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ON THE SPECTRAL RADII OF QUASI-TREE GRAPHS AND QUASI-UNICYCLIC GRAPHS WITH K PENDANT VERTICES*

XIANYA GENG† AND SHUCHAO LI†

Abstract. A connected graph $G = (V, E)$ is called a quasi-tree graph if there exists a vertex $u_0 \in V(G)$ such that $G - u_0$ is a tree. A connected graph $G = (V, E)$ is called a quasi-unicyclic graph if there exists a vertex $u_0 \in V(G)$ such that $G - u_0$ is a unicyclic graph. Set $\mathscr{T}(n, k) := \{G : G$ is a n-vertex quasi-tree graph with k pendant vertices$\}$, and $\mathscr{T}(n, d_0, k) := \{G : G \in \mathscr{T}(n, k)$ and there is a vertex $u_0 \in V(G)$ such that $G - u_0$ is a tree and $d_G(u_0) = d_0\}$. Similarly, set $\mathscr{U}(n, k) := \{G : G$ is a n-vertex quasi-unicyclic graph with k pendant vertices$\}$, and $\mathscr{U}(n, d_0, k) := \{G : G \in \mathscr{U}(n, k)$ and there is a vertex $u_0 \in V(G)$ such that $G - u_0$ is a unicyclic graph and $d_G(u_0) = d_0\}$. In this paper, the maximal spectral radii of all graphs in the sets $\mathscr{T}(n, k)$, $\mathscr{T}(n, d_0, k)$, $\mathscr{U}(n, k)$, and $\mathscr{U}(n, d_0, k)$, are determined. The corresponding extremal graphs are also characterized.

Key words. Quasi-tree graph, Quasi-unicyclic graph, Eigenvalues, Pendant vertex, Spectral radius.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with n vertices and let $A(G)$ be its adjacency matrix. Since $A(G)$ is symmetric, its eigenvalues are real. Without loss of generality, we can write them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. The *characteristic polynomial* of G is just $\det(\lambda I - A(G))$, and is denoted by $\phi(G; \lambda)$. The largest eigenvalue $\lambda_1(G)$ is called the *spectral radius* of G, denoted by $\rho(G)$. If G is connected, then $A(G)$ is irreducible and by the Perron-Frobenius theory of non-negative matrices, $\rho(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho(G)$. We shall refer to such an eigenvector as the *Perron vector* of G.

A connected graph $G = (V, E)$ is called a *quasi-tree graph*, if there exists a vertex $u_0 \in V(G)$ such that $G - u_0$ is a tree. The concept of quasi-tree graph was first

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introduced in [18, 19]. A connected graph $G = (V, E)$ is called a *quasi-unicyclic graph*, if there exists a vertex $u_0 \in V(G)$ such that $G - u_0$ is a unicyclic graph. The concept of quasi-unicyclic graph was first introduced in [9]. For convenience, set $\mathscr{T}(n, k) := \{G : G$ is a n-vertex quasi-tree graph with k pendant vertices$\}$, and $\mathscr{T}(n, d_0, k) := \{G : G \in \mathscr{T}(n, k)$ and there is a vertex $u_0 \in V(G)$ such that $G - u_0$ is a tree and $d_G(u_0) = d_0\}$. Similarly, set $\mathscr{U}(n, k) := \{G : G$ is a n-vertex quasi-unicyclic graph with k pendant vertices$\}$, and $\mathscr{U}(n, d_0, k) := \{G : G \in \mathscr{U}(n, k)$ and there is a vertex $u_0 \in V(G)$ such that $G - u_0$ is a unicyclic graph and $d_G(u_0) = d_0\}$.

The investigation of the spectral radius of graphs is an important topic in the theory of graph spectra. For results on the spectral radius of graphs, one may refer to [1, 2, 3, 4, 5, 6, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 23, 24] and the references therein.

Wu, Xiao and Hong [22] determined the spectral radius of trees on k pendant vertices. Guo, Gutman and Petrović [11, 20] determined the graphs with the largest spectral radius among all the unicyclic and bicyclic graphs with n vertices and k pendant vertices. The present authors determined the graph with maximum spectral radius among n-vertex tricyclic graphs with k pendant vertices; see [7]. In light of the information available for the spectral radii of trees and unicyclic graphs, it is natural to consider other classes of graphs, and the quasi-tree graphs (respectively, quasi-unicyclic graphs) are a reasonable starting point for such an investigation.

In this article, we determine the maximal spectral radii of all graphs in the set $\mathscr{T}(n, k)$, $\mathscr{T}(n, d_0, k)$, $\mathscr{U}(n, k)$, and $\mathscr{U}(n, d_0, k)$ respectively. The corresponding extremal graphs are also characterized.

