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NUMERICAL RANGES OF AN OPERATOR ON AN INDEFINITE INNER PRODUCT SPACE*

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Dedicated to Chandler Davis

Abstract. For $n \times n$ complex matrices A and an $n \times n$ Hermitian matrix S, we consider the *S-numerical range* of A and the *positive S-numerical range* of A defined by

$$W_S(A) = \left\{ \frac{\langle Av, v\rangle_S}{\langle v, v\rangle_S} : v \in \mathbb{C}^n, \langle v, v\rangle_S \neq 0 \right\}$$

and

$$W_S^+(A) = \{ \langle Av, v\rangle_S : v \in \mathbb{C}^n, \langle v, v\rangle_S = 1 \},$$

respectively, where $\langle u, v\rangle_S = v^* S u$. These sets generalize the classical numerical range, and they are closely related to the joint numerical range of three Hermitian forms and the cone generated by it. Using some theory of the joint numerical range we can give a detailed description of $W_S(A)$ and $W_S^+(A)$ for arbitrary Hermitian matrices S. In particular, it is shown that $W_S^+(A)$ is always convex and $W_S(A)$ is always p-convex for all S. Similar results are obtained for the sets

$$V_S(A) = \left\{ \frac{\langle Av, v\rangle}{\langle Sv, v\rangle} : v \in \mathbb{C}^n, \langle Sv, v\rangle \neq 0 \right\}, \quad V_S^+(A) = \{ \langle Av, v\rangle : v \in \mathbb{C}^n, \langle Sv, v\rangle = 1 \},$$

where $\langle u, v\rangle = v^* u$. Furthermore, we characterize those linear operators preserving $W_S(A)$, $W_S^+(A)$, $V_S(A)$, or $V_S^+(A)$. Possible generalizations of our results, including their extensions to bounded linear operators on an infinite dimensional Hilbert or Krein space, are discussed.

Key words. Field of values, numerical range, generalized numerical range, Krein space, convexity, linear preserver, indefinite inner product space.

AMS(MOS) subject classification. 15A60, 15A63, 15A45, 46C20, 52A30

1. Introduction.

Let $\mathbf{M}_n$ be the algebra of $n \times n$ complex matrices. The (*classical*) *numerical range* of $A \in \mathbf{M}_n$ is defined as

$$W(A) := \{ \langle Av, v\rangle : v \in \mathbb{C}^n, \langle v, v\rangle = 1 \} = \left\{ \frac{\langle Av, v\rangle}{\langle v, v\rangle} : v \in \mathbb{C}^n,\ \langle v, v\rangle \neq 0 \right\},$$

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where $\langle u, v\rangle := v^*u$ denotes the Euclidean inner product in $\mathbb{C}^n$. The numerical range is a useful tool for studying matrices and operators, and it has been investigated extensively. There are many generalizations of the concept motivated by both pure and applied topics. For the background of this subject we refer the reader to, e.g., [HJ, Chapter 1] and [AL].

Let $\mathbf{H}_n$ denote the set of all $n \times n$ Hermitian matrices. By replacing the Euclidean inner product with another inner product $\langle u, v\rangle_S$ in the definition of $W(A)$, where $S \in \mathbf{H}_n$ is positive definite and $\langle u, v\rangle_S := v^*Su$, one obtains the *S-numerical range* of A:

$$W_S(A) := \left\{ \frac{\langle Av, v\rangle_S}{\langle v, v\rangle_S} : v \in \mathbb{C}^n,\ \langle v, v\rangle_S \neq 0 \right\},$$

which coincides with the *positive S-numerical range* of A:

$$W_S^+(A) := \{\langle Av, v\rangle_S : v \in \mathbb{C}^n,\ \langle v, v\rangle_S = 1\}.$$

Since S is positive definite, $S = X^*X$ for some nonsingular X, and it is easy to show that $W_S(A) = W_S^+(A) = W(XAX^{-1})$. In particular, if $S^{\frac{1}{2}}$ denotes the (unique) positive definite matrix that satisfies $(S^{\frac{1}{2}})^2 = S$, then $W_S(A) = W(S^{\frac{1}{2}}AS^{-\frac{1}{2}})$, where $S^{-\frac{1}{2}}$ denotes the inverse of $S^{\frac{1}{2}}$. Hence many properties of the classical numerical range (such as the compactness, convexity, etc.) can be extended to $W_S(A) = W_S^+(A)$ if S is positive definite. However, if S is nonsingular and indefinite, i.e., if $\mathbb{C}^n$ becomes an indefinite inner product space with respect to its indefinite inner product $\langle\cdot,\cdot\rangle_S$, then $W_S(A) \neq W_S^+(A)$ in general, and the properties of these sets might be quite different from those of the classical numerical range $W(A)$ and certainly worth investigating.

It is not difficult to verify that for any $S \in \mathbf{H}_n$,

$$W_S(A) = W_S^+(A) \cup W_{-S}^+(A).$$

Some authors use $W_S^+(A)$ as the definition for a numerical range of A in indefinite inner product spaces. For example, Bayasgalan [B] has done so and has shown that the set $W_S^+(A)$ is convex if S is nonsingular and indefinite. However, we think that the lack of symmetry in the definition of $W_S^+(A)$ may limit its usefulness. In fact in [B], the author later switches to the set $W_S(A)$ and shows that its closure contains the spectrum of A if A is positive definite. In this paper, we shall study both $W_S(A)$ and $W_S^+(A)$.

Since $W_S(A)$ and $W_S^+(A)$ are well-defined even if S is singular, one might be interested to learn the general structure and geometrical shape of these sets for various kinds of $S \in \mathbf{H}_n$. In fact, the convexity of $W_S^+(A)$ has been proven for all positive semidefinite matrices S in [GP] under a slightly different setup.

In addition to $W_S(A)$ and $W_S^+(A)$, we also consider the sets

$$V_S(A) := \left\{ \frac{\langle Av, v\rangle}{\langle Sv, v\rangle} : v \in \mathbb{C}^n, \langle Sv, v\rangle \neq 0 \right\}$$

and

$$V_S^+(A) := \{\langle Av, v\rangle : v \in \mathbb{C}^n, \langle Sv, v\rangle = 1\},$$

which have been studied by other researchers (e.g., see [GP]). Evidently, we have

$$W_S(A) = V_S(SA), \quad W_S^+(A) = V_S^+(SA), \quad \text{and} \quad V_S(A) = V_S^+(A) \cup -V_{-S}^+(A).$$

To study the sets $W_S(A), W_S^+(A), V_S(A)$ and $V_S^+(A)$, we use an approach which is different from that in [B] and [GP]. Our work is based on the realization that these sets are closely related to the *joint numerical range* of k Hermitian matrices $H_1, \ldots, H_k$ defined by

$$W(H_1, \ldots, H_k) := \{(\langle H_1 v, v\rangle, \ldots, \langle H_k v, v\rangle) : v \in \mathbb{C}^n, \langle v, v\rangle = 1\},$$

with $k = 3$. This joint numerical range is another generalization of the classical numerical range and is well studied (e.g., see [AT], [BL], [C], [Ju] and their references). More precisely, the relationship between the S-numerical ranges and the joint numerical range is illustrated in the following result, the verification of which is straightforward.

