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MATRIX FUNCTIONS PRESERVING SETS OF GENERALIZED NONNEGATIVE MATRICES*

ABED ELHASHASH† AND DANIEL B. SZYLD‡

Abstract. Matrix functions preserving several sets of generalized nonnegative matrices are characterized. These sets include PFn, the set of $n \times n$ real eventually positive matrices; and WPFn, the set of matrices $A \in \mathbb{R}^{n\times n}$ such that A and its transpose have the Perron-Frobenius property. Necessary conditions and sufficient conditions for a matrix function to preserve the set of $n \times n$ real eventually nonnegative matrices and the set of $n \times n$ real exponentially nonnegative matrices are also presented. In particular, it is shown that if $f(0) \neq 0$ and $f'(0) \neq 0$ for some entire function f, then such an entire function does not preserve the set of $n \times n$ real eventually nonnegative matrices. It is also shown that the only complex polynomials that preserve the set of $n \times n$ real exponentially nonnegative matrices are $p(z) = az + b$, where $a, b \in \mathbb{R}$ and $a \geq 0$.

Key words. Matrix functions, Generalization of nonnegative matrices, Eventually nonnegative matrices, Eventually positive matrices, Exponentially nonnegative matrices, Eventually exponentially nonnegative matrices, Perron-Frobenius property, Strong Perron-Frobenius property.

AMS subject classifications. 15A48.

1. Introduction. Several authors have studied the problem of characterizing matrix functions preserving nonnegative matrices. Micchelli and Willoughby [15] presented a characterization of matrix functions preserving a certain subset of nonnegative matrices, namely, the set of nonnegative symmetric positive semidefinite matrices. Bharali and Holtz devoted article [3] to characterizing even and odd entire functions that preserve all nonnegative symmetric matrices, thus, dropping the condition of positive-semidefiniteness required in [15]. In addition, the authors in [3] presented characterizations of matrix functions preserving certain structured nonnegative matrices (block triangular and circulant matrices). In this paper, we study matrix functions preserving sets of generalized nonnegative matrices such as the set of real $n \times n$ eventually nonnegative matrices, the set of real $n \times n$ exponentially nonnegative matrices, PFn and WPFn. We define these sets and their corresponding matrices below.

We say that a rectangular matrix A is *nonnegative* (respectively, *positive*) if all

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†Department of Mathematics, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104-2816, USA (aae36@drexel.edu).
‡Department of Mathematics, Temple University, 1805 N. Broad Street, Philadelphia, Pennsylvania 19122-6094, USA (szyld@temple.edu).

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its entries are nonnegative (respectively, positive), which is symbolically denoted by $A \geq 0$ (respectively, $A > 0$). In particular, this nomenclature and notation hold for (column) vectors. We call a real $n \times n$ matrix A *eventually nonnegative* (respectively, *eventually positive*) if there is a nonnegative integer k_0 such that $A^k \geq 0$ (respectively, $A^k > 0$) for all integers $k \geq k_0$. Matrix functions preserving the set of real $n \times n$ eventually positive matrices are studied and completely characterized in Section 2 whereas those preserving the set of real $n \times n$ eventually nonnegative matrices are studied in Section 4 where separate necessary and sufficient conditions are presented.

A matrix $A \in \mathbb{R}^{n\times n}$ that is positive, nonnegative, nonnilpotent eventually nonnegative, or eventually positive satisfies the property $Av = \rho(A)v$ where v is a nonnegative nonzero vector and $\rho(A)$ is the spectral radius of A, i.e., A has a nonnegative (right) eigenvector whose corresponding eigenvalue is the spectral radius of A; see, e.g., [2], [6], [9], [12], [13], [16], [19], [21]. The latter property satisfied by such a matrix A is known as the *Perron-Frobenius property* and the eigenvector v is known as the *(right) Perron-Frobenius eigenvector* of A. If in addition to this property the (right) Perron-Frobenius eigenvector v is positive and the spectral radius of A is a simple eigenvalue and the only eigenvalue of maximum modulus, then we say that A has the *strong Perron-Frobenius property.* It is known that positive matrices and eventually positive matrices possess the strong Perron-Frobenius property; see, e.g., [2], [12], [16], [19], [21]. We denote the transpose of a matrix A (respectively, a vector v) by A^T (respectively, v^T). If the matrix A^T has the Perron-Frobenius property, then its right Perron-Frobenius eigenvector is called a *left Perron-Frobenius eigenvector* of A.

We define now two other sets of matrices considered in this paper; in sections 2 and 3. We denote by WPFn the set of matrices $A \in \mathbb{R}^{n\times n}$ such that A and A^T possess the Perron-Frobenius property. Similarly, we denote by PFn the set of matrices $A \in \mathbb{R}^{n\times n}$ such that A and A^T possess the strong Perron-Frobenius property. It has been shown that

$$\begin{aligned} \text{PFn} &= \{\text{Eventually positive matrices in } \mathbb{R}^{n\times n}\} \\ &\subset \{\text{Nonnilpotent eventually nonnegative matrices in } \mathbb{R}^{n\times n}\} \\ &\subset \text{WPFn}, \end{aligned}$$

where all the containments between sets in this statement are proper; see [6].

In addition to the three sets mentioned above, we consider in Section 5 the sets of *exponentially nonnegative* and *eventually exponentially nonnegative* matrices in $\mathbb{R}^{n\times n}$ as yet other generalizations of nonnegative matrices. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *exponentially nonnegative* if the matrix

$$e^{tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

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is nonnegative for all real scalars $t \geq 0$ and is said to be *eventually exponentially nonnegative* if the matrix e^{tA} is nonnegative for all real scalars $t \geq t_0$ for some real scalar $t_0 \geq 0$. For these matrices, we only consider the case of $n \geq 3$, and leave the case $n = 2$ for a combinatorial study; see our comments in Section 6.

