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SOME SUBSPACES OF THE PROJECTIVE SPACE $\mathrm{PG}(\bigwedge^K V)$ RELATED TO REGULAR SPREADS OF $\mathrm{PG}(V)$*

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Abstract. Let V be a $2m$-dimensional vector space over a field $\mathbb{F}$ ($m \geq 2$) and let $k \in \{1, \ldots, 2m-1\}$. Let $A_{2m-1,k}$ denote the Grassmannian of the $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$ and let e_{gr} denote the Grassmann embedding of $A_{2m-1,k}$ into $\mathrm{PG}(\bigwedge^k V)$. Let S be a regular spread of $\mathrm{PG}(V)$ and let X_S denote the set of all $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$ which contain at least one line of S. Then we show that there exists a subspace Σ of $\mathrm{PG}(\bigwedge^k V)$ for which the following holds: (1) the projective dimension of Σ is equal to $\binom{2m}{k} - 2 \cdot \binom{m}{k} - 1$; (2) a $(k-1)$-dimensional subspace α of $\mathrm{PG}(V)$ belongs to X_S if and only if $e_{gr}(\alpha) \in \Sigma$; (3) Σ is generated by all points $e_{gr}(p)$, where p is some point of X_S.

Key words. Regular spread, Grassmannian, Grassmann embedding, Klein correspondence.

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1. The main result. Let V be a $2m$-dimensional vector space over a field $\mathbb{F}$ ($m \geq 2$) and let $\mathrm{PG}(V)$ denote the projective space associated to V. For every $k \in \{1, \ldots, 2m-1\}$, let $A_{2m-1,k}$ denote the following point-line geometry.

- The points of $A_{2m-1,k}$ are the $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$.
- The lines of $A_{2m-1,k}$ are the sets $L(\pi_1, \pi_2)$ of $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$ which contain a given $(k-2)$-dimensional subspace π_1 and are contained in a given k-dimensional subspace π_2 ($\pi_1 \subseteq \pi_2$).
- Incidence is containment.

The geometry $A_{2m-1,k}$ is called the *Grassmannian of the* $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$. Obviously, $A_{2m-1,k} \cong A_{2m-1,2m-k}$ and the geometry $A_{2m-1,1} \cong A_{2m-1,2m-1}$ is isomorphic to the (point-line system of) the projective space $\mathrm{PG}(2m-1, \mathbb{F})$.

For every point $p = \langle \bar{v}_1, \ldots, \bar{v}_k \rangle$ of $A_{2m-1,k}$, let $e_{gr}(p)$ denote the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k \rangle$ of $\mathrm{PG}(\bigwedge^k V)$. The map e_{gr} defines an embedding of the geometry $A_{2m-1,k}$ into the projective space $\mathrm{PG}(\bigwedge^k V)$ which is called the *Grassmann embedding* of $A_{2m-1,k}$. The image of e_{gr} is a so-called *Grassmann variety* $\mathcal{G}_{2m-1,k}$ of $\mathrm{PG}(\bigwedge^k V)$.

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A spread of PG(V) is a set of lines of PG(V) partitioning the point-set of PG(V). In Section 2, we will define a nice class of spreads of PG(V) which are called *regular spreads*.

The following is the main result of this note.

THEOREM 1.1. *Let S be a regular spread of the projective space* PG(V). *Let $k \in \{1, \ldots, 2m-1\}$. Let X_S denote the set of all $(k-1)$-dimensional subspaces of* PG(V) *which contain at least one line of S. Then there exists a subspace Σ of* PG($\bigwedge^k V$) *for which the following holds:*

(1) *The projective dimension of Σ is equal to $\binom{2m}{k} - 2 \cdot \binom{m}{k} - 1$.*
(2) *A $(k-1)$-dimensional subspace α of* PG(V) *belongs to X_S if and only if $e_{gr}(\alpha) \in \Sigma$.*
(3) *Σ is generated by all points $e_{gr}(p)$, where p is some element of X_S.*

In Theorem 1.1 and elsewhere in this paper, we take the convention that $\binom{n}{z} = 0$ for every $n \in \mathbb{N}$ and every $z \in \mathbb{Z} \setminus \{0, \ldots, n\}$.

Some special cases. (1) If $k = 1$, then by Theorem 1.1(1), $\Sigma = \emptyset$. Indeed, in this case we have $X_S = \emptyset$.

(2) If $k = 2$, then by Theorem 1.1, $\dim(\Sigma) = m^2 - 1$ and $X_S = S$ consists of all lines L of PG(V) for which $e_{gr}(L) \in \Sigma \cap \mathcal{G}_{2m-1,2}$. For a discussion of the special case $k = m = 2$, see Section 4.

(3) If $k = m$, then by Theorem 1.1(1), Σ has co-dimension 2 in PG($\bigwedge^m V$).

(4) If $k \in \{m+1, \ldots, 2m-1\}$, then by Theorem 1.1, $\Sigma =$ PG($\bigwedge^k V$) and X_S consists of all $(k-1)$-dimensional subspaces of PG(V).

2. Regular spreads.

2.1. Definition. Let PG($3, \mathbb{F}$) be a 3-dimensional projective space over a field $\mathbb{F}$. A *regulus* of PG($3, \mathbb{F}$) is a set $\mathcal{R}$ of mutually disjoint lines of PG($3, \mathbb{F}$) satisfying the following two properties:

- If a line L of PG($3, \mathbb{F}$) meets three distinct lines of $\mathcal{R}$, then L meets every line of $\mathcal{R}$;
- If a line L of PG($3, \mathbb{F}$) meets three distinct lines of $\mathcal{R}$, then every point of L is incident with (exactly) one line of $\mathcal{R}$.

Any three mutually disjoint lines L_1, L_2, L_3 of PG($3, \mathbb{F}$) are contained in a unique regulus which we will denote by $\mathcal{R}(L_1, L_2, L_3)$.

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Let $n \in \mathbb{N} \setminus \{0,1,2\}$ and $\mathbb{F}$ a field. Recall that a *spread* of the projective space $\mathrm{PG}(n,\mathbb{F})$ is a set of lines which determines a partition of the point set of $\mathrm{PG}(n,\mathbb{F})$. A spread S is called *regular* if the following two conditions are satisfied:

- (R1) If π is a 3-dimensional subspace of $\mathrm{PG}(n,\mathbb{F})$ containing two distinct elements of S, then the elements of S contained in π determine a spread of π;
- (R2) If L_1, L_2 and L_3 are three distinct lines of S which are contained in some 3-dimensional subspace, then $\mathcal{R}(L_1,L_2,L_3) \subseteq S$.

