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THE COMBINATORIALLY SYMMETRIC P-MATRIX COMPLETION PROBLEM*

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Abstract. An n-by-n real matrix is called a P-matrix if all its principal minors are positive. The P-matrix completion problem asks which partial P-matrices have a completion to a P-matrix. Here, we prove that every partial P-matrix with combinatorially symmetric specified entries has a P-matrix completion. The general case, in which the combinatorial symmetry assumption is relaxed, is also discussed.

Key words. P-matrix, completion problem, combinatorial symmetry

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An n-by-n real matrix is called a P-matrix (P_0-matrix) if all its principal minors are positive (nonnegative), see, e.g., [HJ2] for a brief discussion of these classical notions. This class generalizes many other important classes of matrices (such as positive definite, M-matrices, and totally positive), has useful structure (such as inverse closure, inheritance by principal submatrices, and wedge type eigenvalue restrictions), and arises in applications (such as the linear complementarity problem, and issues of local invertibility of functions).

A *partial matrix* is a rectangular array in which some entries are specified, while the remaining unspecified entries are free to be chosen from an indicated set (such as the real numbers or the complex numbers). A *completion* of a partial matrix is a specific choice of values for the unspecified entries resulting in a conventional matrix. A *matrix completion problem* asks whether a given partial matrix has a completion of a desired type; for example, the positive definite completion problem asks which (square) partial (Hermitian) matrices have a positive definite completion [GJSW]. The positive definite completion problem has received considerable attention.

Here, we consider (for the first time) the *P-matrix completion problem* under the assumptions that the partial matrix is square, all diagonal entries are specified, and the data is *combinatorially symmetric* (the *j,i* entry is specified if and only if the *i,j* entry is specified). Further, since the property of being a P-matrix is inherited by principal submatrices, it is necessary that the partial matrix be a *partial P-matrix*, i.e., every fully specified principal submatrix

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must itself be a P-matrix. Of all these assumptions, the only one that is truly restrictive is the combinatorial symmetry. The general case, in which this assumption is relaxed, is commented upon later.

In each of the completion problems: positive definite, M-matrix, inverse M-matrix, and totally positive, there are significant combinatorial restrictions (as well as the necessity of inheritance) on partial matrices, even when combinatorial symmetry is assumed, in order to ensure a desired completion. For example, the condition on partial positive definite matrices necessary to ensure a positive definite completion (without further knowledge of the data) is that the undirected graph of the symmetric data be chordal [GJSW]. Interestingly, it is shown here that, in the case of P-matrix completions, there are no combinatorial restrictions other than the symmetry assumption. Every combinatorially symmetric partial P-matrix has a P-matrix completion. (Thus, in contrast to the positive definite completion problem, the relaxation of symmetry alone dramatically changes the solution.) However, when the combinatorial symmetry assumption is relaxed, the conclusion no longer holds, and the question of which directed graphs for the specified entries ensure that a partial P-matrix has a P-matrix completion is, in general, open. All 3-by-3 partial P-matrices have P-matrix completions, but we exhibit a 4-by-4 partial P-matrix with just one unspecified entry and no P-matrix completion. Finally, we note that our main result does not generalize to the P_0 case. We give an example of a combinatorially symmetric partial P_0-matrix that has no P_0 completion.

Let A be an n-by-n partial P-matrix with one pair of symmetrically placed unspecified entries. By permutation similarity it can be assumed without loss of generality that the unspecified entries are a_{1n} and a_{n1}. Then, A is of the form:

$$A = \begin{bmatrix} a_{11} & a_{12}^T & ? \\ a_{21} & A_{22} & a_{23} \\ ? & a_{32}^T & a_{33} \end{bmatrix}$$

in which A_{22} is $(n-2)$-by-$(n-2)$ and $a_{12}, a_{21}, a_{23}, a_{32} \in R^{n-2}$. Define

$$A(x,y) \equiv \begin{bmatrix} a_{11} & a_{12}^T & x \\ a_{21} & A_{22} & a_{23} \\ y & a_{32}^T & a_{33} \end{bmatrix}$$

and denote $A(0,0)$ by A_0. For $\emptyset \neq \alpha, \beta \subseteq \{1, 2, \ldots, n\} = N$, let $A[\alpha, \beta]$ be the submatrix of A lying in the rows indexed by α and the columns indexed by β. Abbreviate $A[\alpha,\alpha]$ to $A[\alpha]$. Define $C \equiv \{\alpha \subseteq N : 1, n \in \alpha\}$ and let $A_\alpha = A(x,y)[\alpha]$. Note that since $1, n \in \alpha$ for all $\alpha \in C$, x and y are unspecified entries in every A_α, $\alpha \in C$.