2. Preliminaries. Denote the cycle, the path, and the star on n vertices by C_n, P_n, and $K_{1,n-1}$ respectively. Let $G - x$ or $G - xy$ denote the graph that arises from G by deleting the vertex $x \in V(G)$ or the edge $xy \in E(G)$. Similarly, $G + xy$ is a graph that arises from G by inserting an edge $xy \notin E(G)$, where $x, y \in V(G)$. A *pendant vertex* of G is a vertex of degree 1. The k paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have *almost equal lengths* if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$. For $v \in V(G)$, $d_G(v)$ denotes the degree of vertex v and $N_G(v)$ denotes the set of all neighbors of vertex v in G.

Let G' be a subgraph of G with $v \in V(G')$. We denote by T the connected component containing v in the graph obtained from G by deleting the neighbors of v

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in G'. If T is a tree, we call T a *pendant tree* of G. Then v is called the *root* of T, or the *root-vertex* of G. Throughout this paper, we assume that T does not include the root.

In this section, we list some known results which will be needed in this paper.

LEMMA 2.1 ([17, 22]). *Let G be a connected graph and $\rho(G)$ be the spectral radius of $A(G)$. Let u, v be two vertices of G. Suppose $v_1, v_2, \ldots, v_s \in N_G(v)\backslash N_G(u)(1 \leq s \leq d_G(v))$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $A(G)$, where x_i corresponds to $v_i(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and inserting the edges uv_i $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

LEMMA 2.2 ([12]). *Let G, G', G'' be three mutually disjoint connected graphs. Suppose that u, v are two vertices of G, u' is a vertex of G' and u'' is a vertex of G''. Let G_1 be the graph obtained from G, G', G'' by identifying, respectively, u with u' and v with u''. Let G_2 be the graph obtained from G, G', G'' by identifying vertices u, u', u''. Let G_3 be the graph obtained from G, G', G'' by identifying vertices v, u', u''. Then either $\rho(G_1) < \rho(G_2)$ or $\rho(G_1) < \rho(G_3)$.*

Let G be a connected graph, and $uv \in E(G)$. The graph $G_{u,v}$ is obtained from G by subdividing the edge uv, i.e., inserting a new vertex w and edges wu, wv in $G - uv$. Hoffman and Smith define an *internal path* of G as a walk $v_0v_1 \ldots v_s$ $(s \geq 1)$ such that the vertices $v_0, v_1, \ldots, v_s$ are distinct, $d_G(v_0) > 2, d_G(v_s) > 2$, and $d_G(v_i) = 2$, whenever $0 < i < s$. And s is called the length of the internal path. An internal path is closed if $v_0 = v_s$.

Let W_n be the tree on n vertices obtained from a path P_{n-4} (of length $n-5$) by attaching two new pendant edges to each end vertex of P_{n-4}, respectively. In [15], Hoffman and Smith obtained the following result:

LEMMA 2.3 ([15]). *Let uv be an edge of the connected graph G on n vertices.*

(i) *If uv does not belong to an internal path of G, and $G \neq C_n$, then $\rho(G_{u,v}) > \rho(G)$;*
(ii) *If uv belongs to an internal path of G, and $G \neq W_n$, then $\rho(G_{u,v}) < \rho(G)$.*

LEMMA 2.4. *Let G_1 and G_2 be two graphs.*

(i) ([16])*If G_2 is a proper spanning subgraph of a connected graph G_1. Then $\phi(G_2; \lambda) > \phi(G_1; \lambda)$ for $\lambda \geq \rho(G_1)$;*
(ii) ([4, 5])*If $\phi(G_2; \lambda) > \phi(G_1; \lambda)$ for $\lambda \geq \rho(G_2)$, then $\rho(G_1) > \rho(G_2)$;*

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(iii) ([15])*If G_2 is a proper subgraph of a connected graph G_1, then $\rho(G_2) < \rho(G_1)$.*

LEMMA 2.5 ([10, 16]). *Let v be a vertex in a non-trivial connected graph G and suppose that two paths of lengths k, $m\,(k \geq m \geq 1)$ are attached to G by their end vertices at v to form $G^*_{k,m}$. Then $\rho(G^*_{k,m}) > \rho(G^*_{k+1,m-1})$.*

REMARK 2.6. If the m vertices of a graph G can be partitioned into k disjoint paths of almost equal lengths, then a simple arithmetic argument shows that either $k|m$ and all the paths have m/k vertices, or $k \nmid m$ and then the paths have length $\lfloor m/k \rfloor$ or $\lfloor m/k \rfloor + 1$ and there are $m - k \cdot \lfloor \frac{m}{k} \rfloor$ paths of the latter length.

3. Spectral radius of quasi-tree graphs with k pendant vertices. In this section, we shall determine the spectral radii of graphs in $\mathscr{T}(n, d_0, k)$ and $\mathscr{T}(n, k)$, respectively. Note that for the set $\mathscr{T}(n, d_0, k)$, when $d_0 = 1$, $\mathscr{T}(n, 1, k)$ is just the set of all n-vertex trees with k pendant vertices. In [22] the maximal spectral radius of all the graphs in the set $\mathscr{T}(n, 1, k)$ is determined. So, we consider the case of $d_0 \geq 2$ in what follows.