In the following, we review some definitions and properties of matrix functions. For further details, see, e.g., [7], [8], [10], [14]. If A is a matrix in $\mathbb{C}^{n\times n}$, then we denote by $J(A)$ the Jordan canonical form of A, i.e., for some nonsingular matrix $V \in \mathbb{C}^{n\times n}$, we have the Jordan decomposition $A = VJ(A)V^{-1}$, where $J(A) = diag(J_{k_1}(\lambda_1), \ldots, J_{k_m}(\lambda_m))$, each $J_{k_l}(\lambda_l)$ is an elementary Jordan block corresponding to an eigenvalue λ_l of size k_l, and the λ_l's are not necessarily distinct. Moreover, for any $\lambda \in \mathbb{C}$, we denote the null space of $A - \lambda I$ by $E_\lambda(A)$. The spectrum of a matrix A is denoted by $\sigma(A)$. Let f be a complex-valued function holomorphic on an open connected set $\Omega \subseteq \mathbb{C}$ that contains the spectrum of A, and let Γ be a closed rectifiable curve in Ω that is homologous to zero in Ω and that winds around all the eigenvalues of A only once. We extend f to a matrix function so that $f(A)$ is defined by

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)(zI - A)^{-1}dz,$$

where the integral is taken entry-wise, i.e., the (l, j)-entry of the matrix $f(A)$ is $\frac{1}{2\pi i}\int_\Gamma f(z)e_l^T(zI - A)^{-1}e_j dz$ with e_l and e_j being the lth and jth canonical vectors of $\mathbb{C}^n$, respectively. In the following lemma, we summarize some properties of matrix functions.

Lemma 1.1. *Let $A \in \mathbb{C}^{n\times n}$, let f be a holomorphic function defined on an open connected set $\Omega \subseteq \mathbb{C}$ that contains the spectrum of A, and let Γ be a closed rectifiable curve in Ω that is homologous to zero in Ω and that winds around all the eigenvalues of A only once. Then, the following statements hold:*

$$(i)\quad f(A) = V\left[\frac{1}{2\pi i}\int_\Gamma f(z)\left(zI - J(A)\right)^{-1}dz\right]V^{-1} = Vf(J(A))V^{-1},$$

$$(ii)\quad f(J(A)) = diag(f(J_{k_1}(\lambda_1)), \ldots, f(J_{k_m}(\lambda_m))),$$

$$(iii)\quad f(J_{k_l}(\lambda_l)) = \begin{bmatrix} f(\lambda_l) & f'(\lambda_l) & \cdots & \frac{f^{(k_l-1)}(\lambda_l)}{(k_l-1)!} \\ 0 & f(\lambda_l) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f'(\lambda_l) \\ 0 & \cdots & 0 & f(\lambda_l) \end{bmatrix},$$

$$(iv)\quad f(A^T) = f(A)^T,$$

$$(v)\quad E_{\lambda_l}(A) \subseteq E_{f(\lambda_l)}(f(A)),$$

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where nonsingular matrix $V \in \mathbb{C}^{n\times n}$ satisfies $A = VJ(A)V^{-1}$. Moreover, $f(A) = q(A)$ where $q(z)$ is the Hermite polynomial (see, e.g., [10], [11], [14]) that interpolates f and its derivatives at the eigenvalues of A. Furthermore, if f has a power series

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1.1}$$

that converges for $|z| < R$ and $\rho(A) < R$, then $f(A) = \sum_{k=0}^{\infty} a_k A^k$.

2. Matrix functions preserving PFn. In this section, we completely characterize matrix functions preserving the set of real $n \times n$ eventually positive matrices, PFn. For $n = 1$, these functions are simply functions f which are holomorphic on an open set $\Omega \subseteq \mathbb{C}$ containing the positive real axis and which have the property that $f(z)$ is a positive scalar whenever z is a positive scalar. For $n \geq 2$, we present our main result in Theorem 2.5 below, but we begin first with some preliminary results. The following lemma is a well-known property; see, e.g., [4, Theorem 26], [9].

LEMMA 2.1. *Let λ and μ be two distinct eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$. Let $v \in \mathbb{R}^n$ be a right eigenvector corresponding to λ and $w \in R^n$ be a left eigenvector corresponding to μ, then v and w are orthogonal.*

LEMMA 2.2. *Let f be an entire function that preserves PFn. Then, for all matrices $A \in PFn$, whenever $\lambda \in \sigma(A)$ and $\lambda \neq \rho(A)$, we have $f(\lambda) \neq f(\rho(A))$.*

Proof. Pick $\lambda \in \sigma(A)$ with $\lambda \neq \rho(A)$ and suppose to the contrary that $f(\lambda) = f(\rho(A))$. We consider here two cases: case (i) $f(\lambda) = f(\rho(A)) = \rho(f(A))$ and case (ii) $f(\lambda) = f(\rho(A)) \neq \rho(f(A))$. Suppose that case (i) is true. Since $\rho(A)$ and λ are distinct eigenvalues of A and by Lemma 1.1 (v), it follows that $E_{\rho(f(A))}(f(A))$ contains two linearly independent eigenvectors. But, since $f(A) \in PFn$, $E_{\rho(f(A))}(f(A))$ must be one-dimensional. Thus, case (i) leads to a contradiction. If case (ii) is true, then by Lemma 1.1 (v) $E_{\rho(A)}(A) \subseteq E_{f(\rho(A))}(f(A)) = E_{f(\lambda)}(f(A))$. Hence, $E_{f(\lambda)}(f(A))$ must contain a positive right eigenvector v corresponding to $f(\lambda)$. Moreover, $f(A)$ being in PFn must have a positive left eigenvector w corresponding to $\rho(f(A))$. By Lemma 2.1, v and w must be orthogonal, a contradiction since both v and w are positive. Therefore, $f(\lambda) \neq f(\rho(A))$. □