LEMMA 1.1. *For any $A \in \mathbf{M}_n$ and $S \in \mathbf{H}_n$, let $SA = H + iG$ with $H, G \in \mathbf{H}_n$, and define*

$$K(H,G,S) := \bigcup_{\alpha \geq 0} \alpha W(H,G,S) = \{(\langle Hv, v\rangle, \langle Gv, v\rangle, \langle Sv, v\rangle) : v \in \mathbb{C}^n\}. \tag{1}$$

Then,
(a) $x+iy \in W_S(A)$ if and only if either $(x,y,1)$ or $-(x,y,1)$ lies in $K(H,G,S)$,
(b) $x + iy \in W_S^+(A)$ if and only if $(x,y,1) \in K(H,G,S)$.

It is easy to see that the same statements (a) and (b) of Lemma 1.1 are true when $A = H + iG$ for $W_S(A)$ replaced by $V_S(A)$ and $W_S^+(A)$ by $V_S^+(A)$.

The idea of slicing the set $K(H,G,S)$ to generate the numerical range of an operator also appeared in [Fo] and a number of references ([BNa], [D1], [D2], and some of their references) are relevant to our study. In fact, Brickman [Br] has pointed out that the joint numerical ranges of three Hermitian forms have been of interest since the time of Hausdorff.

The set $K(H,G,S)$ defined in formula (1) is clearly a cone in $\mathbb{R}^3$ (recall that a subset K of a real linear space is called a *cone* if $\alpha x \in K$ whenever $\alpha > 0$ and $x \in K$). It is known (e.g., see [AT]) that, for any $H, G, S \in \mathbf{H}_n$, the set $W(H,G,S)$ is always convex if $n > 2$, and it is a (possibly degenerate) elliptical shell (i.e., the surface of an ellipsoid) if $n = 2$. Quite simply one can deduce the following result.

LEMMA 1.2. *For any $H, G, S \in \mathbf{H}_n$, the set $K(H,G,S)$ defined in (1) is a convex cone.*

By using Lemmas 1.1 and 1.2 and other auxiliary results, we shall give detailed descriptions of the sets $W_S(A), W_S^+(A), V_S(A)$ and $V_S^+(A)$ for arbitrary Hermitian matrices S in Section 2. The results are used to extend some properties of the classical numerical range.

Linear operators L on $\mathbf{M}_n$ preserving $W_S(A)$, $W_S^+(A)$, $V_S(A)$, or $V_S^+(A)$ will be characterized in Section 3. The study of linear operators on a matrix space with preserver properties is currently an active research area in matrix theory. The readers are referred to [Pi] for a recent survey on the subject. The proofs of our results rely heavily on the geometrical properties of the S-numerical ranges as developed in Section 2. This indicates that a good understanding of the geometrical properties of S-numerical ranges is important for the study of related problems and further development of the subject.

In the last section, we consider extension of our results to bounded linear operators on infinite dimensional Hilbert or Krein spaces. We believe that the approach of studying numerical ranges of operators on an indefinite inner product space via the joint numerical range may lead to a deeper understanding and more insights of the subject. Several open problems are presented along this direction.

2. Geometrical Structure of the S-numerical Ranges. As shall be seen (cf. Theorems 2.3 and 2.4), $W_S^+(A)$ and $V_S^+(A)$ are always convex, but $W_S(A)$ and $V_S(A)$ are not. Nevertheless, $W_S(A)$ and $V_S(A)$ do enjoy some nice geometrical properties. The following describes one such property. We say that a nonempty subset X of $\mathbb{R}^m$ is *p-convex* if for any distinct pair of points $x, y \in X$, either

$$\{\alpha x + (1-\alpha)y : 0 \le \alpha \le 1\} \ \subset \ X$$

or

$$\{\alpha x + (1-\alpha)y : \alpha \le 0 \text{ or } 1 \le \alpha\} \ \subset \ X.$$

In other words, either the closed line segment $[x, y]$ joining any two points x and y in X is contained in X, or the line that passes through x and y, less the relative interior of $[x, y]$, is contained in X. We need the following two lemmas to prove Theorems 2.3 and 2.4 below.

LEMMA 2.1. *Let K be a nonempty convex cone in $\mathbb{R}^{m+1}$ and define*

$$P^+ := \{x \in \mathbb{R}^m : (x,1) \in K\}, \quad P^- := \{x \in \mathbb{R}^m : -(x,1) \in K\}.$$

Then P^+ and P^- are convex. Moreover, if both P^+ and P^- are nonempty and if $P := P^+ \cup P^-$ is not a singleton set, then both P^+ and P^- are unbounded and P is p-convex.

Proof. P^+ and P^- are both clearly convex. Suppose P^+ and P^- are nonempty, and $P^+ \cup P^-$ is not a singleton set. Then there exist points $x \in P^+$ and $y \in P^-$ with $x \neq y$. Being a convex cone, K must contain the line segment L joining $(x, 1)$ and $-(y, 1)$, and hence the set $\cup_{\beta>0} \beta L$ belongs to K. Since

$$(\alpha x + (1-\alpha)y, 1) \in \bigcup_{\beta>0} \beta L \quad \forall \alpha \geq 1,$$

and

$$-(\alpha x + (1-\alpha)y, 1) \in \bigcup_{\beta>0} \beta L \quad \forall \alpha \leq 0,$$

it follows that both P^+ and P^- are unbounded, and that $P = P^+ \cup P^-$ is p-convex. □

LEMMA 2.2. *Let $S \in \mathbf{H}_n$ have at least one positive eigenvalue. Then*

$$\operatorname{span}\{v \in \mathbb{C}^n : v^*Sv > 0\} \ = \ \mathbb{C}^n.$$

Proof. It is clear that the set $E := \{v \in \mathbb{C}^n : v^*Sv > 0\}$ is nonempty. Suppose $\operatorname{span} E \neq \mathbb{C}^n$. Take $v \in E$ and a nonzero $w \in E^\perp$, where the orthogonal complement is taken with respect to the usual Euclidean inner product. Then for a large enough real number α,

$$(\alpha v + w)^* S(\alpha v + w) = |\alpha|^2 v^*Sv + 2\mathrm{Re}(\alpha w^*Sv) + w^*Sw > 0,$$

which implies $\alpha v + w \in E$, a contradiction. □

We say that two matrices $A, B \in \mathbf{M}_n$ are simultaneously congruent to $\widehat{A}$ and $\widehat{B}$, respectively, if there exists a nonsingular matrix $X \in \mathbf{M}_n$ such that $X^*AX = \widehat{A}$ and $X^*BX = \widehat{B}$. Suppose S is singular and X is nonsingular such that $X^*SX = S_1 \oplus 0_k$ with $S_1 \in \mathbf{H}_{n-k}$. Then $X^*SAX = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}$, where $B_1 \in \mathbf{M}_{n-k}$ and $B_2 \in \mathbb{C}^{(n-k)\times k}$. This justifies our assumption on S and SA in part (c) of the following theorem.

THEOREM 2.3. *Let $A \in \mathbf{M}_n$ and $S \in \mathbf{H}_n$. Then,*
(a) $W_S^+(A) = \emptyset$ if and only if S is negative semidefinite; $W_S(A) = \emptyset$ if and only if $S = 0$.
(b) $W_S(A) = \{\lambda\}$ if and only if $S \neq 0$ and $SA = \lambda S$ for some $\lambda \in \mathbb{C}$; $W_S^+(A) = \{\lambda\}$ if and only if S has at least one positive eigenvalue and $SA = \lambda S$ for some $\lambda \in \mathbb{C}$.
(c) Suppose the conditions in (a) and (b) do not hold. Let S and SA be simultaneously congruent to $S_1 \oplus 0_k$ and $\begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}$, respectively, where $k > 0$, $S_1 \in \mathbf{H}_{n-k}$ is nonsingular, $B_1 \in \mathbf{M}_{n-k}$ and $B_2 \in \mathbb{C}^{(n-k)\times k}$ (for $k = 0$ take $S_1 = S$).