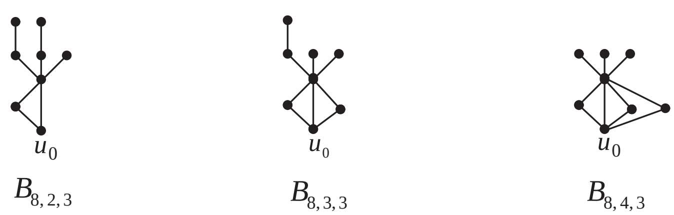

Figure 1. Graphs $B_{8,2,3}, B_{8,3,3}, B_{8,4,3}$.

Let $B_{n,d_0,k}$ be an n-vertex graph obtained from K_{1,d_0-1} and an isolated vertex u_0 by inserting all edges between K_{1,d_0-1} and u_0, and attaching k paths with almost equal lengths to the center of K_{1,d_0-1}. For example, graphs $B_{8,2,3}$, $B_{8,3,3}$, $B_{8,4,3}$ are depicted in Figure 1. Note that for any $G \in \mathscr{T}(n, d_0, k)$, we have $k + d_0 \leq n - 1$.

THEOREM 3.1. *Let $G \in \mathscr{T}(n, d_0, k)$ with $d_0 \geq 2$, $k > 0$. Then*

$$\rho(G) \leq \rho(B_{n,d_0,k})$$

and the equality holds if and only if $G \cong B_{n,d_0,k}$.

Proof. Choose $G \in \mathscr{T}(n, d_0, k)$ such that $\rho(G)$ is as large as possible. Let $V(G) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$ and $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})^T$ be the Perron vector of

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$A(G)$, where x_i corresponds to the vertex u_i for $0 \le i \le n-1$. Assume that $G' := G - u_0$ is a tree. Choose a vertex $u_1 \in V(G')$ such that $d_{G'}(u_1)$ is as large as possible.

Note that G has pendant vertices, hence by Lemma 2.2, there exists exactly one pendant tree, say T, attached to a vertex, say u_2, of G.

First, we establish the following sequence of facts.

Fact 1. *Each vertex u of T has degree $d(u) \le 2$.*

Proof. Suppose to the contrary that there exists one vertex u_i of T such that $d(u_i) > 2$. Denote $N(u_i) = \{z_1, z_2, \ldots, z_t\}$ and $N(u_2) = \{w_1, w_2, \ldots, w_s\}$. Assume that z_1, w_3 belong to the path joining u_i and u_2, and that w_1, w_2 belong to some cycle in G. Let

$$G^* = \begin{cases} G - \{u_i z_3, \ldots, u_i z_t\} + \{u_2 z_3, \ldots, u_2 z_t\}, & \text{if } x_2 \ge x_i, \\ G - \{u_2 w_1, u_2 w_4, \ldots, u_2 w_s\} + \{u_i w_1, u_i w_4, \ldots, u_i w_s\}, & \text{if } x_2 < x_i. \end{cases}$$

Then $G^* \in \mathscr{T}(n, d_0, k)$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Thus G is a graph with k paths attached to u_2. □

Fact 2. *k paths attached to u_2 have almost equal lengths.*

Proof. Denote the k paths attached to u_2 by $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$, then we will show that $|l_i - l_j| \le 1$ for $1 \le i, j \le k$. If there exist two paths, say P_{l_1} and P_{l_2}, such that $|l_1 - l_2| \ge 2$, denote $P_{l_1} = u_2 v_1 v_2 \ldots v_{l_1}$ and $P_{l_2} = u_2 w_1 w_2 \ldots w_{l_2}$. Let

$$G^* = G - \{v_{l_1 - 1} v_{l_1}\} + \{w_{l_2} v_{l_1}\}.$$

Then $G^* \in \mathscr{T}(n, d_0, k)$. By Lemma 2.5, we have $\rho(G^*) > \rho(G)$, a contradiction. Thus k paths attached to u_2 have almost equal lengths. □

By Facts 1 and 2, G is a graph with k paths with almost equal lengths attached to u_2 of G.

Fact 3. *$u_1 = u_2$.*

Proof. Suppose that $u_1 \ne u_2$. Since G' is a tree, there is an unique path P_m ($m \ge 2$) connecting u_1 and u_2 in G'. By the choice of u_1, $d_{G'}(u_1) \ge d_{G'}(u_2) \ge k+2$, there is a vertex $u_3 \in d_{G'}(u_1)$ such that $u_3 \notin P_m$. Let $v_1 \in N_{G'}(u_2)$ and $v_1 \in V(T)$. Set

$$G^* = \begin{cases} G - u_2 v_1 + u_1 v_1, & \text{if } x_1 \ge x_2, \\ G - u_1 u_3 + u_2 u_3, & \text{if } x_1 < x_2. \end{cases}$$

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Then $G^* \in \mathscr{T}(n, d_0, k)$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $u_1 = u_2$. $\square$