2.2. Classification of regular spreads. Let $n \in \mathbb{N}\setminus\{0,1\}$ and let $\mathbb{F},\mathbb{F}'$ be fields such that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$. Let V' be an n-dimensional vector space over $\mathbb{F}'$ with basis $\{\bar{e}_1,\dots,\bar{e}_n\}$. We denote by V the set of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{e}_1,\dots,\bar{e}_n\}$. Then V can be regarded as an n-dimensional vector space over $\mathbb{F}$. We denote the projective spaces associated with V and V' by $\mathrm{PG}(V)$ and $\mathrm{PG}(V')$, respectively. Since every 1-dimensional subspace of V is contained in a unique 1-dimensional subspace of V', we can regard the points of $\mathrm{PG}(V)$ as points of $\mathrm{PG}(V')$. So, $\mathrm{PG}(V)$ can be regarded as a sub-(projective)-geometry of $\mathrm{PG}(V')$. Any subgeometry of $\mathrm{PG}(V')$ which can be obtained in this way is called a *Baer-$\mathbb{F}$-subgeometry* of $\mathrm{PG}(V')$. Notice also that every subspace π of $\mathrm{PG}(V)$ generates a subspace π' of $\mathrm{PG}(V')$ of the same dimension as π.

The following lemma is known (and easy to prove).

LEMMA 2.1. *Every point p of $\mathrm{PG}(V')$ not contained in $\mathrm{PG}(V)$ is contained in a unique line of $\mathrm{PG}(V')$ which intersects $\mathrm{PG}(V)$ in a line of $\mathrm{PG}(V)$, i.e. there exists a unique line L of $\mathrm{PG}(V)$ for which $p \in L'$.*

The line L in Lemma 2.1 is called the line of $\mathrm{PG}(V)$ *induced* by p.

Suppose now that $\mathbb{F}'$ is a separable (and hence also Galois) extension of $\mathbb{F}$ and let ψ denote the unique nontrivial element in $Gal(\mathbb{F}'/\mathbb{F})$. For every vector $\bar{x} = \sum_{i=1}^n k_i\bar{e}_i$ of V', we define $\bar{x}^\psi := \sum_{i=1}^n k_i^\psi \bar{e}_i$. For every point $p = \langle \bar{x} \rangle$ of $\mathrm{PG}(V')$, we define $p^\psi := \langle \bar{x}^\psi \rangle$ and for every subspace π of $\mathrm{PG}(V')$ we define $\pi^\psi := \{p^\psi \,|\, p \in \pi\}$. The subspace π^ψ is called *conjugate to* π with respect to ψ. Notice that if π is a subspace of $\mathrm{PG}(V)$, then ${\pi'}^\psi = \pi'$.

The following proposition is taken from Beutelspacher and Ueberberg [1, Theorem 1.2] and generalizes a result from Bruck [2]. See also the discussion in Section 4.

PROPOSITION 2.2 ([1]).

(a) *Let $t \in \mathbb{N}\setminus\{0,1\}$ and let $\mathbb{F},\mathbb{F}'$ be fields such that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$. Regard $\mathrm{PG}(2t-1,\mathbb{F})$ as a Baer-$\mathbb{F}$-subgeometry of $\mathrm{PG}(2t-1,\mathbb{F}')$. Let π be*

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a $(t-1)$-dimensional subspace of $\mathrm{PG}(2t-1,\mathbb{F}')$ disjoint from $\mathrm{PG}(2t-1,\mathbb{F})$. Then the set S_π of all lines of $\mathrm{PG}(2t-1,\mathbb{F})$ which are induced by the points of π is a regular spread of $\mathrm{PG}(2t-1,\mathbb{F})$.

(b) *Suppose $t\in\mathbb{N}\setminus\{0,1\}$ and that $\mathbb{F}$ is a field. If S is a regular spread of the projective space $\mathrm{PG}(2t-1,\mathbb{F})$, then there exists a quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ such that the following holds if we regard $\mathrm{PG}(2t-1,\mathbb{F})$ as a Baer-$\mathbb{F}$-subgeometry of $\mathrm{PG}(2t-1,\mathbb{F}')$:*

- (i) *If $\mathbb{F}'$ is a separable field extension of $\mathbb{F}$, then there are precisely two $(t-1)$-dimensional subspaces π of $\mathrm{PG}(2t-1,\mathbb{F}')$ disjoint from $\mathrm{PG}(2t-1,\mathbb{F})$ for which $S=S_\pi$.*
- (ii) *If $\mathbb{F}'$ is a non-separable field extension of $\mathbb{F}$, then there is exactly one $(t-1)$-dimensional subspace π of $\mathrm{PG}(2t-1,\mathbb{F}')$ disjoint from $\mathrm{PG}(2t-1,\mathbb{F})$ for which $S=S_\pi$.*

Remark 2.3. In Proposition 2.2(bi), the two $(t-1)$-dimensional subspaces π_1 and π_2 of $\mathrm{PG}(2t-1,\mathbb{F}')$ disjoint from $\mathrm{PG}(2t-1,\mathbb{F})$ for which $S=S_{\pi_1}=S_{\pi_2}$ are conjugate with respect to the unique nontrivial element ψ of $Gal(\mathbb{F}'/\mathbb{F})$. For, a line L of $\mathrm{PG}(2t-1,\mathbb{F})$ belongs to S_{π_1} if and only if L' intersects π_1, i.e., if and only if $L'={L'}^{\psi}$ intersects π_1^{ψ}.

3. Proof of the Main Theorem.

3.1. An inequality. Let $\mathbb{F}$ and $\mathbb{F}'$ be two fields such that $\mathbb{F}'$ is a quadratic extension of $\mathbb{F}$. Let δ be an arbitrary element of $\mathbb{F}'\setminus\mathbb{F}$ and let μ_1,μ_2 be the unique elements of $\mathbb{F}$ such that $\delta^2=\mu_1\delta+\mu_2$. Then $\mu_2\neq 0$. Let $m\geq 1$ and let V' be a $2m$-dimensional vector space over $\mathbb{F}'$ with basis $\{\bar{e}_1^*,\bar{e}_2^*,\dots,\bar{e}_{2m}^*\}$. We denote by V the set of all $\mathbb{F}$-linear combinations of the elements of $\{\bar{e}_1^*,\bar{e}_2^*,\dots,\bar{e}_{2m}^*\}$. Then V can be regarded as a $2m$-dimensional vector space over $\mathbb{F}$. We denote the projective spaces associated with V and V' by $\mathrm{PG}(V)$ and $\mathrm{PG}(V')$, respectively. The projective space $\mathrm{PG}(V)$ can be regarded in a natural way as a subgeometry of $\mathrm{PG}(V')$. Every subspace α of $\mathrm{PG}(V)$ then generates a subspace α' of $\mathrm{PG}(V')$ of the same dimension as α.

Now, let π be an $(m-1)$-dimensional subspace of $\mathrm{PG}(V')$ disjoint from $\mathrm{PG}(V)$. Then there exist vectors $\bar{e}_1,\bar{f}_1,\dots,\bar{e}_m,\bar{f}_m$ such that $\pi=\langle\bar{e}_1+\delta\bar{f}_1,\bar{e}_2+\delta\bar{f}_2,\dots,\bar{e}_m+\delta\bar{f}_m\rangle$.