(i) If $B_2 \neq 0$, then $W_S^+(A) = W_S(A) = \mathbb{C}$.

(ii) If $B_2 = 0$ and $S_1 > 0$, then $W_S^+(A) = W_S(A) = W(S_1^{-\frac{1}{2}} B_1 S_1^{-\frac{1}{2}})$.

(iii) If $B_2 = 0$ and $S_1 < 0$, then $W_S(A)$ can be determined by the result of (ii) above and the general fact that $W_S(A) = W_{-S}(A)$.

(iv) If $B_2 = 0$ and S_1 is indefinite, then $W_S^+(A)$ is an unbounded convex set, and $W_S(A) = W_{S_1}(S_1^{-1}B_1)$ is the union of two unbounded convex sets and is p-convex.

Consequently, $W_S^+(A)$ is always convex, and $W_S(A)$ is convex unless condition (c)(iv) holds.

Proof. Let $SA = H + iG$ with $H, G \in \mathbf{H}_n$. We first deal with $W_S^+(A)$. Since $W_S^+(A)$ is essentially the intersection of $K(H,G,S)$ with the plane $\mathbb{R}^2 \times \{1\}$ (Lemma 1.1), and $K(H,G,S)$ is a convex cone (Lemma 1.2), it is always convex.
Notice that S is negative semidefinite if and only if the equation $v^*Sv = 1$ has no solution $v \in \mathbb{C}^n$. Hence (a) holds.
Suppose $W_S^+(A) = \{\lambda\}$. Then S has at least one positive eigenvalue by (a). Since, by Lemma 1.1, $K(H,G,S) \cap \left(\mathbb{R}^2 \times \{1\}\right)$ is a singleton set $\{p\}$ where $p = (\mathrm{Re}\,\lambda, \mathrm{Im}\,\lambda, 1)$, the convex cone $K(H,G,S)$ is contained in the line passing through p and the origin. It then follows that $v^*(H + iG - \lambda S)v = 0$ for all $v \in \mathbb{C}^n$, and hence $SA = H + iG = \lambda S$. Conversely, if $SA = \lambda S$ and S has at least one positive eigenvalue, then it follows immediately from the definition that $\emptyset \neq W_S^+(A) = \{\lambda\}$. Hence (b) holds.
Next assume that S and A satisfy the hypotheses in (c). We first suppose $B_2 \neq 0$. By our assumption, S and hence S_1 have some positive eigenvalues. By Lemma 2.2 there exists $u \in \mathbb{C}^{n-k}$ such that $u^*S_1u > 0$ and $u^*B_2 \neq 0$. Choose $v \in \mathbb{C}^k$ such that $u^*B_2v \neq 0$, and consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(\alpha) := \frac{u^*B_1u + \alpha u^*B_2v}{u^*S_1u} = \frac{(X(u+\alpha v))^*SA(X(u+\alpha v))}{(X(u+\alpha v))^*S(X(u+\alpha v))}.$$

Note that $f(\alpha) \in W_S^+(A)$ for all α. As $u^*B_2v \neq 0$, the range of f is $\mathbb{C}$ and hence (c)(i) holds.
If $B_2 = 0$ and $S_1 > 0$, then

$$\begin{aligned} W_S^+(A) &= \left\{\frac{(Xv)^*SA(Xv)}{(Xv)^*S(Xv)} : v \in \mathbb{C}^n,\ (Xv)^*S(Xv) > 0\right\} \\ &= \left\{\frac{v^*B_1v}{v^*S_1v} : 0 \neq v \in \mathbb{C}^{n-k}\right\} = W\left(S_1^{-\frac{1}{2}} B_1 S_1^{-\frac{1}{2}}\right). \end{aligned}$$

Hence (c)(ii) holds.
Suppose S_1 is indefinite. Then S is indefinite, and hence $K(H,G,S)$ contains some points above the xy-plane and some points below it. Moreover, since

$H+iG$ is not a (complex) scalar multiple of S by our assumption, the convex cone $K(H,G,S)$ is not a line passing through the origin. Hence $W_S^+(A)$, which is essentially $K(H,G,S)\cap\left(\mathbb{R}^2\times\{1\}\right)$, is unbounded. Therefore the assertion in (c)(iv) holds.

Now we consider $W_S(A)$. As $W_S(A)=W_S^+(A)\cup W_{-S}^+(A)$, parts (a), (b), (c)(i), and (c)(ii) follow immediately from the results on $W_S^+(A)$. Part (c)(iii) follows easily from the definition. It remains to prove (c)(iv). Let S and SA satisfy the hypotheses in (c)(iv). Let $B_1=H_1+iG_1$ where $H_1,G_1\in\mathbf{H}_{n-k}$. Since S_1 is indefinite, $K:=K(H_1,G_1,S_1)$ contains some points above the xy-plane and some points below. As H_1+iG_1 is not a (complex) scalar multiple of S_1 by assumption, the convex cone K is not a line passing through the origin, and hence it contains some points $(x,y,1)$ and $-(x',y',1)$ with $(x,y)\neq(x',y')$. Note that by Lemma 1.1, $(x+iy)\in W_S(A)=W_{S_1}(S_1^{-1}B_1)$ if and only if $(x,y,1)\in K$ or $-(x,y,1)\in K$. By Lemma 2.1, (c)(iv) holds. □

We make several remarks in connection with Theorem 2.3.

1. The sets $W_S^+(A)$ and $W_S(A)$ are nonempty and bounded if and only if (b) or (c)(ii) (or (c)(iii) for the case of $W_S(A)$) of the above theorem holds. In these cases both sets are closed.

2. Moreover, the set $W_S^+(A)$ is always convex, and $W_S(A)$ fails to be convex only if we have case (c)(iv), when it is p-convex. With this exception noted, the previous theorem thus extends the classical convexity result of Hausdorff and Toeplitz of 77 years ago to be (almost) true for $W_S^+(A)$ or $W_S(A)$ in indefinite inner product spaces as well.

3. The sets $W_S(A)$ and $W_S^+(A)$ may not be closed in case (c)(iv). For example, if $S=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $A=\begin{pmatrix}-1&0\\2&1\end{pmatrix}$ then $W_S(A)=\{z\in\mathbb{C}:\mathrm{Re}(z)\neq 0\}$ and $W_S^+(A)=\{z\in\mathbb{C}:\mathrm{Re}(z)>0\}$ are not closed.

4. It is well-known for the classical numerical range that $W(A)=\{\lambda\}$ if and only if $A=\lambda I$. Theorem 2.3(b) can be viewed as a generalization of this fact.

For $V_S(A)$ and $V_S^+(A)$, we have the following analogous result.

THEOREM 2.4. *Let $A\in\mathbf{M}_n$ and $S\in\mathbf{H}_n$. Then,*

(a) $V_S^+(A)=\emptyset$ if and only if S is negative semidefinite; $V_S(A)=\emptyset$ if and only if $S=0$.