FACT 4. *u_1 is adjacent to each vertex of $G' - T$.*

Proof. We first show that there does not exist an internal path of $G - T$ with length greater than 1 unless the path lies on a cycle of length 3. Otherwise, if there is an internal path with length greater than 1 such that this path does not lie on a cycle of length 3, then let $w_1 w_2 \ldots w_l$ be such an internal path, and assume that v_m is a pendant vertex in T. Let

$$G^* = G - w_1w_2 - w_2w_3 + w_1w_3 + v_mw_2.$$

Then $G^* \in \mathscr{T}(n, d_0, k)$ with $\rho(G) < \rho(G^*)$ by Lemmas 2.3(ii) and 2.4(iii), a contradiction. Hence, there does not exist an internal path of $G - T$ with length greater than 1 unless the path lies on a cycle of length 3.

Next we suppose that $u_1u_i \notin E(G)$ for some $u_i \in V(G') \setminus V(T)$. Since G' is a tree, there is an unique path connecting u_1 and u_i in G'. Let u_1, u_4, u_5 be the first three vertices on the path connecting u_1 and u_i in G' (possibly $u_5 = u_i$), then $u_1u_4, u_4u_5 \in E(G)$ and $u_1u_5 \notin E(G)$. Assume that v_1 is in both $N_{G'}(u_1)$ and $V(T)$.

If $x_1 \geq x_4$, let $G^* = G - u_4u_5 + u_1u_5$; if $x_1 < x_4$, let $G^* = G - u_1v_1 + u_4v_1$. In either case, $G^* \in \mathscr{T}(n, d_0, k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $u_1u_i \in E(G)$ for all $u_i \in V(G') \setminus V(T)$. $\square$

By Fact 4, we have $N_G(u_0) \subseteq N_G(u_1)$, and since there does not exist an internal path of $G - T$ with length greater than 1 unless the paths lies on a cycle of length 3. So we can obtain $u_0u_1 \in E(G)$. As $G \in \mathscr{T}(n, d_0, k)$, u_0 must be adjacent to each vertex of $V(G') \setminus V(T)$. Together with Remark 2.6, we obtain $G \cong B_{n,d_0,k}$.

This completes the proof of Theorem 3.1. $\square$

For the set of graphs $\mathscr{T}(n, k)$, when $k = n-1$, $\mathscr{T}(n, n-1) = \{K_{n,n-1}\}$ and when $k = n-2$, $\mathscr{T}(n, n-2) = \{H_t : H_t$ is obtained from an edge v_1v_2 by appending t (resp. $n-2-t$) pendant edges to v_1 (resp. v_2), where $0 < t < n-2\}$. By Lemma 2.2, H_1 is the unique graph in $\mathscr{T}(n, n-2)$ with maximal spectral radius. Hence, we need only consider the case of $1 \leq k \leq n-3$.

Let $C_{n,k}$, $(1 \leq k \leq n-3)$ be a graph obtained from $K_{1,n-2}$ and an isolated vertex u_0 by inserting edges to connecting u_0 with the center of $K_{1,n-2}$ and $n-k-2$

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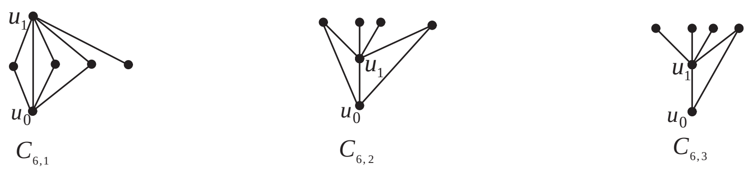

Figure 2. Graphs $C_{6,1}, C_{6,2}$ and $C_{6,3}$.

pendant vertices of $K_{1,n-2}$. For example, graphs $C_{6,1}, C_{6,2}$ and $C_{6,3}$ are depicted in Figure 2.

THEOREM 3.2. *Let $G \in \mathscr{T}(n,k)$ with $1 \le k \le n-3$. Then*

$$\rho(G) \le \rho(C_{n,k})$$

and the equality holds if and only if $G \cong C_{n,k}$.

Proof. Choose $G \in \mathscr{T}(n,k)$ such that $\rho(G)$ is as large as possible. Let $V(G) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex u_i $(0 \le i \le n-1)$. Assume $G - u_0$ is a tree. Denote $G_0 = G - u_0$.

Note that G has pendant vertices, hence in view of Lemma 2.2, there exists exactly one pendant tree, say T, attached to a vertex, say u_1, of G. Similar to the proof of Facts 1 and 2 in Theorem 3.1, we obtain that G is a graph having k paths with almost equal lengths attached to u_1. We establish the following sequence of facts.