COROLLARY 2.3. *Let f be an entire function that preserves PFn. Then, for all matrices $A \in PFn$, whenever $\lambda \in \sigma(A)$ and $\lambda \neq \rho(A)$, we have $|f(\lambda)| < |f(\rho(A))|$.*

Proof. Suppose to the contrary that there is an eigenvalue λ of A such that $\lambda \neq \rho(A)$ and $|f(\lambda)| \geq |f(\rho(A))|$. We choose $\lambda_0 \in \sigma(A)$ such that $|f(\lambda_0)| \geq |f(\lambda)|$ for all $\lambda \in \sigma(A)$ satisfying $\lambda \neq \rho(A)$ and $|f(\lambda)| \geq |f(\rho(A))|$. Since $f(A) \in PFn$, it follows that $|f(\lambda_0)| = f(\lambda_0) = \rho(f(A)) > 0$. Hence, $f(A)$ has a positive left

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eigenvector w corresponding to $f(\lambda_0)$. Moreover, by Lemma 1.1 (v), $E_{\rho(A)}(A) \subseteq E_{f(\rho(A))}(f(A))$. Hence, $f(A)$ has a right positive eigenvector v corresponding to $f(\rho(A))$. By Lemma 2.2, $f(\lambda_0) \neq f(\rho(A))$. Thus, by Lemma 2.1, v and w are orthogonal, which is a contradiction. Therefore, the result follows. $\square$

COROLLARY 2.4. *Let f be an entire function that preserves PFn. Then, for all $A \in PFn$, $f(\rho(A)) = \rho(f(A))$.*

In what follows, we characterize matrix functions that preserve PFn when $n \geq 2$. The case $n = 1$ is trivial. A function f, which is holomorphic on an open set in $\mathbb{C}$ that contains the positive reals, preserves $PF1$ if and only if $f(0, \infty) \subseteq (0, \infty)$.

THEOREM 2.5. *Let f be an entire function. Then, f preserves PFn ($n \geq 2$) if and only if $f(0, \infty) \subseteq (0, \infty)$; the restriction of f to positive reals, denoted by $f|_{(0,\infty)}$, is increasing; and if $n = 2$ (respectively, $n \geq 3$) $|f(z)| \leq f(|z|)$ for all $z \in \mathbb{R}$ (respectively, for all $z \in \mathbb{C}$).*

Proof. We prove this theorem first for $n = 2$. Let A be a matrix in $PF2$ with eigenvalues counted with multiplicity λ_1, λ_2 and let $A = VJ(A)V^{-1}$ be the Jordan decomposition of A. Since A is in $PF2$, it follows that $\lambda_1 = |\lambda_1| > |\lambda_2|$. Moreover, λ_2 must be real since non-real eigenvalues of a real matrix occur in pairs. Hence,

$$A = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1} \tag{2.1}$$

with $\lambda_1 = |\lambda_1| > |\lambda_2|$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. For any function f holomorphic on a domain containing λ_1 and λ_2 and by Lemma 1.1 parts (i) – (iii), we have:

$$f(A) = V \begin{bmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{bmatrix} V^{-1}. \tag{2.2}$$

Now, let f be any holomorphic function on a domain that contains the real line. If f preserves $PF2$ and A is any matrix in $PF2$, then A has the form (2.1) with $\lambda_1 = |\lambda_1| > |\lambda_2|$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f(A)$ has the form (2.2). By Corollary 2.4, $f(\lambda_1) = |f(\lambda_1)| > |f(\lambda_2)|$. Since λ_1 could be an arbitrary positive real satisfying $\lambda_1 > |\lambda_2|$, $\lambda_2 \in \mathbb{R}$ while A still being in $PF2$, it follows that f maps positive reals to positive reals. Moreover, by choosing arbitrary $\lambda_1 > \lambda_2 > 0$ in form (2.1) we still get a matrix A in $PF2$ for which $f(\lambda_1) > f(\lambda_2)$. Thus, $f|_{(0,\infty)}$ is increasing. Furthermore, let s be a real scalar. If $s > 0$, then $f(s) > 0$, and thus, $|f(s)| \leq f(|s|)$. On the other hand, if $s \leq 0$, then let $\lambda_1 = |s| + \epsilon$ for some positive scalar ϵ and let $\lambda_2 = s$. For the latter choice of λ_1 and λ_2, a matrix A in form (2.1) is in $PF2$ and its image $f(A)$ in form (2.2) is also in $PF2$, and therefore, f must satisfy $|f(\lambda_2)| < f(\lambda_1)$, or equivalently, $|f(s)| < f(|s| + \epsilon)$. Since ϵ was an arbitrary positive scalar and by the continuity of f, it follows that $|f(s)| \leq f(|s|)$. Hence, we prove necessity. As for the sufficiency, it is straightforward.