(b) $V_S(A)=\{\lambda\}$ if and only if $S\neq 0$ and $A=\lambda S$ for some $\lambda\in\mathbb{C}$; $V_S^+(A)=\{\lambda\}$ if and only if S has at least one positive eigenvalue and $A=\lambda S$ for some $\lambda\in\mathbb{C}$.

(c) Suppose the conditions in (a) and (b) do not hold. Let S and A be simultaneously congruent to $S_1\oplus 0_k$ and $\begin{pmatrix}A_1&A_2\\A_3&A_4\end{pmatrix}$, respectively, where $0\le k<n$, $S_1\in\mathbf{H}_{n-k}$ is nonsingular, and $A_1\in\mathbf{M}_{n-k}$ (for $k=0$ take $S_1=S$ and $A_1=A$).

(i) If $S_1>0$ and A_2,A_3 and A_4 are zero, then $V_S^+(A)=V_S(A)=$

$W(S_1^{-\frac{1}{2}}A_1S_1^{-\frac{1}{2}})$.

(ii) If $S_1 > 0$ and at least one of A_2, A_3, A_4 is nonzero, then both $V_S^+(A)$ and $V_S(A)$ are unbounded and convex.

(iii) If $S_1 < 0$, then $V_S(A)$ can be determined by the results of (i), (ii) above and the fact that $V_S(A) = -V_{-S}(A)$.

(iv) If S_1 is indefinite, then $V_S^+(A)$ is unbounded and convex, whereas $V_S(A)$ is unbounded and p-convex.

Consequently, $V_S^+(A)$ is always convex, and $V_S(A)$ is convex unless (c)(iv) holds.

Proof. We first deal with $V_S^+(A)$. We only need to prove part (c), as a proof of the other parts is similar to that of Theorem 2.3. Let S and A satisfy the hypotheses in (c). If A_2, A_3 and A_4 are zero, then clearly $V_S^+(A) = V_{S_1}^+(A_1) = W(S_1^{-\frac{1}{2}}A_1S_1^{-\frac{1}{2}})$ which is compact. Conversely, suppose $V_S^+(A)$ is bounded. Without loss of generality we let $S = S_1 \oplus 0_k$ and $A = (a_{ij})$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$ and let $1 \le i \le (n-k) < j \le n$. Consider the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(\alpha) := \frac{(e_i + \alpha e_j)^* A (e_i + \alpha e_j)}{(e_i + \alpha e_j)^* S (e_i + \alpha e_j)} = \frac{a_{ii} + \alpha a_{ij} + \bar{\alpha} a_{ji} + |\alpha|^2 a_{jj}}{e_i^* S_1 e_i}.$$

Notice that $g(\alpha) \in V_S^+(A)$ and hence g is bounded uniformly for all $\alpha \in \mathbb{C}$. One can then deduce that $a_{ij} = a_{ji} = a_{jj} = 0$ for all $1 \le i \le (n-k) < j \le n$. Hence $A = A_1 \oplus 0_k$. This proves (c) for the set $V_S^+(A)$.
The result for $V_S(A)$ can be easily deduced from the fact that $V_S(A) = V_S^+(A) \cup -V_{-S}^+(A)$. □

Concurrent with our work, Grigorieff and Plato [GP, Lemma 2.1, Theorem 2.2] have given an even more detailed description of $V_S^+(A)$ for the case $S \ge 0$ than that in (c)(i) and (c)(ii) above.

It is well-known that $W(A) \subset \mathbb{R}$ for the classical numerical range if and only if A is Hermitian. The next corollary extends this result. We call a matrix $A \in \mathbf{M}_n$ *S-Hermitian* if $SA \in \mathbf{H}_n$.

COROLLARY 2.5. *Let $A \in \mathbf{M}_n$ and $S \in \mathbf{H}_n$. Then,*
(a) $\emptyset \ne W_S(A) \subset \mathbb{R}$ (or $\emptyset \ne V_S(A) \subset \mathbb{R}$, respectively) if and only if $S \ne 0$ and A is S-Hermitian (or A is Hermitian, respectively);
(b) $\emptyset \ne W_S^+(A) \subset \mathbb{R}$ (or $\emptyset \ne V_S^+(A) \subset \mathbb{R}$, respectively) if and only if S has at least one positive eigenvalue and A is S-Hermitian (or A is Hermitian, respectively).

Proof. We consider only $W_S^+(A)$. The proofs for $W_S(A), V_S(A)$ and $V_S^+(A)$ are similar. Let $SA = H + iG$, where $H, G \in \mathbf{H}_n$, and let $K(H, G, S)$ be defined as in (1). Suppose $\emptyset \ne W_S^+(A) \subset \mathbb{R}$. Then S has at least one positive eigenvalue by Theorem 2.3(a), and by Lemma 1.1(b), the set $P := K(H, G, S) \cap \left(\mathbb{R}^2 \times \{1\}\right)$ is nonempty and is contained in the xz-plane of $\mathbb{R}^3$.

This and the fact that $K(H,G,S)$ is a convex cone (Lemma 1.2) imply that $K(H,G,S)$ itself is contained in the xz-plane of $\mathbb{R}^3$: for if $K(H,G,S)$ contains some point $p := (x,y,z)$ with $y \neq 0$, and $p' := (x',0,1) \in P$, then $K(H,G,S)$ contains all points of the form $(\alpha x + \beta x',\ \alpha y,\ \alpha z + \beta) = \alpha p + \beta p'$ with $\alpha, \beta \geq 0$ and, since there exist $\alpha, \beta \geq 0$ that satisfy $\alpha y \neq 0$ and $\alpha z + \beta = 1$, it follows that P is not contained in the xz-plane — a contradiction. Consequently we have $v^* G v = 0$ for all $v \in \mathbb{C}^n$. Thus $G = 0$, or equivalently A is S-Hermitian. By reversing the argument, we obtain the converse statement. ☐

3. Linear Maps Preserving $W_S(A), W_S^+(A), V_S(A)$, or $V_S^+(A)$. This section is devoted to the study of those linear operators L on $\mathbf{M}_n$ that satisfy one of the following linear preserver properties

$$W_S^+(L(A)) = W_S^+(A) \quad \text{for all } A \in \mathbf{M}_n, \tag{2}$$

$$W_S(L(A)) = W_S(A) \quad \text{for all } A \in \mathbf{M}_n, \tag{3}$$

$$V_S^+(L(A)) = V_S^+(A) \quad \text{for all } A \in \mathbf{M}_n, \tag{4}$$

$$V_S(L(A)) = V_S(A) \quad \text{for all } A \in \mathbf{M}_n, \tag{5}$$

where $S \in \mathbf{H}_n$ is a given matrix. To make the study more sensible, we impose some mild conditions on S. For example, in (2) and (4) we shall assume that S has at least one positive eigenvalue; for otherwise $W_S^+(A) = V_S^+(A) = \emptyset$ for all A and thus L can be of any form. Similarly we shall assume that $S \neq 0$ in (3) and (5). As can be seen, the proofs in this section rely heavily on the results concerning the geometrical properties of $W_S(A), W_S^+(A), V_S(A)$ and $V_S^+(A)$ obtained in the previous section.