FACT 1. *The pendant tree T contained in G is a star.*

Proof. As T is a tree obtained by attaching k paths with almost equal lengths to the vertex u_1. Then it is sufficient to show that the length of each path is 1. Suppose to the contrary that $v_1 v_2 \ldots v_t$ where $v_1 = u_1$ is such a path of length $t-1 > 1$. Let

$$G' = G - v_1v_2 - v_2v_3 + v_1v_3 + u_0v_2 + u_1v_2.$$

Then $G' \in \mathscr{T}(n,k)$. By Lemmas 2.3(ii) and 2.4(iii), we have $\rho(G) < \rho(G')$, a contradiction. Hence the length of each path is 1. So we have T is a star. □

FACT 2. *u_1 is adjacent to each vertex of $G_0 - T$.*

Proof. We first show that there does not exist an internal path of $G - T$ with length greater than 1 unless the paths lies on a cycle of length 3. Otherwise, assume

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that $w_1w_2\dots w_l$ is an internal path with length greater than 1 such that this path does not lie on a cycle of length 3. Let

$$G' = G - w_1w_2 - w_2w_3 + w_1w_3 + u_0w_2 + u_1w_2.$$

Then $G' \in \mathscr{T}(n,k)$. By Lemmas 2.3(ii) and 2.4(iii), we have $\rho(G) < \rho(G')$, a contradiction. Hence, there does not exist an internal path of $G - T$ with length greater than 1 unless the paths lies on a cycle of length 3.

Now suppose that $u_1u_i \notin E(G)$ for some $u_i \in V(G_0) \setminus V(T)$. Since G_0 is a tree, there is an unique path connecting u_1 and u_i in G_0. Let u_1, u_4, u_5 be the first three vertices on the path connecting u_1 and u_i in G_0 (possibly $u_5 = u_i$), then $u_1u_4, u_4u_5 \in E(G)$ and $u_1u_5 \notin E(G)$. Denote $v_1 \in N_{G_0}(u_1)$, and $v_1 \in V(T)$.

If $x_1 \geq x_4$, let $G^* = G - u_4u_5 + u_1u_5$; if $x_1 < x_4$, let $G^* = G - u_1v_1 + u_4v_1$. Then in either case, $G^* \in \mathscr{T}(n,k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $u_1u_i \in E(G)$ for all $u_i \in V(G_0) \setminus V(T)$. ◻

By Fact 2, we have $N_G(u_0) \subseteq N_G(u_1)$, and since there does not exist an internal path of $G - T$ with length greater than 1 unless the paths lies on a cycle of length 3. So we can obtain $u_0u_1 \in E(G)$. As $G \in \mathscr{T}(n,k)$, u_0 must be adjacent to each vertex of $V(G_0) \setminus V(T)$, together with Remark 2.6, we obtain $G \cong C_{n,k}$.

This completes the proof of Theorem 3.2. ◻

4. Spectral radius of quasi-unicyclic graphs with k pendant vertices. In this section, we determine the spectral radii of graphs in $\mathscr{U}(n, d_0, k)$ and $\mathscr{U}(n,k)$, respectively. Note that $\mathscr{U}(n,1,k)$ is just the set of all n-vertex unicyclic graphs with k pendant vertices. In [11] the maximal spectral radius of all the graphs in the set $\mathscr{U}(n,1,k)$ is determined. So, we consider the case of $d_0 \geq 2$ in what follows. Note that for any n-vertex quasi-unicyclic graph with k pendant vertices, we have $k \leq n-4$ when $d_0 = 2, 3$, and $k + d_0 \leq n-1$ when $d_0 > 3$.

In order to formulate our results, we need to define some quasi-unicyclic graphs as follows. Graphs U_1, U_2, U_3, U_4 and U_5 are depicted in Figure 3, where the order of U_3 (respectively, U_4, U_5) is $d_0 + 1$ (respectively, $d_0 + 2$, $d_0 + 3$).

Let $U^1_{n,2,k}$ (respectively, $U^2_{n,2,k}$) be an n-vertex graph obtained from U_1 (respectively, U_2) by attaching k paths with almost equal lengths to the vertex u in U_1 (respectively, U_2). For $d_0 \geq 3$, let $U^1_{n,d_0,k}$ (respectively, $U^2_{n,d_0,k}, U^3_{n,d_0,k}$) be an n-vertex

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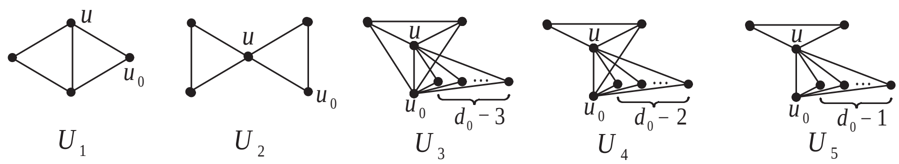

Figure 3. Graphs U_1, U_2, U_3, U_4 and U_5.

graph obtained from U_3 (respectively, U_4, U_5) by attaching k paths with almost equal lengths to the vertex u in U_3 (respectively, U_4, U_5). For example, $U^1_{9,2,2}$, $U^1_{9,3,2}$, $U^2_{9,3,2}$, $U^3_{9,3,2}$, $U^1_{9,4,2}$, $U^2_{9,4,2}$ and $U^3_{9,4,2}$ are depicted in Figure 4.