Lemma 3.1. *$\{\bar{e}_1,\bar{f}_1,\bar{e}_2,\bar{f}_2,\dots,\bar{e}_m,\bar{f}_m\}$ is a basis of V.*

Proof. If this were not the case, then there exist $a_1,b_1,\dots,a_m,b_m\in\mathbb{F}$ with $(a_1,b_1,\dots,a_m,b_m)\neq(0,0,\dots,0,0)$ such that $a_1\bar{e}_1+b_1\bar{f}_1+\cdots+a_m\bar{e}_m+b_m\bar{f}_m=\bar{o}$. Now, put $k_i:=a_i+\frac{b_i}{\mu_2}\delta$ for every $i\in\{1,\dots,m\}$. Then $(k_1,\dots,k_m)\neq(0,\dots,0)$

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since $(a_1, b_1, \dots, a_m, b_m) \neq (0, 0, \dots, 0, 0)$. Since $k_1(\bar{e}_1 + \delta\bar{f}_1) + \cdots + k_m(\bar{e}_m + \delta\bar{f}_m) = \delta(a_1\bar{f}_1 + \frac{b_1}{\mu_2}\bar{e}_1 + \frac{\mu_1}{\mu_2}b_1\bar{f}_1 + \cdots + a_m\bar{f}_m + \frac{b_m}{\mu_2}\bar{e}_m + \frac{\mu_1}{\mu_2}b_m\bar{f}_m)$, the subspace π is not disjoint from $\mathrm{PG}(V)$, a contradiction. So, $\{\bar{e}_1, \bar{f}_1, \dots, \bar{e}_m, \bar{f}_m\}$ is a basis of V. □

Now, let $k \in \{1, \dots, 2m\}$. Let W_1 denote the subspace of $\bigwedge^k V$ generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$ where $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ are k linearly independent vectors of V such that $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\rangle'$ meets π. (If there are no such vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$, then $W_1 = 0$.) We will prove by induction on m that $\dim(W_1) \geq \binom{2m}{k} - 2 \cdot \binom{m}{k}$.

If $k = 1$, then $W_1 = 0$ since $\pi \cap \mathrm{PG}(V) = \emptyset$. Hence, $\dim(W_1) = 0 = \binom{2m}{1} - 2 \cdot \binom{m}{1}$.

Suppose $k = 2m$. Since $\pi \subseteq \langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{2m}\rangle'$ for every $2m$ linearly independent vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{2m}$ of V, we have $W_1 = \bigwedge^{2m} V$ and hence $\dim(W_1) = 1 = \binom{2m}{2m} - 2 \cdot \binom{m}{2m}$.

In the sequel, we may suppose that $m \geq 2$ and $k \in \{2, \dots, 2m-1\}$. Put $U = \langle \bar{e}_2, \bar{f}_2, \dots, \bar{e}_m, \bar{f}_m\rangle$. Every vector χ of $\bigwedge^k V$ can be written in a unique way as

$$\bar{e}_1 \wedge \bar{f}_1 \wedge \alpha(\chi) + \bar{e}_1 \wedge \beta(\chi) + \bar{f}_1 \wedge \gamma(\chi) + \delta(\chi),$$

where $\alpha(\chi) \in \bigwedge^{k-2} U$, $\beta(\chi) \in \bigwedge^{k-1} U$, $\gamma(\chi) \in \bigwedge^{k-1} U$ and $\delta(\chi) \in \bigwedge^k U$. [Here, $\bigwedge^0 U = \mathbb{F}$ and $\bigwedge^{2m-1} U = 0$.] Let θ denote the linear map from $W_1 \subseteq \bigwedge^k V$ to $\bigwedge^{k-1} U$ mapping χ to $\gamma(\chi)$. Then by the rank-nullity theorem,

$$\dim(W_1) = \dim(ker(\theta)) + \dim(Im(\theta)). \tag{3.1}$$

Lemma 3.2. *We have* $\dim(ker(\theta)) \geq \binom{2m-2}{k-2} + \binom{2m-1}{k} - 2 \cdot \binom{m}{k}$.

Proof. (a) If $\bar{v}_3, \dots, \bar{v}_k$ are $k-2$ linearly independent vectors of U, then $\langle \bar{e}_1, \bar{f}_1, \bar{v}_3, \dots, \bar{v}_k\rangle'$ meets π and hence $\bar{e}_1 \wedge \bar{f}_1 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_k \in W_1$. It follows that $\bar{e}_1 \wedge \bar{f}_1 \wedge \bigwedge^{k-2} U \subseteq ker(\theta)$.

(b) Let Z_1 denote the subspace of $\bigwedge^{k-1} U$ generated by all vectors $\bar{v}_2 \wedge \bar{v}_3 \wedge \cdots \wedge \bar{v}_k$ where $\bar{v}_2, \dots, \bar{v}_k$ are $k-1$ linearly independent vectors of U such that $\langle \bar{v}_2, \dots, \bar{v}_k\rangle'$ meets $\langle \bar{e}_2 + \delta\bar{f}_2, \dots, \bar{e}_m + \delta\bar{f}_m\rangle$. By the induction hypothesis, $\dim(Z_1) \geq \binom{2m-2}{k-1} - 2 \cdot \binom{m-1}{k-1}$. Clearly, $\bar{e}_1 \wedge Z_1 \subseteq ker(\theta)$.

(c) Suppose $k \leq 2m-2$. Let Z_2 denote the subspace of $\bigwedge^k U$ generated by all vectors $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k$, where $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ are k linearly independent vectors of U such that $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\rangle'$ meets $\langle \bar{e}_2 + \delta\bar{f}_2, \dots, \bar{e}_m + \delta\bar{f}_m\rangle$. By the induction hypothesis, $\dim(Z_2) \geq \binom{2m-2}{k} - 2 \cdot \binom{m-1}{k}$. Clearly, $Z_2 \subseteq ker(\theta)$.

By (a), (b), (c) and the decomposition $\bigwedge^k V = \left(\bar{e}_1 \wedge \bar{f}_1 \wedge \bigwedge^{k-2} U\right) \oplus \left(\bar{e}_1 \wedge \bigwedge^{k-1} U\right) \oplus \left(\bar{f}_1 \wedge \bigwedge^{k-1} U\right) \oplus \left(\bigwedge^k U\right)$, we have $\dim(ker(\theta)) \geq \binom{2m-2}{k-2} + \binom{2m-2}{k-1} - 2 \cdot$

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$\binom{m-1}{k-1}+\binom{2m-2}{k}-2\cdot\binom{m-1}{k}=\binom{2m-2}{k-2}+\binom{2m-1}{k}-2\cdot\binom{m}{k}$. Notice that this inequality remains valid if $k=2m-1$ since $\binom{2m-2}{k}-2\cdot\binom{m-1}{k}=0$ in this case. ◻

Lemma 3.3. *We have* $Im(\theta)=\bigwedge^{k-1}U$. *Hence,* $\dim(Im(\theta))=\binom{2m-2}{k-1}$.