3.1. Statement of Results, Examples, and Remarks. By the result in [P], a linear operator L on $\mathbf{M}_n$ satisfies

$$W(L(A)) = W(A) \quad \text{for all } A \in \mathbf{M}_n$$

if and only if L is of the form $A \mapsto X^*AX$ or $A \mapsto X^*A^tX$ for some unitary matrix X. It turns out that the linear operators L that satisfy (4) have a very similar structure as shown in the following theorem.

THEOREM 3.1. *Suppose $S \in \mathbf{H}_n$ has at least one positive eigenvalue. A linear operator L on $\mathbf{M}_n$ satisfies $V_S^+(L(A)) = V_S^+(A)$ for all $A \in \mathbf{M}_n$ if and only if L is of the form*
(a) $A \mapsto X^*AX$ for some nonsingular $X \in \mathbf{M}_n$ satisfying $X^*SX = S$, or
(b) $A \mapsto X^*A^tX$ for some nonsingular $X \in \mathbf{M}_n$ satisfying $X^*S^tX = S$.

The structure of those linear operators that satisfy (5) is slightly more complicated. For example, if $n = 2k$ and $S = I_k \oplus -I_k$, then the linear operator L defined by $L(A) := -X^*AX$ with $X = \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix}$ satisfies (5). More generally, we have the following result for (5).

THEOREM 3.2. *Suppose $0 \neq S \in \mathbf{H}_n$. A linear operator L on $\mathbf{M}_n$ satisfies $V_S(L(A)) = V_S(A)$ for all $A \in \mathbf{M}_n$ if and only if there exists $\mu = \pm 1$ such that*

L is of the form
(a) $A \mapsto \mu X^*AX$ for some nonsingular $X \in \mathbf{M}_n$ satisfying $X^*SX = \mu S$, or
(b) $A \mapsto \mu X^*A^tX$ for some nonsingular $X \in \mathbf{M}_n$ satisfying $X^*S^tX = \mu S$.

For linear operators L to satisfy (2) and (3), the conditions are much more complicated. For example, if $S = I_{n-k} \oplus 0_k$, then (2) and (3) are satisfied by any linear operator L on $\mathbf{M}_n$ of the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mapsto \begin{pmatrix} A_1 & A_2 \\ \phi_a(A) & \phi_b(A) \end{pmatrix},$$

where $A_1 \in \mathbf{M}_{n-k}$, and $\phi_a : \mathbf{M}_n \to \mathbb{C}^{k\times(n-k)}$, $\phi_b : \mathbf{M}_n \to \mathbf{M}_k$ are arbitrary linear transformations. As can be easily seen, such complications will not arise when S is nonsingular. In fact, if we impose the assumption that S is nonsingular, the statements of our results for (2) and (3) and the proofs would be much simpler. For the sake of completeness, however, we shall present the results and proofs for (2) and (3) for the general case. When stating our theorems, we shall use Φ_X and Ψ_X to denote the following linear operators on $\mathbf{M}_n$ defined by

$$\Phi_X(A) := X^*AX, \qquad \Psi_X(A) := X^{-1}AX,$$

respectively, where $X \in \mathbf{M}_n$ is nonsingular. Clearly Φ_X and Ψ_X are the congruence transform and the similarity transform induced by X. We shall use τ to denote the transpose operator on $\mathbf{M}_n$, i.e., $\tau(A) := A^t$.

THEOREM 3.3. *Let $S \in \mathbf{H}_n$ have at least one positive eigenvalue and satisfy $R^*SR = S_1 \oplus 0_k$ for some nonsingular $R \in \mathbf{M}_n$ and a nonsingular $S_1 \in \mathbf{H}_{n-k}$ $(0 \le k < n)$.*
A linear operator L on $\mathbf{M}_n$ satisfies $W_S^+(L(A)) = W_S^+(A)$ for all $A \in \mathbf{M}_n$ if and only if $\Psi_R \circ L \circ \Psi_{R^{-1}}$ is of the form

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mapsto \begin{pmatrix} \phi_1(A_1)+\phi_2(A_2) & \phi_3(A_2) \\ \phi_a(A) & \phi_b(A) \end{pmatrix} \tag{6}$$

in which the block matrix partitioning is compatible with that of $S_1 \oplus 0_k = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $\phi_1, \phi_2, \phi_3, \phi_a$ and ϕ_b are linear with ϕ_3 nonsingular, and
(a) $\phi_1 = \Psi_X$ for some nonsingular $X \in \mathbf{M}_{n-k}$ satisfying $\Phi_X(S_1) = S_1$, or
(b) $\phi_1 = \Psi_X \circ \tau$ for some nonsingular $X \in \mathbf{M}_{n-k}$ satisfying $\Phi_X \circ \tau(S_1^{-1}) = S_1$, and there are no further restrictions on ϕ_2, ϕ_a, ϕ_b.

THEOREM 3.4. *Let $0 \ne S \in \mathbf{H}_n$ be such that $R^*SR = S_1 \oplus 0_k$ for some nonsingular $R \in \mathbf{M}_n$ and a nonsingular $S_1 \in \mathbf{H}_{n-k}$ $(0 \le k < n)$.*
A linear operator L on $\mathbf{M}_n$ satisfies $W_S(L(A)) = W_S(A)$ for all $A \in \mathbf{M}_n$ if and only if $\Psi_R \circ L \circ \Psi_{R^{-1}}$ is of the form in (6), where $\phi_1, \phi_2, \phi_3, \phi_a$ and ϕ_b are linear with ϕ_3 nonsingular, and there exists $\mu = \pm 1$ with
(a) $\phi_1 = \Psi_X$ for some nonsingular $X \in \mathbf{M}_{n-k}$ satisfying $\Phi_X(S_1) = \mu S_1$, or

(b) $\phi_1 = \Psi_X \circ \tau$ *for some nonsingular* $X \in \mathbf{M}_{n-k}$ *satisfying* $\Phi_X \circ \tau(S_1^{-1}) = \mu S_1$, *and there are no further restrictions on* ϕ_2, ϕ_a, ϕ_b.

Several remarks are in order in connection with Theorems 3.1 – 3.4.

1. From Theorems 3.1 and 3.2, we see that apart from the trivial cases of $S = 0$ for $V_S(\cdot)$ and $S \leq 0$ for $V_S^+(\cdot)$, the linear preservers of $V_S(\cdot)$ or $V_S^+(\cdot)$ must be nonsingular, even for singular S.
2. If S in Theorem 3.3 or 3.4 is nonsingular, we may take $R = I$ and $S = S_1$, so that linear preservers L of $W_S^+(\cdot)$ (or of $W_S(\cdot)$) are indeed the ϕ_1 described in (a) and (b) of Theorem 3.3 (or in (a) and (b) of Theorem 3.4). In this case, L must be nonsingular. But for singular S, the operator L could be singular.
3. In Theorems 3.3 and 3.4, one can always choose R so that $R^*SR = I_r \oplus -I_s \oplus 0_t$, the inertia matrix of S.
4. In Theorems 3.2 and 3.4, μ may assume the value -1 if and only if S has balanced inertia, i.e., if S has the same number of positive and negative eigenvalues.
5. If $S = I$, then all of our Theorems 3.1 – 3.4 reduce to the classical result in [P].

3.2. Proofs. We shall first prove the sufficiency parts of Theorems 3.1 and 3.2. Next we shall show that linear preservers of $V_S^+(\cdot)$ or $V_S(\cdot)$ are preservers of rank-1 Hermitian matrices. Then we can apply a result in [L] on linear preservers of rank-1 Hermitian matrices to prove the necessity parts of Theorems 3.1 and 3.2. Finally, we shall apply Theorems 3.1 and 3.2 to prove Theorems 3.3 and 3.4.