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For $n \geq 3$, a matrix A in PFn would have a Jordan decomposition of the form

$$A = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & B \end{bmatrix} V^{-1},$$

where $B = diag\,(J_{k_2}(\lambda_2), J_{k_3}(\lambda_3), \ldots, J_{k_n}(\lambda_n))$ and $\lambda_1 = |\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. We note that for $n \geq 3$ (unlike the case when $n = 2$) the eigenvalues $\lambda_2, \ldots, \lambda_n$ may be nonreal. If f is any entire function, then the matrix $f(A)$ is given by

$$f(A) = V \begin{bmatrix} f(\lambda_1) & 0 \\ 0 & f(B) \end{bmatrix} V^{-1},$$

and from this point on the proof proceeds in a very similar manner to the case when $n = 2$. $\square$

3. Matrix functions preserving WPFn. A holomorphic function f defined on nonnegative reals preserves WPF1 if and only if $f([0, \infty)) \subseteq [0, \infty)$. Also, by using proof techniques similar to those used in the proof of Theorem 2.5, one could easily show that a holomorphic function f defined on nonnegative reals preserves WPF2 if and only if $|f(x)| \leq f(y)$ whenever x and y are real numbers satisfying $|x| \leq y$. In the following theorem, we consider the nontrivial case and characterize entire functions preserving WPFn ($n \geq 3$).

THEOREM 3.1. *Let f be an entire function. Then, f preserves WPFn ($n \geq 3$) if and only if $|f(z)| \leq f(|z|)$ for all $z \in \mathbb{C}$.*

Proof. Suppose that f is an entire function such that $|f(z)| \leq f(|z|)$ for all $z \in \mathbb{C}$ and let A be a matrix in WPFn. Consider the closed disc in the complex plane centered at the origin of radius equal to $\rho(A)$. Applying the maximum modulus principle to that closed disc, we obtain

$$\max\ \{|f(z)|\ :\ |z| \leq \rho(A)\}\ =\ \max\ \{|f(z)|\ :\ |z| = \rho(A)\}.$$

Thus, if λ is any eigenvalue of A, then

$$|f(\lambda)| \leq\ \max\ \{|f(z)|\ :\ |z| = \rho(A)\} \leq\ \max\ \{f(|z|)\ :\ |z| = \rho(A)\} = f(\rho(A)).$$

Since the eigenvalues of $f(A)$ are the images under f of the eigenvalues of A, it follows that the spectral radius of $f(A)$ is in fact $f(\rho(A))$. Furthermore, if v a right (respectively, a left) Perron-Frobenius eigenvector of A, then it is a right (respectively, a left) Perron-Frobenius eigenvector of $f(A)$, and hence, $f(A)$ is in WPFn. Therefore, f preserves WPFn.

Conversely, suppose that f is an entire function that preserves WPFn. Let q_1 be a positive vector in $\mathbb{R}^n$ of unit length. Extend q_1 to an orthonormal basis $\{q_1, q_2, \ldots, q_n\}$

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of $\mathbb{R}^n$ and construct the orthogonal matrix $Q = [q_1\ q_2 \cdots\ q_n]$. Let t be a nonnegative scalar, let $a, b \in \mathbb{R}$ be such that $\sqrt{a^2 + b^2} = t$, and let $C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Construct a real $n \times n$ matrix M as follows:

$$M = \begin{cases} [t] \oplus \underbrace{C \oplus \cdots \oplus C}_{\frac{n-1}{2} \text{ copies}} & \text{if } n \text{ is odd,} \\ \\ [t] \oplus [t] \oplus \underbrace{C \oplus \cdots \oplus C}_{\frac{n-2}{2} \text{ copies}} & \text{if } n \text{ is even.} \end{cases}$$

Consider matrices A in WPFn of the form $A = QMQ^T$. Then, we have

$$f(A) = Qf(M)Q^T = \begin{cases} Q \left[[f(t)] \oplus \underbrace{f(C) \oplus \cdots \oplus f(C)}_{\frac{n-1}{2} \text{ copies}} \right] Q^T & \text{if } n \text{ is odd,} \\ \\ Q \left[[f(t)] \oplus [f(t)] \oplus \underbrace{f(C) \oplus \cdots \oplus f(C)}_{\frac{n-2}{2} \text{ copies}} \right] Q^T & \text{if } n \text{ is even.} \end{cases}$$

Note that the spectrum of C is $\{a + ib, a - ib\}$ and that the spectrum of A is $\{t, a + ib, a - ib\}$. Moreover, note that each of the columns of Q other than the first column q_1 must have a positive entry and a negative entry because the first column q_1 is positive and it is orthogonal to the remaining columns. Since f preserves WPFn and since q_1 is the only column of Q that does not have two entries with opposite signs, it follows that $f(t)$ is the spectral radius of $f(A)$. Thus, $|f(a \pm ib)| \le f(t)$. But, the scalars a, b, and t were arbitrary scalars satisfying $\sqrt{a^2 + b^2} = t$. Hence, the entire function f must satisfy the inequality $|f(z)| \le f(|z|)$ for all $z \in \mathbb{C}$. ◻

Corollary 3.2. *If f is an entire function such that $f^{(k)}(0) \ge 0$ for all $k \ge 0$, then f preserves WPFn.*

Proof. Since f is an entire function, it can be written as a power series as in (1.1). Moreover, $a_k = \dfrac{f^{(k)}(0)}{k!} \ge 0$ for all $k \ge 0$ because $f^{(k)}(0) \ge 0$ for all $k \ge 0$. Thus, $|f(z)| \le \displaystyle\sum_{k=0}^{\infty} a_k |z|^k = f(|z|)$ for all $z \in \mathbb{C}$. Therefore, by Theorem 3.1, f preserves WPFn. ◻

The proof of the following theorem is very similar to that of Theorem 3.1, and thus, will be omitted.

