{ip} < x_1 t_{j1} + \cdots + (x_k + \varepsilon) t_{jk} + \cdots + x_p t_{jp}, j \neq i$. Define $\widehat{X} = (x_1, \ldots, x_k + \varepsilon, \ldots, x_p)^t$. Since $t_{ik} > 0$, hence $\min T(X) = x_1 t_{i1} + x_2 t_{i2} + \cdots + x_p t_{ip} < x_1 t_{i1} + \cdots + (x_k + \varepsilon) t_{ik} + \cdots + x_p t_{ip} = \min T(\widehat{X})$. But $X \prec_\ell \widehat{X}$, a contradiction. □

Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator. Without loss of generality, we assume that $[T] = [T^p/T^n/\widetilde{T}]$, where all entries of T^p (resp., T^n) are positive (resp., negative) and each row of $\widetilde{T}$ has nonnegative and non positive entries.

Corollary 4.3. *Let T and $\widetilde{T}$ be as above. Then T preserves $\prec_\ell$ if and only if $C(T) = C(\widetilde{T})$ and $\widetilde{T}$ preserves $\prec_\ell$, where $C(T)$ is as in Definition 4.1.*

Proof. Let T preserve $\prec_\ell$. By Lemma 4.2, $C(T) = C(\widetilde{T})$. Thus, if $X \in \mathbb{R}^p$, then $\max T(X) = \max \widetilde{T}(X)$ and $\min T(X) = \min \widetilde{T}(X)$. Therefore $\widetilde{T}$ preserves $\prec_\ell$. Conversely, let $C(T) = C(\widetilde{T})$. Then $\max T(X) = \max \widetilde{T}(X)$ and $\min T(X) = \min \widetilde{T}(X)$. Since $\widetilde{T}$ preserves $\prec_\ell$, T preserves $\prec_\ell$. □

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Definition 4.4. Let $T:\mathbb{R}^p\to\mathbb{R}^n$ be a linear operator. Define

$$\Delta = \mathrm{Conv}(\{(P_i,N_i),(N_i,P_i): 1\le i\le n\}),$$

where P_i, N_i be as in (1.2). Let $E(T)=\{(P_i,N_i):(P_i,N_i)\ \text{ is a corner of }\ \Delta\}$. Let $1\le i\le n$, define $[i]=\{j:1\le j\le n, P_i=P_j\ \text{ and }\ \mathrm{N_i=N_j}\}$.

Lemma 4.5. *Let $T\colon\mathbb{R}^p\to\mathbb{R}^n$ be a linear preserver of $\prec_\ell$ and let $C(T),E(T)$ be as in Definitions 4.1, 4.4, respectively. If $(P_r,N_r)\in E(T)$ for some $1\le r\le n$, then there exists $k\in[r]$ such that $T_k\in C(T)$.*

Proof. Suppose there exist $1\le r\le n$ such that $(P_r,N_r)\in E(T)$. Then there exists $m\le M$ such that

$$P_r m + N_r M < P_j m + N_j M,\ \ j\notin[r]. \tag{4.1}$$

Let $X\in\mathbb{R}^p$ such that $\min(X)=m$ and $\max(X)=M$. Then there exists $1\le k\le n$ such that $\min TX=\sum_{l=1}^p t_{kl}x_l$. Hence

$$P_r m + N_r M \le P_k m + N_k M \le \sum_{l=1}^p t_{kl}x_l = \min T(X). \tag{4.2}$$

Define $Y\in\mathbb{R}^p$ by $y_l=m$, if $t_{rl}>0$ and $y_l=M$, if $t_{rl}\le 0$. Obviously $Y\prec_\ell X$. Since T preserves $\prec_\ell$, $TY\prec_\ell TX$ which implies that

$$P_k m + N_k M \le \sum_{l=1}^p t_{kl}x_l = \min TX \le \min TY \le P_r m + N_r M. \tag{4.3}$$

Now, by (4.2) and (4.3), we have $P_r m+N_r M = P_k m + N_k M$. Thus by (4.1), $k\in[r]$ and $\min TX=\sum_{l=1}^p t_{kl}x_l$. Hence $T_k\in C(T)$ for some $k\in[r]$. □

Next we state the main result in this paper.