Proof. It suffices to prove that every vector of the form $\bar{g}_2\wedge\bar{g}_3\wedge\cdots\wedge\bar{g}_k$ belongs to $Im(\theta)$, where $\bar{g}_2,\bar{g}_3,\ldots,\bar{g}_k$ are $k-1$ distinct elements of $\{\bar{e}_2,\bar{f}_2,\ldots,\bar{e}_m,\bar{f}_m\}$. Without loss of generality, we may suppose that $\bar{g}_2\in\{\bar{e}_2,\bar{f}_2\}$. Since $\langle(\bar{e}_1+\bar{e}_2)+\delta(\bar{f}_1+\bar{f}_2)\rangle$ belongs to π, $(\bar{e}_1+\bar{e}_2)\wedge(\bar{f}_1+\bar{f}_2)\wedge\bar{g}_3\wedge\cdots\wedge\bar{g}_k\in W_1$ and hence $\bar{e}_2\wedge\bar{g}_3\wedge\cdots\wedge\bar{g}_k\in Im(\theta)$. Since $\langle(\bar{e}_1+\delta\bar{f}_1)+\delta(\bar{e}_2+\delta\bar{f}_2)\rangle=\langle(\bar{e}_1+\mu_2\bar{f}_2)+\delta(\bar{f}_1+\bar{e}_2+\mu_1\bar{f}_2)\rangle$ belongs to π, $(\bar{e}_1+\mu_2\bar{f}_2)\wedge(\bar{f}_1+\bar{e}_2+\mu_1\bar{f}_2)\wedge\bar{g}_3\wedge\cdots\wedge\bar{g}_k\in W_1$ and hence $\bar{f}_2\wedge\bar{g}_3\wedge\cdots\wedge\bar{g}_k\in Im(\theta)$ (recall $\mu_2\neq 0$). ◻

Corollary 3.4. *We have* $\dim(W_1)\geq\binom{2m}{k}-2\cdot\binom{m}{k}$.

Proof. By equation (3.1) and Lemmas 3.2, 3.3, we have that $\dim(W_1)\geq\binom{2m-2}{k-1}+\binom{2m-2}{k-2}+\binom{2m-1}{k}-2\cdot\binom{m}{k}=\binom{2m-1}{k-1}+\binom{2m-1}{k}-2\cdot\binom{m}{k}=\binom{2m}{k}-2\cdot\binom{m}{k}$. ◻

3.2. Proof of Theorem 1.1. We continue with the notation introduced in Section 3.1. We suppose here that $m\geq 2$ and $k\in\{1,\ldots,2m-1\}$. Let S be the spread of $\mathrm{PG}(V)$ induced by the points of π (recall Proposition 2.2(a)) and let X_S denote the set of all $(k-1)$-dimensional subspaces of $\mathrm{PG}(V)$ which contain at least one line of S.

Lemma 3.5. *A* $(k-1)$*-dimensional subspace* α *of* $\mathrm{PG}(V)$ *contains a line of* S *if and only if* α' *meets* π.

Proof. Suppose α contains a line L of S. Since α' contains the line L' which meets π, α' must also meet π.

Conversely, suppose that α' meets π and let p be an arbitrary point in the intersection $\alpha'\cap\pi$. Then in the subspace α' there exists a unique line L' through p which meets α in a line L (recall Lemma 2.1). Since L is a line of $\mathrm{PG}(V)$, we must necessarily have $L\in S$. So, α contains a line of S. ◻

Corollary 3.6. *If* $k\in\{m+1,m+2,\ldots,2m-1\}$, *then* X_S *consists of all* $(k-1)$*-dimensional subspaces of* $\mathrm{PG}(V)$.

Let W_2 denote the subspace of $\bigwedge^k V$ consisting of all vectors $\chi\in\bigwedge^k V$ satisfying $(\bar{e}_1+\delta\bar{f}_1)\wedge(\bar{e}_2+\delta\bar{f}_2)\wedge\cdots\wedge(\bar{e}_m+\delta\bar{f}_m)\wedge\chi=0$.

Lemma 3.7.

(1) *The subspace* $\mathrm{PG}(W_1)$ *is generated by all points* $e_{gr}(\alpha)$ *where* α *is some element of* X_S.

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(2) *A* $(k-1)$*-dimensional subspace* α *of* $\mathrm{PG}(V)$ *belongs to* X_S *if and only if* $e_{gr}(\alpha) \in \mathrm{PG}(W_2)$.
(3) $\mathrm{PG}(W_1) \subseteq \mathrm{PG}(W_2)$.

Proof. Claim (1) is an immediate corollary of Lemma 3.5 and the definition of the subspace W_1. By Lemma 3.5, a $(k-1)$-dimensional subspace $\alpha = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle$ of $\mathrm{PG}(V)$ belongs to X_S if and only if π meets $\alpha' = \langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k \rangle'$, i.e. if and only if $(\bar{e}_1 + \delta \bar{f}_1) \wedge (\bar{e}_2 + \delta \bar{f}_2) \wedge \cdots \wedge (\bar{e}_m + \delta \bar{f}_m) \wedge \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_k = 0$, i.e. if and only if $e_{gr}(\alpha) \in \mathrm{PG}(W_2)$. Claim (3) follows directly from Claims (1) and (2). □

Lemma 3.8. *We have* $\dim(W_2) \leq \binom{2m}{k} - 2 \cdot \binom{m}{k}$.

Proof. If $k \in \{m+1, \ldots, 2m-1\}$, then $W_2 = \bigwedge^k V$ and hence $\dim(W_2) = \binom{2m}{k} = \binom{2m}{k} - 2 \cdot \binom{m}{k}$. We may therefore suppose that $k \in \{1, \ldots, m\}$.