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THEOREM 3.3. *Let f be an entire function. Then, $|f(z)| \leq f(|z|)$ for all $z \in \mathbb{C}$ if and only if f preserves the set of nonnilpotent matrices in WPFn ($n \geq 3$).*

4. Matrix functions preserving the set of real $n \times n$ eventually nonnegative matrices. In this section we consider matrix functions that preserve the set of real $n \times n$ eventually nonnegative matrices. In separate results, we give necessary conditions and sufficient conditions.

We begin with results corresponding to sufficient conditions.

THEOREM 4.1. *If f is an entire function such that $f(0) = 0$ and $f^{(k)}(0) \geq 0$ for $k = 1, 2, \ldots$, then f preserves the set of real eventually nonnegative matrices of order n for $n = 1, 2, \ldots$*

Proof. Suppose that f is an entire function satisfying the hypothesis of this theorem. Then f has a power series expansion given by (1.1) for all $z \in \mathbb{C}$. Let A be any real eventually nonnegative matrix of order n. Then, for any positive integer m we have

$$(f(A))^m = \left(\lim_{N \to \infty} \sum_{k=1}^{N} a_k A^k \right)^m = \lim_{N \to \infty} \left(\sum_{k=1}^{N} a_k A^k \right)^m .$$

Using the multinomial theorem to expand the expression $\left(\sum_{k=1}^{N} a_k A^k \right)^m$, one obtains a polynomial in the matrix A of the form $\sum c_k A^k$, where all the coefficients c_k are nonnegative and $k \geq m$. Thus, if k_0 is the smallest positive integer such that $A^k \geq 0$ for all $k \geq k_0$, then

$$\left(\sum_{k=1}^{N} a_k A^k \right)^m \geq 0 \quad \text{for all} \quad m \geq k_0.$$

And hence, $(f(A))^m \geq 0$ for all $m \geq k_0$. □

We note here that the converse of Theorem 4.1 is not true. For example, the entire function $f(z) = 1$ preserves the set of real eventually nonnegative matrices of order n for $n = 1, 2, \ldots$, yet $f(0) \neq 0$. On the other hand, the requirement that $f(0) = 0$ in Theorem 4.1 may not be dropped as we show in Remark 4.3 below. We begin with the following lemma.

LEMMA 4.2. *Let α and β be two complex scalars such that $\alpha \neq 0$ and let i and j be two distinct integers in the set $\{1, 2, \ldots, n\}$. Then the $n \times n$ matrix $\alpha I + \beta e_i e_j^T$ is real eventually nonnegative if and only if α is a real positive scalar and β is a real nonnegative scalar.*

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Proof. Suppose that the matrix $\alpha I + \beta e_i e_j^T$ is real eventually nonnegative. Then, there exists k_0 such that $\left(\alpha I + \beta e_i e_j^T\right)^m \geq 0$ for all $m \geq k_0$, or equivalently, for all $m \geq k_0$ we have

$$\sum_{k=0}^{m} \left(\begin{array}{c} m \\ k \end{array} \right) \beta^k \alpha^{m-k} \left(e_i e_j^T\right)^k = \alpha^m I + m\alpha^{m-1}\beta \; e_i e_j^T \geq 0.$$

Thus, $\alpha^m \geq 0$ for all $m \geq k_0$. Since α is a nonzero complex scalar it follows that α is a real positive scalar. Similarly, since $m\alpha^{m-1}\beta \geq 0$ for $m \geq k_0$ we conclude that $\beta \geq 0$. Conversely, if $\alpha > 0$ and $\beta \geq 0$ then clearly the matrix $\alpha I + \beta e_i e_j^T$ is nonnegative and thus real eventually nonnegative. ∎

We point out here that it may not be concluded from Lemma 4.2 that an eventually nonnegative triangular matrix with a positive diagonal must necessarily be nonnegative. For example, consider the upper triangular matrix $A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Since A^2 and A^3 are nonnegative matrices, it follows that $A^k \geq 0$ for $k \geq 2$ yet matrix A has a negative entry.

Remark 4.3. *The requirement in Theorem 4.1 that $f(0) = 0$ may not be dropped. For example, if we consider the entire function $f(z) = e^z$, then*

$$f^{(k)}(0) = \frac{d^k}{dz^k}\left[e^z\right]|_{z=0} = e^z|_{z=0} = e^0 = 1 \geq 0 \quad \textit{for} \quad k = 1, 2, 3, \ldots,$$

but $f(0) = 1 \neq 0$. Applying the function f to the nilpotent matrix $-e_1 e_2^T$ that has -1 in the $(1,2)$-entry and zeroes everywhere else, one obtains $f(-e_1 e_2^T) = I - e_1 e_2^T$, which is not eventually nonnegative by Lemma 4.2.

We consider now results on necessary conditions.