Theorem 4.6. *Let T and $E(T)$ be as in Definition 4.4. Then T preserves $\prec_\ell$ if and only if $\mathcal{P}_p(\alpha,\beta)$ is a submatrix of $[T]$ for all $(\alpha,\beta)\in E(T)$.*

Proof. Let T be a preserver of $\prec_\ell$ and let $(P_r,N_r)\in E(T)$. Then there exists $m\le M$ such that $P_r m+N_r M < P_j m + N_j M,\ j\notin[r]$. Choose ε_0 small enough so that for all $0<\varepsilon<\varepsilon_0$,

$$P_r(m-\varepsilon)+N_r(M+\varepsilon) < P_j(m-\varepsilon)+N_j(M+\varepsilon),\qquad j\notin[r],$$

If $j\in[r]$, then $P_j=P_r$ and $N_j=N_r$. Thus

$$P_r(m-\varepsilon)+N_r(M+\varepsilon) \le P_j(m-\varepsilon)+N_j(M+\varepsilon),\qquad 1\le j\le n. \tag{4.4}$$

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Let $0 < \varepsilon < \varepsilon_0$, be fixed and let $X^\varepsilon = (x_1^\varepsilon, \ldots, x_p^\varepsilon)^t \in \mathbb{R}^p$ with $\min X^\varepsilon = m - \varepsilon$ and $\max X^\varepsilon = M + \varepsilon$. As in the proof of Lemma 4.5, there exists $k \in [r]$ such that

$$P_r(m-\varepsilon) + N_r(M+\varepsilon) = \min T(X^\varepsilon) = \sum_{l=1}^{p} t_{kl} x_l^\varepsilon.$$

Fix $i \neq j \in \{1, \ldots, p\}$ and define $Y^\varepsilon = (y_1^\varepsilon, \ldots, y_p^\varepsilon)^t \in \mathbb{R}^p$ such that $y_i^\varepsilon = m - \varepsilon$, $y_j^\varepsilon = M + \varepsilon$ and $y_l^\varepsilon = \gamma_l$, $\ m - \varepsilon < \gamma_l < M + \varepsilon, l \neq i, j$. Since $X^\varepsilon \sim_\ell Y^\varepsilon$, $TX^\varepsilon \sim_\ell TY^\varepsilon$, there exists $q \in [r]$ such that $t_{qi}(m-\varepsilon) + t_{qj}(M+\varepsilon) + \sum_{l \neq i,j} \gamma_l t_{ql} = P_r(m-\varepsilon) + N_r(M+\varepsilon)$. Since $0 < \varepsilon < \varepsilon_0$ and $m - \varepsilon \leq \gamma_l \leq M + \varepsilon$, $l \neq r, s$ are arbitrary, it is easy to show that there exists $s \in [r]$ such that $t_{si} = P_r$ and $t_{sj} = N_r$ and $t_{sl} = 0, l \neq i, j$. Therefore $[T]$ has $\mathcal{P}_p(P_r, N_r)$ as a submatrix.

Conversely, Let $E(T) = \{(P_{i_1}, N_{i_1}), \ldots, (P_{i_s}, N_{i_s})\}$. Then up to a row permutation $[T] = [\mathcal{P}_p(P_{i_1}, N_{i_1})/\ldots/\mathcal{P}_p(P_{i_s}, N_{i_s})/Q]$.

Let $\widehat{T}$ be the operator on $\mathbb{R}^p$ such that $[\widehat{T}] = [\mathcal{P}_p(P_{i_1}, N_{i_1})/\ldots/\mathcal{P}_p(P_{i_k}, N_{i_k})]$. Let $T_i \in Q$ and suppose there exists $X \in \mathbb{R}^p$ such that

$$\min T(X) = \sum_{l=1}^{p} t_{il} x_l \leq \sum_{l=1}^{p} t_{jl} x_l, 1 \leq j \leq n.$$

Obviously, $P_i m + N_i M \leq \sum_{l=1}^p t_{il} x_l \leq \sum_{l=1}^p t_{jl} x_l, 1 \leq j \leq n$, where $m = \min X$ and $M = \max X$. We know that $(P_i, N_i) \in \Delta$ and Δ is convex. Hence there is $1 \leq k \leq n$ such that $(P_k, N_k) \in E(T)$ and $P_k m + N_k M \leq P_i m + N_i M$. As in the proof of Lemma 4.5, $\min TX = P_k m + N_k M$. Then $\min \widehat{T} X \leq \min TX$. But we know that $\min T(X) \leq \min \widehat{T} X$ and thus $\min \widehat{T} X = \min TX$. Similarly, $\max \widehat{T} X = \max TX$. Therefore, T is a preserver of $\prec_\ell$ if and only if $\widehat{T}$ preserves $\prec_\ell$. By Lemma 2.5 each $\mathcal{P}_p(P_{i_l}, N_{i_l})$ is a preserver of $\prec_\ell, 1 \leq l \leq k$. Hence $\widehat{T}$ is a preserver of $\prec_\ell$ and the theorem is proved. ☐

Next we state necessary conditions for $T : \mathbb{R}^p \to \mathbb{R}^n$ to be a linear preserver of $\prec_\ell$. We use the notation of Theorem 2.3 in the following corollary.