Let T denote the set of all $(m-k)$-tuples $(i_1, \ldots, i_{m-k})$, where $i_1, \ldots, i_{m-k} \in \{1, \ldots, m\}$ satisfies $i_1 < i_2 < \cdots < i_{m-k}$. We take the convention here that if $k = m$, then $|T| = 1$ and T consists of the unique "0-tuple". If $\tau \in T$, then $\chi \in W_2$ implies that

$$\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_{m-k}} \wedge (\bar{e}_1 + \delta \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m + \delta \bar{f}_m) \wedge \chi = 0. \tag{3.2}$$

We can write (3.2) as

$$(\alpha_\tau + \delta \beta_\tau) \wedge \chi = 0, \tag{3.3}$$

where

$$\alpha_\tau + \delta \beta_\tau = \frac{\bar{e}_{i_1} \wedge \cdots \wedge \bar{e}_{i_{m-k}} \wedge (\bar{e}_1 + \delta \bar{f}_1) \wedge \cdots \wedge (\bar{e}_m + \delta \bar{f}_m)}{\delta^{m-k}},$$

$$\alpha_\tau, \beta_\tau \in \bigwedge^{2m-k} V.$$

Equation (3.3) is equivalent with

$$\begin{cases} \alpha_\tau \wedge \chi &= 0, \\ \beta_\tau \wedge \chi &= 0. \end{cases} \tag{3.4}$$

Consider now a basis B of $\bigwedge^{2m-k} V$ which consists only of vectors of the form $\bar{g}_1 \wedge \bar{g}_2 \wedge \cdots \wedge \bar{g}_{2m-k}$, where $\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_{2m-k} \in \{\bar{e}_1, \bar{f}_1, \ldots, \bar{e}_m, \bar{f}_m\}$.

The $2 \cdot \binom{m}{m-k} = 2 \cdot \binom{m}{k}$ equations determined by (3.4) are linearly independent if and only if the $2 \cdot \binom{m}{k}$ vectors α_τ, β_τ $(\tau \in T)$ are linearly independent.

Suppose there exist $k_\tau, l_\tau \in \mathbb{F}$ $(\tau \in T)$ such that

$$\sum_{\tau \in T} (k_\tau \alpha_\tau + l_\tau \beta_\tau) = 0. \tag{3.5}$$

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Take an arbitrary $\tau^* = (i_1, i_2, \ldots, i_{m-k})$ of T. If we write the left hand side of equation (3.5) as a linear combination of the elements of the basis B of $\bigwedge^{2m-k} V$, then the sum of all terms which contain the factor $(\bar{e}_{i_1} \wedge \bar{f}_{i_1}) \wedge (\bar{e}_{i_2} \wedge \bar{f}_{i_2}) \wedge \cdots \wedge (\bar{e}_{i_{m-k}} \wedge \bar{f}_{i_{m-k}})$ must be 0. This implies that $k_{\tau^*}\alpha_{\tau^*} + l_{\tau^*}\beta_{\tau^*} = 0$. Now, the two vectors α_{τ^*} and β_{τ^*} are linearly independent: α_{τ^*} contains a term which is a multiple of $\bar{e}_1 \wedge \bar{e}_2 \wedge \cdots \wedge \bar{e}_m \wedge \bar{f}_{i_1} \wedge \bar{f}_{i_2} \wedge \cdots \wedge \bar{f}_{i_{m-k}}$, while β_{τ^*} does not contain such a term; for every $j \in \{1, \ldots, m\} \setminus \{i_1, \ldots, i_{m-k}\}$, β_{τ^*} contains a term which is a multiple of $\bar{e}_1 \wedge \cdots \wedge \bar{e}_{j-1} \wedge \widehat{\bar{e}_j} \wedge \bar{e}_{j+1} \wedge \cdots \wedge \bar{e}_m \wedge \bar{f}_j \wedge \bar{f}_{i_1} \wedge \bar{f}_{i_2} \wedge \cdots \wedge \bar{f}_{i_{m-k}}$, while α_{τ^*} does not contain such a term. We conclude that $k_{\tau^*} = l_{\tau^*} = 0$. Since τ^* was an arbitrary element of T, we can indeed conclude that the vectors α_τ, β_τ $(\tau \in T)$ are linearly independent.

Since the vectors χ of W_2 satisfy a linear system of $2 \cdot \binom{m}{k}$ linearly independent equations (recall (3.4)), we can indeed conclude that $\dim(W_2) \leq \binom{2m}{k} - 2 \cdot \binom{m}{k}$. $\square$

Theorem 1.1 is now an immediate consequence of Corollary 3.4 and Lemmas 3.7, 3.8.

4. On the classification of the regular spreads of PG$(3, \mathbb{F})$**.** Proposition 2.2(b) plays an essential role in this paper. The proof of Proposition 2.2(b) given in [1] consists of two parts. In [1, Section 3], the case $t = 2$ was treated and subsequently this classification was used in [1, Section 5] to obtain also a classification in the case $t \geq 3$. In the proof for the case $t = 2$, a gap seems to occur. Indeed, in [1, Section 3] the authors tacitly assume that the lines and reguli of a given regular spread determine a Möbius plane. This fact is trivial in the finite case, where one could use a simple counting argument to prove it, but not at all obvious in the infinite case.

The aim of this section is to fill this apparent gap. We give a proof for Proposition 2.2(b) in the case that t is equal to 2. The methods used here will be different from the ones of [1]. Our treatment will be more geometric and based on the Klein correspondence. A discussion of regular spreads of finite 3-dimensional projective spaces can also be found in [3, Section 17.1]. Some of the tools we need here are already in [3], either explicitly or implicitly.

Let V be a 4-dimensional vector space over a field $\mathbb{F}$. For every line $L = \langle \bar{u}_1, \bar{u}_2 \rangle$ of PG(V), let $\kappa(L)$ denote the point $\langle \bar{u}_1 \wedge \bar{u}_2 \rangle$ of PG$(\bigwedge^2 V)$. The image Q of κ is a nonsingular quadric of Witt index 3 of PG$(\bigwedge^2 V)$. If $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ is a basis of V, then the equation of Q with respect to the ordered basis $B^* := (\bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{e}_4, \bar{e}_2 \wedge \bar{e}_3, \bar{e}_2 \wedge \bar{e}_4, \bar{e}_3 \wedge \bar{e}_4)$ of $\bigwedge^2 V$ is equal to $X_1X_6 - X_2X_5 + X_3X_4 = 0$. The bijective correspondence κ between the set of lines of PG(V) and the set of points of Q is often referred to as the *Klein correspondence.* For every point x of PG(V), let $\mathcal{L}_x$ denote the set of lines of PG(V) containing x and for every plane π of PG(V), let

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$\mathcal{L}_\pi$ denote the set of lines of $\mathrm{PG}(V)$ contained in π. The sets $\kappa(\mathcal{L}_x)$ and $\kappa(\mathcal{L}_\pi)$ are generators of Q. Let $\mathcal{M}^+$ [respectively, $\mathcal{M}^-$] denote the set of generators of Q of the form $\kappa(\mathcal{L}_x)$ [respectively, $\kappa(\mathcal{L}_\pi)$] for some point x [respectively, plane π] of $\mathrm{PG}(V)$. Then $\mathcal{M}^+$ and $\mathcal{M}^-$ are the two families of generators of Q, i.e. (i) $\mathcal{M}^+ \cap \mathcal{M}^- = \emptyset$, (ii) $\mathcal{M}^+ \cup \mathcal{M}^-$ consists of all generators of Q, and (iii) two generators of Q belong to the same family $\mathcal{M}^\epsilon$ for some $\epsilon \in \{+,-\}$ if and only if they intersect in a subspace of even co-dimension. Every line of Q is contained in precisely two generators, one generator of $\mathcal{M}^+$ and one generator of $\mathcal{M}^-$.