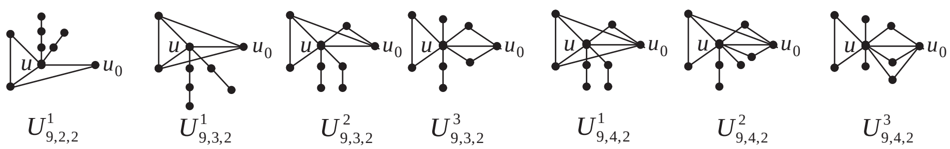

Figure 4. Graphs $U^1_{9,2,2}$, $U^1_{9,3,2}$, $U^2_{9,3,2}$, $U^3_{9,3,2}$, $U^1_{9,4,2}$, $U^2_{9,4,2}$ and $U^3_{9,4,2}$.

THEOREM 4.1. *Let* $G \in \mathscr{U}(n, d_0, k), k > 0$. *Then*

(i) *if* $d_0 = 2$, *then*

$$\rho(G) \le \rho(U^1_{n,2,k})$$

and equality holds if and only if $G \cong U^1_{n,2,k}$.

(ii) *if* $d_0 \ge 3$, *then*

$$\rho(G) \le \{\rho(U^1_{n,d_0,k}), \rho(U^2_{n,d_0,k}), \rho(U^3_{n,d_0,k})\}$$

and equality holds if and only if $G \cong U^1_{n,d_0,k}$ *or,* $G \cong U^2_{n,d_0,k}$ *or,* $G \cong U^3_{n,d_0,k}$.

Proof. Choose $G \in \mathscr{U}(n, d_0, k)$ such that $\rho(G)$ is as large as possible. Let $V(G) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$ and $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex u_i, $(0 \le i \le n-1)$. Assume $G - u_0$ is a unicyclic graph. Denote $G' = G - u_0$. Choose a vertex $u_1 \in V(G')$ such that $d_{G'}(u_1)$ is as large as possible.

Note that G has pendant vertices, hence by Lemma 2.2, there exists exactly one pendant tree, say T, attached to a vertex, say u_2, of G. Similar to the proof of Facts 1 and 2 in Theorem 3.1, G is a graph having k paths with almost equal lengths attached to u_2. We establish the following sequence of facts.

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FACT 1. *The cycle contained in G' is C_3.*

Proof. We first show that there does not exists an internal path of $G - T$ with length greater than 1 unless the paths lies on a cycle of length 3. Otherwise, if there is an internal path with length greater than 1 such that this path does not lie on a cycle of length 3, then let $w_1 w_2 \dots w_l$ be such an internal path, and assume that v_m is a pendant vertex in T. Let

$$G^* = G - w_1w_2 - w_2w_3 + w_1w_3 + v_mw_2.$$

Then $G^* \in \mathscr{U}(n, d_0, k)$ with $\rho(G) < \rho(G^*)$ by Lemmas 2.3(ii) and 2.4(iii), a contradiction. Hence, there does not exist an internal path of G with length greater than 1 unless the paths lies on a cycle of length 3. And then we suppose that this cycle contained in G' is $C_m(m > 3)$. We may assume $u_2u_3 \in E(C_m)$. Since $m > 3$, there is at least a vertex $u_4 \in N(u_2)\backslash N(u_3)$, and there is at least a vertex $u_5 \in N(u_3)\backslash N(u_2)$. Let

$$G^* = \begin{cases} G - u_3u_5 + u_2u_5, & \text{if } x_2 \geq x_3, \\ G - u_2u_4 + u_3u_4, & \text{if } x_2 < x_3. \end{cases}$$

Then, $G^* \in \mathscr{U}(n, d_0, k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $m = 3$. ∎

FACT 2. *The vertex u_1 is in $V(C_3)$.*

Proof. Suppose that $u_1 \notin V(C_3)$ and set $V(C_3) := \{u_3, u_4, u_5\}$. Since G' is a connected graph, there is a unique path $P_k(k \geq 2)$ connecting u_1 with C_3 in G'. We may assume that $u_3 \in P_k$. By the choice of u_1, $d_{G'}(u_1) \geq d_{G'}(u_3) \geq 3$, there is a vertex $u_6 \in N(u_1)$ such that $u_6 \notin P_k$.

If $x_1 \geq x_3$, let $G^* = G - u_3u_4 + u_1u_4$; if $x_1 < x_3$, let $G^* = G - u_1u_6 + u_3u_6$. Then in either case, $G^* \in \mathscr{U}(n, d_0, k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $u_1 \in V(C_3)$. ∎

By Fact 2, we may assume $V(C_3) = \{u_1, u_3, u_4\}$.

FACT 3. $u_1 = u_2$.