Theorem 4.4. *If f is an entire function that preserves real eventually nonnegative matrices of order n, then $f([0,\infty)) \subseteq [0,\infty)$.*

Proof. Suppose that f is an entire function that preserves real eventually nonnegative matrices of order n and suppose that the power series expansion of f is given by (1.1) for all $z \in \mathbb{C}$. Let t be a nonnegative scalar. Then $f(tI)$ must be a real eventually nonnegative matrix. But

$$f(tI) = \sum_{k=0}^{\infty} a_k t^k I^k = \left(\sum_{k=0}^{\infty} a_k t^k \right) I = f(t) I.$$

This latter matrix is real eventually nonnegative if and only if $f(t)$ is real and nonnegative. Hence, $f([0,\infty)) \subseteq [0,\infty)$. ∎

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THEOREM 4.5. *Let f be an entire function that preserves the set of real eventually nonnegative matrices of order n. If $f(t) \neq 0$ for some real scalar $t \geq 0$, then $f(t) > 0$ and $f'(t) \geq 0$. Furthermore, if $f(0) \neq 0$, then $f(0) > 0$ and $f'(0) = 0$.*

Proof. Let t be a nonnegative scalar such that $f(t) \neq 0$ and let (1.1) be the power series expansion of f about 0. Consider the nonnegative matrix $tI + e_1e_2^T$ of order n. We claim that

$$f(tI + e_1e_2^T) = f(t)I + f'(t)e_1e_2^T. \tag{4.1}$$

To see this, note that the scalar t is either zero or positive. For $t = 0$, using the fact that $(e_1e_2^T)^k$ is the zero matrix for all $k \geq 2$, we get

$$f(tI + e_1e_2^T) = f(e_1e_2^T) = \sum_{k=0}^{\infty} a_k(e_1e_2^T)^k = a_0I + a_1e_1e_2^T = f(0)I + f'(0)e_1e_2^T.$$

Similarly, if t is a positive scalar, then the matrix $f(tI + e_1e_2^T)$ is given by

$$\begin{aligned} f(tI + e_1e_2^T) &= \sum_{k=0}^{\infty} a_k \left(tI + e_1e_2^T\right)^k = \sum_{k=0}^{\infty} a_k \left(t^kI + kt^{k-1}e_1e_2^T\right) \\ &= \left(\sum_{k=0}^{\infty} a_kt^k\right) I + \left(\sum_{k=0}^{\infty} ka_kt^{k-1}\right) e_1e_2^T = f(t)I + f'(t)e_1e_2^T. \end{aligned}$$

Thus, for any $t \geq 0$ satisfying the hypothesis of this theorem, the matrix $f(tI+e_1e_2^T)$ is equal to $f(t)I+f'(t)e_1e_2^T$. Since f preserves real $n\times n$ eventually nonnegative matrices and since $f(t) \neq 0$, it follows that the matrix $f(tI+e_1e_2^T) = f(t)I+f'(t)e_1e_2^T$ must be real eventually nonnegative and by Lemma 4.2 we must have $f(t) > 0$ and $f'(t) \geq 0$. In particular, when $t = 0$, we have $f(0) > 0$ and we claim that $f'(0) = 0$ in this case. To prove this claim, consider the nilpotent matrix $\gamma e_1e_2^T$ where γ is any real scalar. Since f preserves real $n \times n$ eventually nonnegative matrices, it follows that the matrix $f(\gamma e_1e_2^T) = a_0I+\gamma a_1e_1e_2^T$ must be real and eventually nonnegative. Since $a_0 = f(0) > 0$, it follows from Lemma 4.2 that $\gamma a_1 \geq 0$. But γ is an arbitrary real scalar. Hence, $a_1 = f'(0) = 0$. □

THEOREM 4.6. *If f is an entire function that preserves the set of real eventually nonnegative matrices of order n and $f'(0) \neq 0$, then $f(0) = 0$.*

Proof. Let f be an entire function satisfying the hypotheses of this theorem and suppose to the contrary that $f(0) \neq 0$. We will construct a matrix $A \in \mathbb{R}^{n\times n}$ such that A is nilpotent (and thus, eventually nonnegative) yet $f(A)$ is not eventually nonnegative. Consider the unit vector $v_1 = \left[\frac{\sqrt{2}}{2}, 0, \cdots, 0, -\frac{\sqrt{2}}{2}\right]^T$ in $\mathbb{R}^n$ and extend the set $\{v_1\}$ to an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$. Build the real orthogonal

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matrix $Q = [v_1\ v_2\ \cdots\ v_n]$ and construct the real matrix $A = QJ_n(0)Q^T$, where $J_n(0)$ is the elementary Jordan block of order n corresponding to the complex number 0. Clearly, $A^k = 0$ for all $k \geq n$ and thus A is eventually nonnegative. Moreover,

$$f(A) = Qf(J_n(0))Q^T = Q \begin{bmatrix} f(0) & f'(0) & \cdots & \frac{f^{n-1}(0)}{(n-1)!} \\ 0 & f(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f'(0) \\ 0 & \cdots & 0 & f(0) \end{bmatrix} Q^T.$$

Since $f(0) \neq 0$, it follows that $f(A)$ must be a nonnilpotent eventually nonnegative matrix in $\mathbb{R}^n$ and thus $f(A) \in$ WPFn. The eigenspace of $f(A)$ corresponding to the only eigenvalue $f(0)$, $E_{f(0)}(f(A))$, is one dimensional because $f'(0) \neq 0$. Therefore, any eigenvector corresponding to the eigenvalue $f(0)$ must be a nonzero scalar multiple of v_1, which is a vector that has a positive entry and a negative entry. But this means that $f(A)$ does not have a Perron-Frobenius eigenvector, which is a contradiction. Hence, $f(0) \neq 0$. □

Corollary 4.7. *If f is an entire function such that $f(0) \neq 0$ and $f'(0) \neq 0$, then f does not preserve the set of real $n \times n$ eventually nonnegative matrices.*

Remark 4.8. *If an entire function f preserves the set of real $n \times n$ eventually nonnegative matrices, then it does not necessarily follow that its derivative f' does so. For example, the function $f(z) = e^z - 1$ preserves the set of $n \times n$ real eventually nonnegative matrices by Theorem 4.1. However, by Remark 4.3, its derivative $f'(z) = e^z$ does not.*

We end this section by presenting a result on preserving the eventual nonnegativity of matrices of lower orders.