LEMMA. *Every partial P-matrix with exactly one symmetrically placed pair of unspecified entries has a P-matrix completion. Moreover, all pairs (x, y),*

$y=-x$ may be used for this completion, for x in an open interval extending to $+\infty$.

Proof. Let A be an n-by-n partial P-matrix with exactly one symmetrically placed pair of unspecified entries, which we assume without loss of generality to be in the $1,n$ and $n,1$ positions. To find a P-matrix completion of $A(x,y)$ we must find x, y such that $\det A_\alpha > 0$ for all $\alpha \in C$ (the remaining principal minors of A are positive by hypothesis). For each $\alpha \in C$, $\alpha = \left\{ i_1 = 1, i_2, \ldots, i_{|\alpha|} = n \right\}$ define

$$\begin{aligned}
a_\alpha &= \det A(x,y)\left[\left\{ i_2, i_3, \ldots, i_{|\alpha|-1} \right\}\right] \\
b_\alpha &= \det A_0 \left[\left\{ i_2, i_3, \ldots, i_{|\alpha|} \right\}; \left\{ i_1, i_2, \ldots, i_{|\alpha|-1} \right\}\right] \\
c_\alpha &= \det A_0 \left[\left\{ i_1, i_2, \ldots, i_{|\alpha|-1} \right\}; \left\{ i_2, i_3, \ldots, i_{|\alpha|} \right\}\right] \\
d_\alpha &= \det A_0 \left[\alpha\right].
\end{aligned}$$

By using Sylvester's identity (see e.g. [HJ1, 0.8.6]) we see that

$$\det A_\alpha = -a_\alpha xy + (-1)^{|\alpha|-1} b_\alpha x + (-1)^{|\alpha|-1} c_\alpha y + d_\alpha.$$

Since $A(x,y)$ is a partial P-matrix, $a_\alpha > 0$ for all $\alpha \in C$. Then x, y can be chosen so that $xy < 0$ and $-a_\alpha xy$ overcomes the other terms of $\det A_\alpha$. In order to have $xy < 0$, choose $y = -x$. The line $y = -x$ intersects the hyperbola $\det A_\alpha = 0$ at the points

$$x_\alpha^{\pm} = \frac{-\left(b_\alpha - c_\alpha\right) \pm \sqrt{\left(b_\alpha - c_\alpha\right)^2 - 4a_\alpha d_\alpha}}{2a_\alpha}.$$

Define $m\left(\alpha\right) = \max\left\{ |x_\alpha^+|, |x_\alpha^-| \right\}$. Then $\det A_\alpha > 0$ for each x such that $|x| > m(\alpha)$. In order to find a P-matrix completion of A we must find a pair x, y that works for all $\alpha \in C$. There is a P-matrix completion of $A(x,y)$ for all x, y such that $|x| > \max_{\alpha \in C} m(\alpha)$ and $y = -x$. □

The lemma can be used sequentially to find a completion of *any* combinatorially symmetric partial P-matrix. The lemma verifies the case in which there is one pair of symmetrically placed unspecified entries. Assume there is a P-matrix completion of every partial P-matrix with $k-1$ pairs of symmetrically placed unspecified entries and let A be a partial P-matrix with k pairs of symmetrically placed unspecified entries. Choose one symmetrically placed pair of unspecified entries of A. Consider all of the maximal principal submatrices, $A\left[\alpha_1\right], A\left[\alpha_2\right], \ldots, A\left[\alpha_p\right]$, that this pair completes. (There are no other unspecified entries in any of these maximal submatrices.) Each $A\left[\alpha_i\right]$, $i = 1, 2, \ldots, p$, is a partial P-matrix by inheritance and, by the lemma, can be completed to a P-matrix with a pair of entries $y_i = -x_i$, x_i in an open, semi-infinite interval extending to $+\infty$. Complete each of these submatrices with the pair $x > \max_{i=1,2,\ldots,p} x_i$ and $y = -x$. Then, we are left with a partial P-matrix with $k-1$ pairs of symmetrically placed unspecified entries which can be completed to a P-matrix by the induction hypothesis. Note that the order of completion is immaterial (as long as combinatorial symmetry is maintained). This proves our main result.