Proof. Suppose that $u_1 \neq u_2$. Since G' is a connected graph, there is a unique path P_m $(m \geq 2)$ connecting u_1 and u_2 in G'. By the choice of u_1, $d_{G'}(u_1) \geq d_{G'}(u_2) \geq k + 2$, there is a vertex $u_5 \in N_{G'}(u_1)$ and $u_5 \notin P_m$. Assume that $v_1 \in N_{G'}(u_2)$ and

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$v_1 \in V(T)$. If $u_2 \in V(C_3) \setminus \{u_1\}$, let

$$G^* = \begin{cases} G - u_2v_1 + u_1v_1, & \text{if } x_1 \geq x_2, \\ G - u_1u_5 + u_2u_5, & \text{if } x_1 < x_2. \end{cases}$$

If $u_2 \in V(G') \setminus V(C_3)$, let

$$G^* = \begin{cases} G - u_2v_1 + u_1v_1, & \text{if } x_1 \geq x_2, \\ G - u_1u_3 + u_2u_3, & \text{if } x_1 < x_2. \end{cases}$$

Then in either case $G^* \in \mathscr{U}(n, d_0, k)$. By Lemma 2.1, we have $\rho(G^*) > \rho(G)$, a contradiction. Hence $u_1 = u_2$. □

Thus we assume that $V(C_3) = \{u_1, u_2, u_3\}$.

Fact 4. *The vertex u_1 is adjacent to each vertex of $V(G') \setminus V(T)$.*

Proof. ¿From Fact 1 we know that there does not exist an internal path of $G-T$ with length greater than 1 unless the paths lies on a cycle of length 3. And then we suppose that $u_1u_i \notin E(G)$ for some $u_i \in V(G') \setminus V(T)$. Since G' is a unicyclic graph, there is a unique path connecting u_1 and u_i in G'. Let u_1, u_4, u_5 be the first three vertices on the path connecting u_1 and u_i in G' (possibly $u_5 = u_i$), then $u_1u_4, u_4u_5 \in E(G)$ and $u_1u_5 \notin E(G)$. Denote $v_1 \in N_{G'}(u_1)$, and $v_1 \in V(T)$.

If $x_1 \geq x_4$, let $G^* = G - u_4u_5 + u_1u_5$; if $x_1 < x_4$, let $G^* = G - u_1v_1 + u_4v_1$. Then in either case, $G^* \in \mathscr{U}(n, d_0, k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $u_1u_i \in E(G)$ for all $u_i \in V(G') \setminus V(T)$. □

Fact 5. $u_0u_1 \in E(G)$.

Proof. Suppose that $u_0u_1 \notin E(G)$. Since $d_G(u_0) \geq 1$, we may assume, without loss of generality, that $u_iu_0 \in E(G)$, where $u_i \in V(G') \setminus \{u_1\}$. Assume there is a vertex $u_4 \in V(T)$.

If $u_i \in \{u_2, u_3\}$, then

$$G^* = \begin{cases} G - u_0u_i + u_1u_i, & \text{if } x_1 \geq x_i, \\ G - u_1u_4 + u_iu_4, & \text{if } x_1 < x_i. \end{cases}$$

If $u_i \in V(G') \setminus \{u_2, u_3\}$, then

$$G^* = \begin{cases} G - u_0u_i + u_1u_i, & \text{if } x_1 \geq x_i, \\ G - u_1u_2 + u_iu_2, & \text{if } x_1 < x_i. \end{cases}$$

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Then in either case, $G^* \in \mathscr{U}(n, d_0, k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $u_0u_1 \in E(G)$. □

By Facts 1-5 and Remark 2.6, if $d_0 = 2$, we obtain that $G \cong U^1_{n,2,k}$ or $G \cong U^2_{n,2,k}$. We know from [20] that $\rho(U^1_{n,2,k}) > \rho(U^2_{n,2,k})$, therefore, Theorem 4.1(i) holds. Similarly, if $d_0 \geq 3$, then we obtain that $G \cong U^1_{n,d_0,k}$ or, $G \cong U^2_{n,d_0,k}$ or, $G \cong U^3_{n,d_0,k}$, therefore, Theorem 4.1(ii) holds.

This completes the proof of Theorem 4.1. □

To conclude this section, we determine the spectral radius of graphs in $\mathscr{U}(n, k)$. Let $B_m (m \geq 3)$ be a graph of order m obtained from C_3 by attaching $m-3$ pendant vertices to a vertex of C_3. For any $G \in \mathscr{U}(n, k)$, we have $k \leq n-3$. When $k = n-3$, $\mathscr{U}(n, n-3) = \{B_n\}$. So we consider only the case of $1 \leq k \leq n-4$ here.

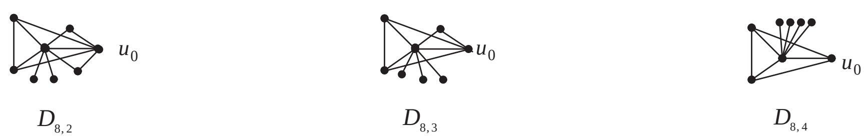

Figure 5. Graphs $D_{8,k}$ for $k = 2, 3, 4$.