LEMMA 3.5. *Let* $S \in \mathbf{H}_n$.
(a) Suppose L is of the form (a) or (b) in Theorem 3.1. Then $V_S^+(L(A)) = V_S^+(A)$ *for all* $A \in \mathbf{M}_n$.
(b) Suppose L is of the form (a) or (b) in Theorem 3.2. Then $V_S(L(A)) = V_S(A)$ *for all* $A \in \mathbf{M}_n$.

Proof. Each statement can be verified directly. As an illustration, we shall show that if L is in the form of (b) in Theorem 3.2 with $\mu = -1$, then $V_S(L(A)) = V_S(A)$ for all $A \in \mathbf{M}_n$. Notice that by assumption $L(A) = -X^*A^tX$ and $-X^*S^tX = S$. Hence

$$\begin{aligned} V_S(L(A)) &= V_{-X^*S^tX}(-X^*A^tX) = \left\{ \frac{v^*(-X^*A^tX)v}{v^*(-X^*S^tX)v} : v^*(-X^*S^tX)v \neq 0 \right\} \\ &= \left\{ \frac{v^*Av}{v^*Sv} : v^*Sv \neq 0 \right\} = V_S(A). \qquad \Box \end{aligned}$$

We now prove the necessity part of Theorem 3.1.

LEMMA 3.6. *Suppose* $S \in \mathbf{H}_n$ *has at least one positive eigenvalue. If* $V_S^+(L(A)) = V_S^+(A)$ *for all* $A \in \mathbf{M}_n$ *then L is of form (a) or (b) in Theorem 3.1.*

Proof. Notice that if $S = R^*DR$ where R is nonsingular and $D = I_r \oplus -I_s \oplus 0_t$, one can easily verify that $V_S^+(A) = V_D^+(R^{-*}AR^{-1})$, where R^{-*} stands for $(R^{-1})^*$. Notice further that $V_S^+(L(A)) = V_S^+(A)$ for all A if and only if $V_D^+(\widehat{L}(A)) = V_D^+(A)$ for all A, where $\widehat{L}(A) := R^{-*}(L(R^*AR))R^{-1}$. Thus we may assume without loss of generality that $S = D = I_r \oplus -I_s \oplus 0_t$ with $r > 0$. Suppose $V_D^+(L(A)) = V_D^+(A)$ for all $A \in \mathbf{M}_n$. We are going to show that L is of the form $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$, where U is nonsingular and satisfies $U^*DU = D$. Then the result will follow.

Consider the usual inner product $\langle \cdot, \cdot \rangle$ in $\mathbf{M}_n$ defined by $\langle A, B\rangle := \operatorname{tr}(AB^*)$. Recall that the dual transformation L^* of L is the linear operator that satisfies $\langle L(A), B\rangle = \langle A, L^*(B)\rangle$ for all $A, B \in \mathbf{M}_n$. Since L satisfies $V_D(L(A)) = V_D(A)$ for all A, it follows that $\langle A, L^*(\Delta_+)\rangle = \langle A, \Delta_+\rangle$ for all A, where

$$\Delta_+ := \{vv^* :\ v \in \mathbb{C}^n, \langle D, vv^*\rangle = 1\}. \tag{7}$$

Consequently we have $L^*(\operatorname{conv}\Delta_+) = \operatorname{conv}\Delta_+$ and, by the linearity of L^*, the restriction of L^* on $\operatorname{conv}\Delta_+$ is one-to-one. Since Δ_+ is the set of all extreme points of $\operatorname{conv}\Delta_+$, it follows that $L^*(\Delta_+) = \Delta_+$. Since $e_1e_1^* \in \Delta_+$, we have $L^*(e_1e_1^*) = uu^* \in \Delta_+$ for some $u \in \mathbb{C}^n$. We claim that L^* maps the set of rank one Hermitian matrices into itself. In fact, if $v \in \mathbb{C}^n$ is such that $\langle D, vv^*\rangle > 0$, then $\mu vv^* \in \Delta_+$ for some $\mu > 0$. Thus $L^*(\mu vv^*) \in \Delta_+$ is a rank one Hermitian matrix. If $v \in \mathbb{C}^n$ is such that $\langle D, vv^*\rangle \le 0$, then consider $v(\mu) = \mu e_1 + v$. For all sufficiently large $\mu > 0$, $\langle D, v(\mu)v(\mu)^*\rangle > 0$. Hence $L^*(v(\mu)v(\mu)^*) = \mu^2L^*(e_1e_1^*) + \mu L^*(e_1v^* + ve_1^*) + L^*(vv^*)$ has rank one for all sufficiently large $\mu > 0$ (and hence for infinitely many μ) by the previous discussion. It follows that $L^*(vv^*)$ has rank one. Using Theorem 4 from [L], we conclude that L^* is of the form $A \mapsto UAU^*$ or $A \mapsto \overline{U}A^tU^t$. One can easily check that consequently L is of form $A \mapsto U^*AU$ or $A \mapsto U^*A^tU$. Finally $\{1\} = V_D^+(D) = V_D^+(L(D)) = V_D^+(U^*DU)$, thus using Theorem 2.4(b) we see that $U^*DU = D$. $\square$

The following proves the necessity part of Theorem 3.2.

LEMMA 3.7. *Suppose* $0 \ne S \in \mathbf{H}_n$. *If* $V_S(L(A)) = V_S(A)$ *for all* $A \in \mathbf{M}_n$, *then* L *is of form (a) or (b) in Theorem 3.2.*

Proof. With an argument similar to that in the proof of Lemma 3.6, we assume without loss of generality that $S = D = I_r \oplus -I_s \oplus 0_t$ with $r + s > 0$. Suppose $V_D(L(A)) = V_D(A)$ for all A. Then L^* satisfies

$$\langle A, L^*(\Delta)\rangle = \langle A, \Delta\rangle \quad \text{for all } A, \tag{8}$$

where $\Delta := \Delta_+ \bigcup \Delta_-$ with Δ_+ defined in (7) and

$$\Delta_- := \{-vv^* :\ v \in \mathbb{C}^n, \langle D, -vv^*\rangle = 1\}.$$

If one of Δ_+ and Δ_- is empty then the other must be nonempty and, by the argument used in the proof of Lemma 3.6, L is of the form (a) or (b) of Theorem 3.2 with $\mu = 1$. Suppose both Δ_+ and Δ_- are nonempty (i.e., both $r > 0$ and $s > 0$). Then it may not be true that Δ is the set of extreme

points of conv Δ, and the proof used for Lemma 3.6 need not work. Hence we proceed with the following argument instead.
As the real linear span of Δ equals $\mathbf{H}_n$, (8) implies that L^*, and hence L as well, maps $\mathbf{H}_n$ onto $\mathbf{H}_n$ and thus L must be invertible. Observe that $0 \notin \langle A, \Delta\rangle$ for any definite matrix A. On the other hand, if $X \in \mathbf{H}_n$ is zero or indefinite then there exists a definite $A \in \mathbf{H}_n$ such that $\langle A, X\rangle = 0$. With (8), we conclude that