THEOREM. *Every combinatorially symmetric partial P-matrix has a P-matrix completion.*

The theorem does not generalize to the case of P_0 matrices.

EXAMPLE. The matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & x \\ -1 & 0 & 0 & -2 \\ -1 & 0 & 0 & -1 \\ y & 1 & 1 & 1 \end{bmatrix}$$

is partial P_0 but has no P_0 completion, as $\det A(x,y)$ is identically -1.

As mentioned above, when the combinatorial symmetry assumption is relaxed, the conclusion of the theorem no longer holds. The question of which directed graphs for the specified entries ensures that a partial P-matrix has a P-matrix completion is, in general, open. However, we do know the following two propositions.

PROPOSITION 1. *Every 3-by-3 partial P-matrix has a P-matrix completion.*

Proof. The combinatorially symmetric case is covered by the lemma above. The only case that remains to be considered is the case in which A is a 3-by-3 partial P-matrix with one unspecified entry. Note that if there are more unspecified entries, values may be assigned to entries making sure the 2-by-2 principal minors are positive, until either one pair of symmetrically placed unspecified entries, or only one unspecified entry remains.
By permutation similarity, it can be assumed without loss of generality that the unspecified entry is in the 3,1 position. It can also be assumed, by positive left diagonal multiplication and diagonal similarity (which preserves P-matrices), that there are ones on the main diagonal and on the super diagonal. Then A is of the form

$$A = \begin{bmatrix} 1 & 1 & c \\ a & 1 & 1 \\ y & b & 1 \end{bmatrix}$$

in which $a, b < 1$ since A is a partial P-matrix. In order to complete A to a P-matrix y must be chosen so that the 1,3 minor is positive (which yields $yc < 1$) and $\det A = 1 + abc - a - b + y(1-c)$ is positive.
There are several cases to consider. If $c \leq 0$, choose $y > 0$ and large enough to make $\det A$ positive. Similarly, if $c > 1$, choose $y < 0$ and large in absolute

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value. If $c = 1$, then $\det A = 1 + ab - e - b = (1-a)(1-b)$ which is positive since $a, b < 1$. So, $y = 0$ will give a P-matrix completion of A. All that remains is the case $0 < c < 1$. If $a, b \geq 0$ or if $ab \leq 0$ choose $y = ab$. This gives $yc = abc < 1$ (since $a, b, c < 1$) and

$$\begin{aligned} \det A &= 1 + abc - a - b + ab(1-c) \\ &= 1 - a - b + ab \\ &= (1-a)(1-b) > 0. \end{aligned}$$

For $a, b < 0$ the term $1 + abc - a - b$ in the determinant of A is positive. So, $y = 0$ will result in a P-matrix completion of A. Thus, every 3-by-3 partial P-matrix has a P-matrix completion. ☐

PROPOSITION 2. *For every $n \geq 4$, there is a partial P-matrix with exactly one unspecified entry for which there is no P-matrix completion.*

Proof. The matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ y & -10 & -1 & 1 \end{bmatrix}$$

is a partial P-matrix with no P-matrix completion. It is easy to check that all 2-by-2 and 3-by-3 principal minors that do not include both rows 1 and 4 are positive. However, the two 3-by-3 determinants that involve y cannot simultaneously be made positive. In order for the determinant of $A[\{1,2,4\}]$ to be positive is must be the case that $y < -\frac{7}{2}$ while a positive determinant for $A[\{1,3,4\}]$ requires that $y > -3$. Thus, there is no P-matrix completion of A. This data can be embedded as a principal submatrix, by putting 1's on the diagonal and 0's in the other specified positions, to produce a partial P-matrix with one unspecified entry and no P-matrix completion for any $n > 4$. ☐

Other completion problems, intermediate between the positive definite completion problem and the combinatorially symmetric P-matrix completion problem are open and may be of interest. For example, when does a sign symmetric, partial P-matrix have a sign symmetric P-matrix completion and what may be said about the infimum of the Frobenius norms of completions of combinatorially symmetric partial P-matrices.

REFERENCES

[GJSW] R. Grone, C. R. Johnson, E. M. Sá, and H. Wolkowicz, Positive Definite Completions of Partial Hermitian Matrices, *Linear Algebra Appl.* 58 (1984), 109-124.

[HJ1] R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[HJ2] R. Horn and C. R. Johnson, *Topics in Matrix Analysis* , Cambridge University Press, Cambridge, 1991.