Proposition 4.9. *Let f be an entire function. Then f preserves the set of real eventually nonnegative matrices of order n if and only if it preserves the set of real eventually nonnegative matrices of order k for $k = 1, 2, \ldots, n$.*

Proof. Suppose that f is an entire function that preserves the set of real eventually nonnegative matrices of order n. Consider the diagonal block matrix $A \in \mathbb{R}^{n\times n}$ of the form $A = B_1 \oplus B_2 \oplus \cdots \oplus B_m$ where each of the diagonal blocks B_k $(1 \leq k \leq m)$ is eventually nonnegative. One could easily see that in this case $f(A) = f(B_1) \oplus f(B_2) \oplus \cdots \oplus f(B_m)$. Since $f(A)$ is eventually nonnegative, it follows that $f(B_k)$ is eventually nonnegative $(1 \leq k \leq m)$. By taking eventually nonnegative diagonal blocks of various sizes, it easily follows that f preserves the set of real eventually nonnegative matrices of order k for all $k \leq n$. The converse is immediate. □

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5. Matrix functions preserving exponentially nonnegative matrices. In order to characterize entire functions preserving exponentially nonnegative matrices, we make use of the equivalence between these matrices and essentially nonnegative matrices, which follows, e.g., from [17, Lemma 3.1] or from [2, Ch. 6, Theorem 3.12]. A matrix $A \in \mathbb{R}^{n\times n}$ $(n \geq 2)$ is said to be *essentially nonnegative* if all its off-diagonal entries are nonnegative; see, e.g., [2], [9].

In this section we consider the case $n \geq 3$, and completely characterize complex polynomials that preserve $n \times n$ exponentially nonnegative matrices (or equivalently, essentially nonnegative matrices). We also give necessary conditions and sufficient conditions for an entire function (with some additional properties in some cases) to preserve the set of $n\times n$ exponentially nonnegative matrices. We will use the following notation. For an integer k, the matrix O_k (respectively, I_k) will denote the zero matrix (respectively, the identity matrix) of order k if the integer $k \geq 1$ and the empty matrix if the integer $k < 1$.

We begin our results by considering polynomials, and later in the section, we discuss general entire functions.

PROPOSITION 5.1. *The entire function $f(z) = z^{2m}$ $(m = 1, 2, \ldots)$ does not preserve the set of $n \times n$ exponentially nonnegative matrices for any integers $n \geq 2$.*

Proof. Consider the matrix A given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \oplus O_{n-2}. \tag{5.1}$$

Then A is an $n \times n$ exponentially nonnegative matrix. Moreover, $A^2 = -A$ and thus $A^{2m} = -A$ for $m = 1, 2, \ldots$, i.e., the matrix A^{2m} has exactly one negative off-diagonal entry, the $(1, 2)$-entry. Hence, the entire function $f(z) = z^{2m}$ $(m = 1, 2, \ldots)$ does not preserve the set of $n \times n$ exponentially nonnegative matrices for any $n \geq 2$ since $f(A) = A^{2m}$ is not exponentially nonnegative. ▯

PROPOSITION 5.2. *The entire function $f(z) = z^{2m+1}$ $(m = 1, 2, \ldots)$ does not preserve the set of $n \times n$ exponentially nonnegative matrices for all integers $n \geq 3$.*

Proof. Consider the matrix B given by

$$B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \oplus O_{n-3}. \tag{5.2}$$

Then B is an $n \times n$ exponentially nonnegative matrix and the Jordan decomposition

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of B is given by $B = VJV^{-1}$, where

$$V = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -\sqrt{2} & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix} \oplus I_{n-3}, \quad J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1-\sqrt{2} & 0 \\ 0 & 0 & -1+\sqrt{2} \end{bmatrix} \oplus O_{n-3},$$

$$\text{and} \quad V^{-1} = \left(\frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 1 & -\sqrt{2} & 1 \\ 1 & \sqrt{2} & 1 \end{bmatrix} \right) \oplus I_{n-3}.$$

Thus, for any positive integer k, the matrix B^k is given by $B^k = VJ^kV^{-1}$. Note that the $(1,3)$-entry of B^k is $\frac{1}{4}\left(2(-1)^{k+1} + \left(-1-\sqrt{2}\right)^k + \left(-1+\sqrt{2}\right)^k\right)$. In particular, if the positive integer k is an odd positive integer of the form $2m+1$ with $m = 1, 2, \ldots$, then the (1,3)-entry of the matrix B^{2m+1} is

$$\begin{aligned} &\frac{1}{4}\left(2 + \left(-1-\sqrt{2}\right)^{2m+1} + \left(-1+\sqrt{2}\right)^{2m+1}\right) \\ &= \frac{1}{4}\left(2 + \sum_{j=0}^{2m+1} \binom{2m+1}{j} (-1)^j \left(\left(-\sqrt{2}\right)^{2m+1-j} + \left(\sqrt{2}\right)^{2m+1-j}\right)\right) \\ &= \frac{1}{4}\left(2 - 2 \sum_{0 \le \text{ odd integer } j \le 2m+1} \binom{2m+1}{j} \left(\sqrt{2}\right)^{2m+1-j}\right) < 0, \end{aligned}$$

where the last equality follows from the fact that $\left(-\sqrt{2}\right)^{2m+1-j} + \left(\sqrt{2}\right)^{2m+1-j} = 0$ if j is even. Hence, the (1,3)-entry of the matrix B^{2m+1} is negative for all integers $m \geq 1$. Therefore, the entire function $f(z) = z^{2m+1}$ $(m = 1, 2, \ldots)$ does not preserve the set of $n \times n$ exponentially nonnegative matrices for any $n \geq 3$ since $f(B) = B^{2m+1}$ is not exponentially nonnegative. ◻

REMARK 5.3. *If $f(z)$ is an entire function as in* (1.1) *that preserves the set of $n \times n$ exponentially nonnegative matrices, then it must map the $n \times n$ elementary Jordan block of 0 to an exponentially nonnegative matrix, i.e., the matrix*

$$f(J_n(0)) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ 0 & \cdots & 0 & a_0 \end{bmatrix}$$

has nonnegative off-diagonal entries. Thus, $a_1, \ldots, a_{n-1}$ must be nonnegative and hence real. In the following lemma, we show that all the a_k's must be real.