$$L^*(\Delta) \subset (\mathbf{H}_+ \cup -\mathbf{H}_+) \setminus \{0_n\},$$

where $\mathbf{H}_+$ denotes the cone of positive semidefinite matrices in $\mathbf{H}_n$. Take any element $E_0 \in \Delta_+$. If $L^*(E_0) \in \mathbf{H}_+$ then we must have $L^*(\Delta_+) \subset \mathbf{H}_+$; for if there is another $E_1 \in \Delta_+$ with $L^*(E_1) \in -\mathbf{H}_+$, then, since Δ_+ is path connected, there must be some $E \in \Delta_+$ such that $L^*(E)$ is either zero or indefinite — a contradiction. Similarly, if $L^*(E_0) \in -\mathbf{H}_+$ then we must have $L^*(\Delta_+) \subset -\mathbf{H}_+$. We consider the case $L^*(\Delta_+) \subset \mathbf{H}_+$ first.
Since $\{1\} = \langle D, \Delta_+\rangle$, by (8) we have $L^*(\Delta_+) \subset \widehat{\mathbf{H}}_+$, where $\widehat{\mathbf{H}}_+ := \{H \in \mathbf{H}_+ : \langle D, H\rangle = 1\}$ is a convex set with Δ_+ as the set of extreme points. It then follows that $L^*(\widehat{\mathbf{H}}_+) \subset \widehat{\mathbf{H}}_+$. As this is true for $(L^*)^{-1}$ as well, we have $L^*(\widehat{\mathbf{H}}_+) = \widehat{\mathbf{H}}_+$ and hence $L^*(\Delta_+) = \Delta_+$. Now we can repeat the argument used in the proof of Lemma 3.6 to show that L is of the form (a) or (b) of Theorem 3.2 with $\mu = 1$.
If $L^*(\Delta_+) \subset -\mathbf{H}_+$ then, using the arguments in the preceding paragraph, we can show that $(-L)^*(\widehat{\mathbf{H}}_+) = \widehat{\mathbf{H}}_+$. Hence L is of the form $A \mapsto -U^*AU$ or $A \mapsto -U^*A^tU$ for some nonsingular U. Since $\{1\} = V_D(D) = V_D(L(D)) = V_D(-U^*DU)$, we have $-U^*DU = D$ by Theorem 2.4(b). Hence L is of the form (a) or (b) of Theorem 3.2 with $\mu = -1$. □

Next we present the proof of Theorem 3.4. The proof of Theorem 3.3 is similar and simpler and shall be omitted.

Proof of Theorem 3.4. We first consider the case of $S = S_1$ being nonsingular. For any linear operator L on $\mathbf{M}_n$, define $\widehat{L}$ by $\widehat{L}(A) = SL(S^{-1}A)$. This establishes a one-to-one correspondence between L and $\widehat{L}$. Since $W_S(A) = V_S(SA)$, we can easily deduce that $V_S(\widehat{L}(A)) = V_S(A)$ for all A if and only if $W_S(L(A)) = W_S(A)$ for all A. Then we can apply Theorem 3.2 to show that $W_S(L(A)) = W_S(A)$ for all A if and only if $L = \phi_1$, where ϕ_1 is of the form described in (a) or (b) of Theorem 3.4. This proves Theorem 3.4 for nonsingular S.
Now suppose that S is singular and that $R^*SR = S_1 \oplus 0_k$, where both $R \in \mathbf{M}_n$ and $S_1 \in \mathbf{H}_{n-k}$ are nonsingular. It is easy to deduce that $W_S(A) = W_S(L(A))$ for all A if and only if

$$V_{S_1\oplus 0_k}((S_1 \oplus 0_k)A) = V_{S_1\oplus 0_k}((S_1 \oplus 0_k)\phi(A)) \tag{9}$$

for all A, where $\phi(A) := R^{-1}L(RAR^{-1})R = \Psi_R \circ L \circ \Psi_{R^{-1}}(A)$. Write $A =$

$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, in which the block partition is compatible with that of $S_1 \oplus 0_k = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$. Let ϕ be of the form

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \mapsto \begin{pmatrix} \sum_{j=1}^4 \phi_{1j}(A_j) & \sum_{j=1}^4 \phi_{2j}(A_j) \\ \phi_a(A) & \phi_b(A) \end{pmatrix}.$$

Then (9) is satisfied for all A if and only if

$$(10)\ V_{S_1 \oplus 0_k} \begin{pmatrix} S_1 A_1 & S_1 A_2 \\ 0 & 0 \end{pmatrix} = V_{S_1 \oplus 0_k} \begin{pmatrix} \sum_{j=1}^4 S_1 \phi_{1j}(A_j) & \sum_{j=1}^4 S_1 \phi_{2j}(A_j) \\ 0 & 0 \end{pmatrix}$$

for all A.
Suppose (10) is satisfied for all A. By putting $A_1 = 0$ and $A_2 = 0$, the left hand side of (10) becomes $\{0\}$. By Theorem 2.4(b), we conclude that $\phi_{13}, \phi_{14}, \phi_{23}$ and ϕ_{24} are all zero. Now take $A_2 = 0$ and $A_1 \in S_1^{-1}\mathbf{H}_{n-k}$. By Corollary 2.5, the left hand side of (10) is contained in $\mathbb{R}$. This and Corollary 2.5 imply that $\phi_{21}(S_1^{-1}\mathbf{H}_{n-k}) = \{0\}$. Since the complex span of $\mathbf{H}_{n-k}$ equals $\mathbf{M}_{n-k}$, this implies that ϕ_{21} is zero. By putting $A_2 = 0$, (10) becomes $V_{S_1}(S_1 A_1) = V_{S_1}(S_1 \phi_{11}(A_1))$, or equivalently, $W_{S_1}(A_1) = W_{S_1}(\phi_{11}(A_1))$ for all A_1. By the result at the beginning of the proof, ϕ_{11} is of the form of ϕ_1 in (a) or (b) of Theorem 3.4. In particular, ϕ_{11} is nonsingular. Finally, let $A_1 \in \mathbf{M}_{n-k}$ be such that $S_1\phi_{11}(A_1) = B_1 > 0$. Suppose there exists a nonzero $B_2 \in \mathbb{C}^{(n-k)\times k}$ such that $\phi_{22}(B_2) = 0$. Take $A_2 = \epsilon B_2$, where $\epsilon > 0$. Then the left hand side of (10) becomes $W_{S_1 \oplus 0_k} \begin{pmatrix} A_1 & \epsilon B_2 \\ 0 & 0 \end{pmatrix}$, which is equal to $\mathbb{C}$ by Theorem 2.3. On the other hand, the right hand side of (10) then becomes $V_{S_1}(B_1 + \epsilon S_1 \phi_{12}(B_2))$. Let $\lambda > 0$ be the smallest eigenvalue of B_1. For ϵ small enough and for any nonzero $v \in \mathbb{C}^{n-k}$, we have

$$|v^*(B_1 + \epsilon S_1 \phi_{12}(B_2))v| \ \geq\ (\lambda - \epsilon \|S_1 \phi_{12}(B_2)\|)\, \|v\|^2 \ >\ 0,$$

where the symbol $\|\cdot\|$ denotes the Euclidean vector norm on $\mathbb{C}^{n-k}$ and the corresponding induced matrix norm on $\mathbf{M}_{n-k}$, respectively. Thus 0 is not contained in the right hand side of equation (10) — a contradiction. Hence ϕ_{22} must be nonsingular. Consequently, ϕ is of the form described in Theorem 3.4, with $\phi_{11} = \phi_1$, $\phi_{12} = \phi_2$ and $\phi_{22} = \phi_3$.
Conversely, if L is of the form described in Theorem 3.4, with ϕ_1 in the form of (a) or (b), then one can easily deduce that (9) is satisfied for all A, which implies $W_S(A) = W_S(L(A))$ for all A. □