Let $D_{n,k} (1 \leq k \leq n-4)$ be a graph obtained from B_{n-1} and an isolated vertex u_0 by inserting all edges between u_0 and three non-pendant vertices and $n-k-4$ pendant vertices of B_{n-1}. For example, graphs $D_{8,2}, D_{8,3}, D_{8,4}$ are depicted in Figure 5. It is easy to see that the graph $D_{n,k}$ defined as above is in $\mathscr{U}(n, k)$.

Theorem 4.2. *Let $G \in \mathscr{U}(n, k)$ with $1 \leq k \leq n-4$. Then*

$$\rho(G) \leq \rho(D_{n,k})$$

and the equality holds if and only if $G \cong D_{n,k}$.

Proof. Choose $G \in \mathscr{U}(n, k)$ such that $\rho(G)$ is as large as possible. Let $V(G) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})^T$ be the Perron vector of $A(G)$, where x_i corresponds to the vertex u_i $(0 \leq i \leq n-1)$. Assume $G - u_0$ is a unicyclic graph. Denote $G' = G - u_0$.

Note that G has pendant vertices, hence in view of Lemma 2.2, there exists exactly one pendant tree, say T, attached to a vertex, say u_1, of G. With a similar method used in the proof of Facts 1 and 2 in Theorem 3.1, we obtain that $G \in \mathscr{U}(n, k)$ and

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G has k paths with almost equal lengths attached to u_1. We establish the following sequence of facts.

FACT 1. *The cycle contained in G' is C_3.*

Proof. We first show that there does not exist an internal path of $G-T$ with length greater than 1 unless the paths lies on a cycle of length 3. Otherwise, if there is an internal path with length greater than 1 such that this path does not lie on a cycle of length 3, and let $w_1w_2\dots w_lw_1$ be such an internal path. Set

$$G^* = G - w_1w_2 - w_2w_3 + w_1w_3 + u_0w_2 + u_1w_2.$$

Then $G^* \in \mathscr{U}(n,k)$. By Lemmas 2.3(ii) and 2.4(iii), we have $\rho(G) < \rho(G^*)$, a contradiction. So, there does not exist an internal path of $G-T$ with length greater than 1 unless the paths lies on a cycle of length 3. And then we suppose that this cycle in G' is $C_m\,(m>3)$. We may assume $u_2u_3 \in E(C_m)$. Since $m>3$, there is at least a vertex $u_4 \in N(u_2)\setminus N(u_3)$, and there is at least one vertex, say u_5, in $N(u_3)\setminus N(u_2)$. Let

$$G^* = \begin{cases} G - u_3u_5 + u_2u_5, & \text{if } x_2 \geq x_3,\\ G - u_2u_4 + u_3u_4, & \text{if } x_2 < x_3.\end{cases}$$

Then, $G^* \in \mathscr{U}(n,k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $m = 3$. $\square$

FACT 2. *T is a star.*

Proof. It is sufficient to show that the length of each path is 1. Suppose to the contrary that $v_1v_2\dots v_k$ is such a path, where $v_1 = u_1$ and $k > 2$. Let

$$G^* = G - v_1v_2 - v_2v_3 + v_1v_3 + u_0v_2 + u_1v_2.$$

Then $G^* \in \mathscr{U}(n,k)$. By Lemmas 2.3(ii) and 2.4(iii), we have $\rho(G) < \rho(G^*)$, a contradiction. Hence the length of each path is 1. So we have T is a star. $\square$

FACT 3. *u_1 is adjacent to each vertex of $V(G')\setminus V(T)$.*

Proof. By Fact 1 (in Theorem 4.2), there does not exist an internal path of $G-T$ with length greater than 1 unless the paths lies on a cycle of length 3. And then we suppose that $u_1u_i \notin E(G)$ for some $u_i \in V(G')\setminus V(T)$. As G is a quasi-unicyclic graph, there is an unique path connecting u_1 and u_i in G'. Let u_1, u_4, u_5 be the

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first three vertices on the path connecting u_1 and u_i in G' (possibly $u_5 = u_i$), then $u_1u_4, u_4u_5 \in E(G)$ and $u_1u_5 \notin E(G)$. Denote $v_1 \in N_{G'}(u_1)$, and $v_1 \in V(T)$.

If $x_1 \geq x_4$, let $G^* = G - u_4u_5 + u_1u_5$; if $x_1 < x_4$, let $G^* = G - u_1v_1 + u_4v_1$. Then in either case, $G^* \in \mathscr{U}(n, k)$, and by Lemma 2.1, $\rho(G) < \rho(G^*)$, a contradiction. Therefore, $u_1u_i \in E(G)$ for all $u_i \in V(G') \setminus V(T)$. This completes the proof of Fact 3. ◻

By Facts 1-3, if we insert an edge e to a connected graph G, then $\rho(G+e) > \rho(G)$ as the adjacent matrix of a connected graph is irreducible. Therefore the proof of Theorem 4.2 is completed. ◻

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