The following three lemmas are known and their proofs are straightforward.

LEMMA 4.1. *Let $\mathcal{R}$ be a regulus of $\mathrm{PG}(V)$. Then there exists a 2-dimensional subspace α of $\mathrm{PG}(\bigwedge^2 V)$ such that $\kappa(\mathcal{R}) = \alpha \cap Q$ is a nonsingular quadric of Witt index 1 of α.*

LEMMA 4.2. *Suppose α is a 3-dimensional subspace of $\mathrm{PG}(\bigwedge^2 V)$ which intersects Q in a nonsingular quadric of Witt index 1 of α. Then the set S of all lines L of $\mathrm{PG}(V)$ for which $\kappa(L) \in \alpha$ is a regular spread of $\mathrm{PG}(V)$.*

LEMMA 4.3. *Suppose α is a 3-dimensional subspace of $\mathrm{PG}(\bigwedge^2 V)$ and that S is a spread of $\mathrm{PG}(V)$ such that $\alpha \cap Q \subseteq \kappa(S)$. Then α intersects Q in a nonsingular quadric of Witt index 1 of α. Moreover, $\alpha \cap Q = \kappa(S)$.*

LEMMA 4.4. *Suppose $\mathbb{F} = \mathbb{F}_2$. Then $\mathrm{PG}(V) = \mathrm{PG}(3,2)$. The following hold:*

(1) *Every spread of $\mathrm{PG}(V)$ is regular.*
(2) *Every regulus of $\mathrm{PG}(V)$ can be extended to a unique spread of $\mathrm{PG}(V)$.*
(3) *If S is a regular spread of $\mathrm{PG}(V)$, then there exists a unique subspace α of dimension 3 of $\mathrm{PG}(\bigwedge^2 V)$ such that $\kappa(S) = \alpha \cap Q$ is a nonsingular quadric of Witt index 1 of α.*

Proof. Claims (1) and (2) are well known and easy to prove. So, we will only give a proof for Claim (3). Suppose S is a (regular) spread of $\mathrm{PG}(V)$ and $\mathcal{R}$ a regulus contained in S. Then by Lemma 4.1 there exists a 2-dimensional subspace β of $\mathrm{PG}(\bigwedge^2 V)$ such that $\kappa(\mathcal{R}) = \beta \cap Q$ is a nonsingular conic of β. Now, by an easy counting argument there are three 3-dimensional subspaces γ_1 through β which intersect Q in a singular quadric of γ_1 (namely the subspaces $\langle \beta, \kappa(M) \rangle$ where M is one of the three lines of $\mathrm{PG}(V)$ meeting each line of $\mathcal{R}$), three 3-dimensional subspaces γ_2 through β which intersect Q in a nonsingular hyperbolic quadric of γ_2 and one 3-dimensional subspace α through β which intersects Q in a nonsingular elliptic quadric of α. Since $\kappa^{-1}(\alpha \cap Q)$ is a spread containing $\mathcal{R}$, $\kappa^{-1}(\alpha \cap Q) = S$ by Claim (2). Hence, $\alpha \cap Q = \kappa(S)$. □

LEMMA 4.5. *Suppose $|\mathbb{F}| \geq 3$. If S is a regular spread of $\mathrm{PG}(V)$, then there*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 20, pp. 354-366, July 2010

exists a unique subspace α of dimension 3 *of* $\mathrm{PG}(\bigwedge^2 V)$ *such that $\kappa(S) = \alpha \cap Q$ is a nonsingular quadric of Witt index 1 of α.*

Proof. Let L_1, L_2, L_3 and L_4 be four distinct lines of S such that $L_4 \notin \mathcal{R}(L_1, L_2, L_3)$. Put $\mathcal{R}_1 = \mathcal{R}(L_1, L_2, L_3)$ and $\mathcal{R}_2 = \mathcal{R}(L_1, L_2, L_4)$. By Lemma 4.1, there exists a 2-dimensional subspace α_i, $i \in \{1, 2\}$, of $\mathrm{PG}(\bigwedge^2 V)$ such that $\kappa(\mathcal{R}_i) = \alpha_i \cap Q$. Since $\mathcal{R}_1 \neq \mathcal{R}_2$, we have $\alpha_1 \neq \alpha_2$. Since $\kappa(L_1)$ and $\kappa(L_2)$ are contained in α_1 and α_2, $\alpha_1 \cap \alpha_2$ is a line and $\alpha := \langle \alpha_1, \alpha_2 \rangle$ is a 3-dimensional subspace of $\mathrm{PG}(\bigwedge^2 V)$.

We prove that every point x of $\alpha \cap Q$ belongs to $\kappa(S)$. Clearly, $\alpha_1 \cap Q = \kappa(\mathcal{R}_1) \subseteq \kappa(S)$ and $\alpha_2 \cap Q = \kappa(\mathcal{R}_2) \subseteq \kappa(S)$. So, we may assume that $x \in (\alpha \cap Q) \setminus (\alpha_1 \cup \alpha_2)$. Let M denote a line through x which meets α_1 in a point y_1 of $(\alpha_1 \cap Q) \setminus \alpha_2$ and let y_2 be the intersection of M with α_2. Since $|\mathbb{F}| \geq 3$, we may suppose that we have chosen M in such a way that y_2 is not the kernel of the quadric $\alpha_2 \cap Q$ of α_2 in the case the characteristic of $\mathbb{F}$ is equal to 2. Then there exists a line $N \subseteq \alpha_2$ through y_2 which intersects $Q \cap \alpha_2$ in two points, say u and v. The plane $\alpha_3 := \langle M, N \rangle$ through M is contained in α and contains the points y_1, u and v of $\kappa(\mathcal{R}_1 \cup \mathcal{R}_2)$. So, there exist three distinct lines U, V and W of $\mathcal{R}_1 \cup \mathcal{R}_2$ such that $\kappa(U)$, $\kappa(V)$ and $\kappa(W)$ belong to α_3. If $\mathcal{R}_3$ denotes the unique regulus of $\mathrm{PG}(V)$ containing U, V and W, then $\kappa(\mathcal{R}_3) = \alpha_3 \cap Q$ by Lemma 4.1. Now, $\mathcal{R}_3 \subseteq S$ since S is regular and $x \in \alpha_3 \cap Q$. So, there exists a line $L \in S$ such that $x = \kappa(L)$. This is what we needed to prove.

By the above, we know that $\alpha \cap Q \subseteq \kappa(S)$. Lemma 4.3 then implies that $\alpha \cap Q = \kappa(S)$ is a nonsingular quadric of Witt index 1 of α. ◻

Now, let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$ (which is unique, up to isomorphism) and let $\overline{V}$ denote a 4-dimensional vector space over $\overline{\mathbb{F}}$ which also has $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$ as basis. We will regard $\mathrm{PG}(V)$ as a subgeometry of $\mathrm{PG}(\overline{V})$ and $\mathrm{PG}(\bigwedge^2 V)$ as a subgeometry of $\mathrm{PG}(\bigwedge^2 \overline{V})$.