COROLLARY 4.7. *Let $T : \mathbb{R}^p \to \mathbb{R}^n$ be a linear operator and let a and b be as given in (1.1). If the following conditions hold, then T is a linear preserver of $\prec_\ell$.*

- *$[T]$ has $[\mathcal{P}_p(a, 0)/\mathcal{P}_p(0, b)/\mathcal{P}_p(a, b)]$ as a submatrix.*
- *$0 \leq P_i \leq a$ and $b \leq N_i \leq 0, \ \ 1 \leq i \leq n$,*

where P_i and N_i, $1 \leq i \leq n$ are as in Definition 1.2.

Proof. It is clear that $E(T) = \{(a, 0), (0, b), (a, b)\}$. Since $[T]$ has $\mathcal{P}_p(a, 0), \mathcal{P}_p(0, b)$ and $\mathcal{P}_p(a, b)$ as submatrices, it follows by Theorem 4.6 that T is a linear preserver of $\prec_\ell$. ☐

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Let $T:\mathbb{R}^p \to \mathbb{R}^n$ be a linear preserver of $\prec_\ell$, and let $[T]=[T^1|T^2|\ldots|T^p]$, where T^i is the i^{th} column of $[T]$. For $i\neq j\in\{1,\ldots,p\}$ define $T^{ij}:\mathbb{R}^2\to\mathbb{R}^n$ such that $[T^{ij}]=[T^i|T^j]$.

LEMMA 4.8. *Let $T:\mathbb{R}^p\to\mathbb{R}^n$ be a linear preserver of $\prec_\ell$, and let T^{ij} be as above. Then T^{ij} is a linear preserver of $\prec_\ell$ for all $i\neq j\in\{1,\ldots,p\}$.*

Proof. Let $i\neq j\in\{1,\ldots,p\}$ and let $x=(x_1,x_2)^t$, $y=(y_1,y_2)^t\in\mathbb{R}^2$ such that $x\prec_\ell y$. Define $X,Y\in\mathbb{R}^p$ such that $X_i=x_1$, $X_j=x_2$, $Y_i=y_1$, $Y_j=y_2$ and $X_k=Y_k=0$, for all $k\neq i,j$. It is obvious that $X\prec_\ell Y$ in $\mathbb{R}^p$ and hence $TX\prec_\ell TY$ in $\mathbb{R}^n$. But $T^{ij}x=x_1T^i+x_2T^j=TX\prec_\ell TY=y_1T^i+y_2T^j=T^{ij}y$. Therefore, T^{ij} is a linear preserver of $\prec_\ell$. ◻

The following example shows that the converse of Lemma 4.8 is not necessarily true.

EXAMPLE 4.9. Assume $[T]=[\mathcal{P}_3(1,-0.5)/\ 0.25\ 0.25\ 0.25]$. Consider $X=(-1,-1,-1)^t$ and $Y=(-1,-1,-0.75)^t$, we know that $X\prec_\ell Y$ and $\min TX<\min TY$. Thus T is not a linear preserver of $\prec_\ell$. However, by Corollary 4.7, for all $i\neq j\in\{1,2,3\}, T^{ij}$ preserves $\prec_\ell$.

5. Additional results. In this section we give short proofs of some Theorems from [6, 9].

THEOREM 5.1. [6] *Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be a linear operator. Then T preserves $\prec_\ell$ if and only if T has the form $T(X)=(aI+bP)X$ for all $X\in\mathbb{R}^2$, where P is the 2×2 permutation matrix not equal to I, and $ab\le 0$.*

Proof. Let T be a preserver of $\prec_\ell$. By Assumption 1.3, $a=1$. By Theorem 2.3, there exist $0\le\alpha\le 1$ and $b\le\beta\le 0$ such that $P(1,\beta)$ and $P(b,\alpha)$ are submatrices of $[T]$. Since $[T]$ is a 2×2 matrix, $\beta=b$ and $\alpha=1$. Therefore, $[T]=\begin{bmatrix}1 & b\\ b & 1\end{bmatrix}$ and hence $T(X)=(I+bP)X$, for all $X\in\mathbb{R}^2$. Conversely, up to a row permutation, $[T]=\mathcal{P}_2(1,b)$ and by Lemma 2.5, T preserves $\prec_\ell$. ◻