4. Further Extensions and Research. For a bounded linear operator A on an infinite dimensional Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ equipped with a sesquilinear form $(u, v) \mapsto \langle u, v\rangle_S := \langle Su, v\rangle$, where S is self-adjoint, one may define

$W_S(A)$ as

$$W_S(A) := \left\{ \frac{\langle Av, v\rangle_S}{\langle v, v\rangle_S} : v \in \mathcal{H},\ \langle v, v\rangle_S \neq 0 \right\},$$

and define $W_S^+(A), V_S(A)$, and $V_S^+(A)$ analogously. By writing SA or A as $H + iG$, where H and G are self-adjoint, and considering the joint numerical range $W(H, G, S)$ and the convex cone $K(H, G, S)$ (the definitions of which are the same as those for the finite dimensional case), one may obtain some results analogous to Theorems 2.3, 2.4, and Corollary 2.5. For instance, we have the following proposition.

PROPOSITION 4.1. *Let A and S be bounded linear operators on $\mathcal{H}$ such that $S = S^*$. Then,*
(a) $W_S^+(A)$ is always convex.
(b) $W_S^+(A) = \emptyset$ if and only if S is negative semidefinite.
$W_S(A) = \emptyset$ if and only if $S = 0$.
(c) $W_S^+(A) = \{\lambda\}$ if and only if there exists $v \in \mathcal{H}$ with $\langle Sv, v\rangle > 0$ and $SA = \lambda S$ for some $\lambda \in \mathbb{C}$.
$W_S(A) = \{\lambda\}$ if and only if $S \neq 0$ and $SA = \lambda S$ for some $\lambda \in \mathbb{C}$.
(d) $\emptyset \neq W_S^+(A) \subset \mathbb{R}$ if and only if there exists $v \in \mathcal{H}$ with $\langle Sv, v\rangle > 0$ and SA is self-adjoint.
$\emptyset \neq W_S(A) \subset \mathbb{R}$ if and only if $S \neq 0$ and SA is self-adjoint.
(e) If S is semidefinite then $W_S(A)$ is convex.
(f) If S is indefinite and $SA \neq \lambda S$ for any $\lambda \in \mathbb{C}$, then $W_S(A)$ is unbounded and p-convex.

It is worth mentioning that one has to be careful when applying the finite dimensional techniques to study infinite dimensional operators. For example, in the infinite dimensional case, even if S is positive definite, S^{-1} may not exist, and therefore one may not be able to obtain a condition such as (c)(ii) of Theorem 2.3.

Next, we turn to another generalization. Given Hermitian matrices (or operators, if the space $\mathbb{C}^n$ is changed to an infinite dimensional Hilbert space) $S, H_1, \ldots, H_k$, one may define:

$$\begin{aligned} V_S(H_1, \ldots, H_k) &:= \{(\langle H_1 v, v\rangle, \ldots, \langle H_k v, v\rangle)/\langle Sv, v\rangle : v \in \mathbb{C}^n,\ \langle Sv, v\rangle \neq 0\}, \\ V_S^+(H_1, \ldots, H_k) &:= \{(\langle H_1 v, v\rangle, \ldots, \langle H_k v, v\rangle) : v \in \mathbb{C}^n,\ \langle Sv, v\rangle = 1\}. \end{aligned}$$

The *joint S-numerical ranges* $W_S(H_1, \ldots, H_k)$ and $W_S^+(H_1, \ldots, H_k)$ can be similarly defined for S-Hermitian matrices (or operators) $H_1, \ldots, H_k$, so that the resulting sets are subsets in $\mathbb{R}^k$. Note that $(x, y) \in W_S(H_1, H_2)$ if and only if $x + iy \in W_S(H_1 + iH_2)$, and thus our results (e.g., Theorem 2.3 and Corollary 2.5) on $W_S(A)$ can be rephrased to hold for $W_S(H_1, H_2)$. Similar conclusions can be drawn for $V_S(H_1, H_2)$ and $V_S(H_1 + iH_2)$, etc. It is known that if $k \geq 3$ and S is indefinite then, in general, $W_S^+(H_1, \ldots, H_k)$ may not be convex and $W_S(H_1,\ldots,H_k)$ may not be p-convex. It would be of interest to study the properties of $W_S(H_1,\ldots,H_k)$, etc., for general $k \geq 3$.

We emphasize that this paper is intended as a starting point for studying the numerical range of operators on an indefinite inner product space using the theory of the joint numerical range and the cone generated by it. There are many open problems whose study may lead to a better understanding of linear operators acting on an indefinite inner product space. We describe a few of them in the following questions.

1. It is known (e.g., see [HJ]) that $W(A)$ is the convex hull of $\sigma(A)$, the spectrum of A, if $A \in \mathbf{M}_n$ is normal. Is there an analogous result for $W_S(A)$ and S-normal matrices A? (Note that A is S-normal if $AS^{-1}A^*S = S^{-1}A^*SA$, see e.g. [GLR, pp. 84–85].)

2. It is known that $A \in \mathbf{M}_n$ is unitary if and only if $W(A) \subset \mathbf{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ and $\sigma(A) \subset \partial\mathbf{D}$, the boundary of $\mathbf{D}$. Can one get a similar characterization for S-unitary matrices? (Note that A is S-unitary if $A^*SA = S$, see, e.g., [GLR, p. 25].)

One item of interest is to verify spectral containment such as $\sigma(A) \subset W_S(A)$ in the finite dimensional or Hilbert space setting. It was proved in [B] that for positive operators A the closure of $W_S(A)$ contains the eigenvalues of A. Clearly if u is an *isotropic* eigenvector for A with $\langle u,u\rangle_S = 0$, this vector u cannot contribute to the set $W_S(A)$. If we call a vector u *anisotropic* if $\langle u,u\rangle_S \neq 0$, and use the term *anisotropic spectrum* $\sigma_a(A)$ to denote the set of eigenvalues of A with anisotropic eigenvectors, then clearly $\sigma_a(A) \subset W_S(A)$ and hence $\overline{\sigma_a(A)} \subset \overline{W_S(A)}$, generalizing the result of [B]. But an eigenvalue might be isotropic and not be repeated for A and still appear in $W_S(A)$. For example, if $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ with $x \neq y$ and $z \neq 0$, then x is an eigenvalue which is isotropic and not repeated for A, and $x \in W_S(A)$. Related to these observations, one may ask the following questions.

3. Since every set $W_S(A)$ contains all of the anisotropic part of the spectrum of A, what shape and significance does their intersection $\bigcap_{S\in\mathbf{H}_n} W_S(A)$ have?

4. How can one classify the matrices and operators A and S for which spectral containment $\sigma(A) \subset W_S(A)$ holds?

Concerning the geometrical properties of the S-numerical ranges, one may ask the following questions.

5. Are the boundary curves of the S-numerical ranges algebraic curves as shown to be true for those of $W(A)$ by Fiedler [F]?

6. How can one plot the S-numerical ranges $V_S(A)$ and $W_S(A)$, possibly using the ideas and algorithm of Johnson for plotting $W(A)$ in [J]?

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