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$ which is contained in $\overline{\mathbb{F}}$. Let $V_{\mathbb{K}}$ denote the set of all $\mathbb{K}$-linear combinations of the elements of $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$. Then $V_{\mathbb{K}}$ can be regarded as a vector space over $\mathbb{K}$. We will regard $\mathrm{PG}(V)$ as a subgeometry of $\mathrm{PG}(V_{\mathbb{K}})$ and $\mathrm{PG}(V_{\mathbb{K}})$ as a subgeometry of $\mathrm{PG}(\overline{V})$. Similarly, we will regard $\mathrm{PG}(\bigwedge^2 V)$ as a subgeometry of $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ and $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ as a subgeometry of $\mathrm{PG}(\bigwedge^2 \overline{V})$. Every subspace α of $\mathrm{PG}(V)$ (respectively $\mathrm{PG}(\bigwedge^2 V)$) then generates a subspace $\alpha_{\mathbb{K}}$ of $\mathrm{PG}(V_{\mathbb{K}})$ (respectively $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$) with the same dimension as α. We define $\overline{\alpha} := \alpha_{\overline{\mathbb{F}}}$ and $\overline{\alpha_{\mathbb{K}}} := \alpha_{\overline{\mathbb{F}}}$.

We denote by $Q_{\mathbb{K}}$ the quadric of $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ whose equation with respect to B^* is equal to $X_1X_6 - X_2X_5 + X_3X_4 = 0$, and put $\overline{Q} := Q_{\overline{\mathbb{F}}}$. Then $Q \subseteq Q_{\mathbb{K}} \subseteq \overline{Q}$. The Klein correspondence between the set of lines of $\mathrm{PG}(V_{\mathbb{K}})$ and the points of $Q_{\mathbb{K}}$ will be denoted by $\kappa_{\mathbb{K}}$. We define $\overline{\kappa} := \kappa_{\overline{\mathbb{F}}}$. Notice that two distinct lines L_1 and L_2 of

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Volume 20, pp. 354-366, July 2010

$\mathrm{PG}(\overline{V})$ meet if and only if the points $\overline{\kappa}(L_1)$ and $\overline{\kappa}(L_2)$ are $\overline{Q}$-collinear.

Now, suppose S is a regular spread of $\mathrm{PG}(V)$. Then by Lemmas 4.4 and 4.5, there exists a unique subspace α of dimension 3 of $\mathrm{PG}(\bigwedge^2 V)$ such that $\kappa(S) = \alpha \cap Q$ is a non-singular quadric of Witt index 1 of α. With respect to a suitable reference system of α, the quadric $\alpha \cap Q$ of α has equation $f(X_0, X_1) + X_2X_3 = 0$, where $f(X_0, X_1)$ is an irreducible quadratic polynomial of $\mathbb{F}[X_0, X_1]$. Now, there exists a unique quadratic extension $\mathbb{K}$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$ such that $f(X_0, X_1)$ is reducible when regarded as a polynomial of $\mathbb{K}[X_0, X_1]$. This quadratic extension $\mathbb{K}$ is independent from the reference system of α with respect to which the equation of $\alpha \cap Q$ is of the form $f(X_0, X_1) + X_2X_3 = 0$. Now, we can distinguish two cases.

(I) The quadratic extension $\mathbb{K}/\mathbb{F}$ is a Galois extension. Let ψ denote the unique element in $Gal(\mathbb{K}/\mathbb{F})$. Then $f(X_0, X_1) = a(X_0 + \delta X_1)(X_0 + \delta^\psi X_1)$ for a certain $a \in \mathbb{F} \setminus \{0\}$ and a certain $\delta \in \mathbb{K} \setminus \mathbb{F}$. It follows that $\alpha_{\mathbb{K}} \cap Q_{\mathbb{K}}$ is a nonsingular quadric of Witt index 2 of $\alpha_{\mathbb{K}}$. If $(X_1, \ldots, X_6)$ are the coordinates of a point p of $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ with respect to the ordered basis B^*, then p^ψ denotes the point of $\mathrm{PG}(\bigwedge^2 V)$ whose coordinates with respect to B^* are equal to $(X_1^\psi, \ldots, X_6^\psi)$. Clearly, $Q_{\mathbb{K}}^\psi = Q_{\mathbb{K}}$.

(II) The quadratic extension $\mathbb{K}/\mathbb{F}$ is not a Galois extension. Then $\mathrm{char}(\mathbb{K}) = 2$ and $f(X_0, X_1) = a(X_0 + \delta X_1)^2$ for some $a \in \mathbb{F} \setminus \{0\}$ and some $\delta \in \mathbb{K} \setminus \mathbb{F}$ satisfying $\delta^2 \in \mathbb{F}$. It follows that $\alpha_{\mathbb{K}} \cap Q_{\mathbb{K}}$ is a singular quadric of $\alpha_{\mathbb{K}}$ having a unique singular point[1].

Now, let X denote the set of all points x of $\overline{Q}$ which are $\overline{Q}$-collinear with every point of $\alpha \cap Q$. Notice that $x \in X$ if and only if $\overline{\kappa}^{-1}(x)$ meets every line $\overline{L}$ where $L \in S$. We prove the following lemma which implies Proposition 2.2(b) in the case $t = 2$.

Lemma 4.6.

(1) *We have* $X \subseteq Q_{\mathbb{K}}$.
(2) *If* $\mathbb{K}/\mathbb{F}$ *is a Galois extension, then* $|X| = 2$. *Moreover, if* $X = \{x_1, x_2\}$, *then* $x_2 = x_1^\psi$.
(3) *If* $\mathbb{K}/\mathbb{F}$ *is not a Galois extension, then* $|X| = 1$.
(4) *If* $x \in X$, *then the points of* Q *which are* $Q_{\mathbb{K}}$*-collinear with* x *are precisely the points of* $\alpha \cap Q$, *or equivalently, the lines of* S *are precisely those lines* L *of* $\mathrm{PG}(V)$ *for which* $L_{\mathbb{K}}$ *meets* $\kappa_{\mathbb{K}}^{-1}(x)$. *The line* $\kappa_{\mathbb{K}}^{-1}(x)$ *of* $\mathrm{PG}(V_{\mathbb{K}})$ *is disjoint from* $\mathrm{PG}(V)$.

[1]With a singular point of a quadric, we mean a point of the quadric with the property that every line though it is a tangent line, i.e. a line which intersects the quadric in either a singleton or the whole line. The tangent hyperplane in a singular point is not defined.