THEOREM 5.2. [6] *Let $p\ge 3$. Then $T:\mathbb{R}^p\to\mathbb{R}^p$ is a linear preserver of left matrix majorization if and only if T is of the form $X\mapsto aPX$ for some $a\in\mathbb{R}$ and some permutation matrix P.*

Proof. By Assumption 1.3, we have $a=1$. Let T be a preserver of $\prec_\ell$. By Theorem 2.3, $b=0$ and $[T]$ has $\mathcal{P}_p(1,0)$ as a submatrix; hence, up to a row permutation, $[T]=\mathcal{P}_p(1,0)=I$. Conversely, by a row permutation, $[T]=\mathcal{P}_p(1,0)$; hence by Lemma 2.5, T preserves $\prec_\ell$. ◻

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THEOREM 5.3. ([9, Theorem 3.1]) *For a linear preserver T of $\mathbb{R}^p$ to $\mathbb{R}^n$ the following assertions hold.*

(a) *If $n < 2p$ and $p \geq 3$, then T is nonnegative.*

(b) *If T is nonnegative, then there exists an $n \times n$ permutation matrix Q such that $[T] = Q[I/W]$, where W is a (possibly vacuous) $(n-p) \times p$ matrix of one of the following forms (i), (ii) or (iii):*

(i) *W is row stochastic;*

(ii) *W is row substochastic and has a zero row;*

(iii) *$W = [(cI)/B]$, where $0 < c < 1$ and B is an $(n-2p) \times p$ row substochastic matrix with row sums at least c.*

(c) *Let Q be an $n \times n$ permutation matrix, and let W be an $(n-p) \times p$ matrix of the form (i), (ii), or (iii) in part (b). Then the operator $X \mapsto Q[X/(WX)]$ from $\mathbb{R}^p$ into $\mathbb{R}^n$ is a nonnegative linear preserver of $\prec_\ell$.*

Proof.

(a) Assume that, if possible, $b < 0$. By Theorem 2.3 $n \geq p(p-1)$. Since $p \geq 3, n \geq 2p$, a contradiction.

(b) Since T is nonnegative, $N_i = 0, 1 \leq i \leq n$, and $0 \leq P_i \leq 1$. By Theorem 2.3, $[T]$ has $\mathcal{P}_p(1,0)$ as its submatrix and therefore up to a row permutation $[T] = [I/W]$. Let $c = \min\{P_i, 1 \leq i \leq n\}$. Then $E(T) = \{(1,0),(c,0)\}$. By Theorem 4.6, $\mathcal{P}_p(c,0)$ is a submatrix of $[T]$. If $c = 1$ then (i) holds; if $c = 0$ then (ii) holds and if $0 < c < 1$, then (iii) holds.

(c) Let $[T] = [I/W]$, where W is an $(n-p) \times p$ matrix of the form (i), (ii), or (iii) in part (b). Then $E(T) = \{(1,0),(c,0)\}$. By Theorem 4.6, T is a nonnegative linear preserver of $\prec_\ell$. □

THEOREM 5.4. ([9, Theorem 4.5]) *Assume $T : \mathbb{R}^p \to \mathbb{R}^n$ is a linear preserver of $\prec_\ell$, $b < 0$ and $2p \leq n < p(p-1)$. Let P_i (resp., N_i) denote the sum of the positive (resp., negative) entries of the i^{th} row of $[T]$. Then, up to a row permutation, $[T] = [I/bI/B]$ and $\min(N_i + bP_i) = b$, $(i = 1, 2, \ldots, n)$.*

Proof. By Theorem 2.3, $\mathcal{P}_p(1,\beta)$ and $\mathcal{P}_p(\alpha,b)$ are submatrices of $[T]$. Since $n < p(p-1)$, $\beta = \alpha = 0$ and $E(T) = \{(1,0),(0,b)\}$, where $E(T)$ is as in Definition4.4. Then up to a row permutation, $[T] = [I/bI/B]$ and $\min\{(bx+y) : (x,y) \in \Delta\} = \min\{(bx+y) : (x,y) \in E(T)\} = b$. Therefore, $\min(N_i + bP_i) = b$, $(i = 1, 2, \ldots, n)$. □

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