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Volume 20, pp. 354-366, July 2010

Proof. (I) Suppose the quadratic extension $\mathbb{K}/\mathbb{F}$ is a Galois extension. Let L_1 and L_2 be two disjoint lines of $\alpha_{\mathbb{K}} \cap Q_{\mathbb{K}}$ and let β_1, β_2 denote the two planes of $Q_{\mathbb{K}}$ through L_1. Then $\overline{\beta_1}$ and $\overline{\beta_2}$ are the two planes of $\overline{Q}$ through $\overline{L_1}$. Let x_i, $i \in \{1,2\}$, denote the unique point of β_i $Q_{\mathbb{K}}$-collinear with every point of L_2. Then x_i is also the unique point of $\overline{\beta_i}$ $\overline{Q}$-collinear with every point of $\overline{L_2}$.

Let $i \in \{1,2\}$. We prove that $x_i \notin \mathrm{PG}(\bigwedge^2 V)$, or equivalently, that $x_i \notin Q$. Suppose this is not the case and consider the hyperplane T of $\mathrm{PG}(\bigwedge^2 V)$ which is tangent to Q at the point x_i. Then $T_{\mathbb{K}}$ is the hyperplane of $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ which is tangent to $Q_{\mathbb{K}}$ at the point x_i. Since $L_1 \cup L_2 \subseteq T_{\mathbb{K}}$, α is a hyperplane of T not containing x_i and hence $\alpha \cap Q$ would be a nonsingular quadric of Witt index 2 of α, clearly a contradiction.

We prove that $X = \{x_1, x_2\}$. Clearly, $\{x_1, x_2\} \subseteq X$. Conversely, suppose that x is a point of X. Since no point of $\overline{L_1}$ is $\overline{Q}$-collinear with every point of L_2, we have $x \notin \overline{L_1}$. Since x is collinear with every point of $\overline{L_1}$, we have $\langle x, \overline{L_i} \rangle = \overline{\beta_i}$ for some $i \in \{1,2\}$. Since x is $\overline{Q}$-collinear with every point of $L_2 \subseteq \overline{L_2}$, we necessarily have $x = x_i$. Hence, $X = \{x_1, x_2\} \subseteq Q_{\mathbb{K}}$. Since x_1 is $Q_{\mathbb{K}}$-collinear with every point of $\alpha \cap Q$, $x_1^{\psi} \neq x_1$ is $Q_{\mathbb{K}}$-collinear with every point of $(\alpha \cap Q)^{\psi} = \alpha \cap Q$. It follows that $x_2 = x_1^{\psi}$.

(II) Suppose the quadratic extension $\mathbb{K}/\mathbb{F}$ is not a Galois extension. Then $\alpha_{\mathbb{K}} \cap Q_{\mathbb{K}}$ is a singular quadric of $\alpha_{\mathbb{K}}$ with a unique singular point x^*. Clearly, $x^* \notin \mathrm{PG}(\bigwedge^2 V)$ and $x^* \notin Q$.

We prove that $X = \{x^*\}$. Clearly, $x^* \in X$. Suppose now that there exists a point $x \in X \setminus \{x^*\}$. Then x is $\overline{Q}$-collinear with every point of $\overline{\alpha} \cap \overline{Q}$ and hence cannot be contained in $\overline{\alpha}$ since $x \neq x^*$. The points of $\overline{Q}$ which are $\overline{Q}$-collinear with x and x^* are contained in a 3-dimensional subspace of $\mathrm{PG}(\bigwedge^2 \overline{V})$, namely the intersection of the tangent hyperplanes to $\overline{Q}$ at the points x and x^*. This 3-dimensional subspace necessarily coincides with $\overline{\alpha}$ and contains the points x and x^*, a contradiction, since $x \notin \overline{\alpha}$. So, we have that $X = \{x^*\} \subseteq Q_{\mathbb{K}}$.

Now, let x be an arbitrary point of X. Then $x \in \mathrm{PG}(\bigwedge^2 V_{\mathbb{K}}) \setminus \mathrm{PG}(\bigwedge^2 V)$. By Lemma 2.1, there exist two distinct points x_1 and x_2 of $\mathrm{PG}(\bigwedge^2 V)$ such that $x \in x_1x_2$. Let ζ denote the orthogonal or symplectic polarity of $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ associated to the quadric $Q_{\mathbb{K}}$. We prove that the points of Q which are $Q_{\mathbb{K}}$-collinear with x are precisely the points of $\alpha \cap Q$. Since $x \in X$, every point of $\alpha \cap Q$ is $Q_{\mathbb{K}}$-collinear with x. Conversely, suppose that y is a point of Q which is $Q_{\mathbb{K}}$-collinear with x. Then $x \in y^{\zeta}$. By Lemma 2.1 applied to the subspace y^{ζ}, we see that $x_1, x_2 \in y^{\zeta}$ and hence $y \in x_1^{\zeta} \cap x_2^{\zeta}$. Now, $x_1^{\zeta} \cap x_2^{\zeta}$ is a 3-dimensional subspace of $\mathrm{PG}(\bigwedge^2 V_{\mathbb{K}})$ which necessarily coincides with $\alpha_{\mathbb{K}}$ since every point of $\alpha \cap Q$ is $Q_{\mathbb{K}}$-collinear with x. So, $y \in \alpha_{\mathbb{K}}$ and hence $y \in Q \cap \alpha$.

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If p would be a point of $\mathrm{PG}(V)$ contained in $\kappa_{\mathbb{K}}^{-1}(x)$, then every line of $\mathrm{PG}(V)$ through p would be contained in the spread S, clearly a contradiction. $\square$

REMARK 4.7. If we go back to Proposition 2.2(b) and regard $\mathrm{PG}(2t-1,\mathbb{F})$ as a subgeometry of $\mathrm{PG}(2t-1,\overline{\mathbb{F}})$, where $\overline{\mathbb{F}}$ is a fixed algebraic closure of $\mathbb{F}$, then Lemma 4.6 implies that there exists a unique quadratic extension $\mathbb{F}'$ of $\mathbb{F}$ contained in $\overline{\mathbb{F}}$ for which the corresponding subgeometry $\mathrm{PG}(2t-1,\mathbb{F}')$ of $\mathrm{PG}(2t-1,\overline{\mathbb{F}})$ satisfies the properties (i) or (ii) of Proposition 2.2(b).

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REFERENCES

[1] A. Beutelspacher and J. Ueberberg. Bruck's vision of regular spreads or What is the use of a Baer superspace? *Abh. Math. Sem. Univ. Hamburg*, 63:37–54, 1993.

[2] Richard H. Bruck. Construction problems of finite projective planes. pp. 426–514 in *Combinatorial Mathematics and its Applications*, (Proc. Conf., Univ. North Carolina, Chapel Hill, N.C., 1967), Univ. North Carolina Press, Chapel Hill, 1969.

[3] James W. P. Hirschfeld. *Finite projective spaces of three dimensions.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1985.