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LEMMA 5.4. *Let $f(z)$ be an entire function as in* (1.1). *If f preserves the set of $n \times n$ exponentially nonnegative matrices ($n \geq 2$), then all the coefficients a_k are real.*

Proof. Since $a_k = \dfrac{f^{(k)}(0)}{k!}$ for $k = 0, 1, 2, \ldots$, it suffices to show that $f^{(k)}(0) \in \mathbb{R}$ for all nonnegative integers k. We prove a stronger condition, namely, the condition that for all nonnegative integers k the k^{th} derivative $f^{(k)}(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. The proof is by induction on k. When $k = 0$ we have to prove that $f(t)$ is a real scalar whenever t is a real scalar. Since f preserves exponentially nonnegative matrices, it follows that the matrix $f(tI) = \sum_{k=0}^{\infty} \left(a_k t^k I\right) = f(t)I$ is an exponentially nonnegative matrix for all $t \in \mathbb{R}$. Hence, $f(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. Suppose now that for all nonnegative integers m such that $0 \leq m \leq k-1$, the m^{th} derivative $f^{(m)}(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. Let t be any fixed real scalar and let h complex scalar variable. Since f is an entire function, the limit of the quotient $\dfrac{f^{(k-1)}(t+h) - f^{(k-1)}(t)}{h}$ as the complex scalar variable $h \to 0$ exists and is equal to $f^{(k)}(t)$. In particular, by letting h approach 0 along the real axis, the quotient $\dfrac{f^{(k-1)}(t+h) - f^{(k-1)}(t)}{h}$ approaches a limit which is a real number since the numerator and the denominator are real. By the uniqueness of the limit, the k^{th} derivative $f^{(k)}(t)$ must be real. Hence, $f^{(k)}(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$ and all nonnegative integers k. □

THEOREM 5.5. *If $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_0$ is any complex polynomial of degree $k \geq 2$ then $p(z)$ does not preserve the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$).*

Proof. Suppose to the contrary that $p(z)$ does preserve the set of $n \times n$ exponentially nonnegative matrices. Then, by Lemma 5.4 all the coefficients a_j ($0 \leq j \leq k$) must be real. Since the degree of the polynomial $p(z)$ is k, it follows that $a_k \neq 0$. There are two cases: case (1) $a_k > 0$ and case (2) $a_k < 0$. If case (1) holds, i.e., $a_k > 0$, then we have two subcases: subcase (i) k is even and subcase (ii) k is odd. Suppose that subcase (i) holds, i.e., k is even. Then let t be a nonnegative scalar variable and consider the exponentially nonnegative matrix tA, where A is given by (5.1). Note that $A^k = -A$ because k is an even positive integer. Hence, if we consider the limit of the (1, 2)-entry of the matrix $p(tA)$ as $t \to \infty$, then we see that $\lim_{t\to\infty} e_1^T p(tA) e_2 = \lim_{t\to\infty} a_k t^k e_1^T A^k e_2 = \lim_{t\to\infty} -a_k t^k e_1^T A e_2 = -\infty$, i.e., the (1,2)-entry, which is an off-diagonal entry, of the matrix $p(tA)$ is negative for all real scalars t sufficiently large, a contradiction. Suppose subcase (ii) holds, i.e., k is odd. Then let t be a nonnegative scalar variable and consider the exponentially nonnegative matrix tB where B is given by (5.2). Then, by an argument similar to that presented in subcase (i), we conclude that for all t sufficiently large the (1, 3)-

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entry, which is an off-diagonal entry, of the matrix $p(tB)$ is negative, a contradiction. Suppose that case (2) holds, i.e., $a_k < 0$. Then let t be a nonnegative scalar variable and consider the exponentially nonnegative matrix tC where $C = I + e_1 e_2^T$. Note that $C^k = I + k e_1 e_2^T$. The sign of the $(1, 2)$-entry of the matrix $p(tC)$ is determined by the $(1, 2)$-entry of the matrix $a_k t^k C^k = a_k t^k (I + k e_1 e_2^T)$ for all t sufficiently large. Hence, for all t sufficiently large, the $(1, 2)$-entry (an off-diagonal entry) of the matrix $p(tC)$ is negative, a contradiction. Therefore, for $n \geq 3$, $p(z)$ does not preserve the set of $n \times n$ exponentially nonnegative matrices. □

THEOREM 5.6. *A complex polynomial $p(z)$ preserves the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$) if and only if $p(z) = az + b$ where $a, b \in \mathbb{R}$ and $a \geq 0$.*

Proof. If $p(z)$ is a polynomial preserving the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$), then it follows from Theorem 5.5 that the degree of $p(z)$ can not exceed 1. Hence, $p(z) = az + b$. Moreover, by Lemma 5.4, we have $a, b \in \mathbb{R}$. Furthermore, if $a < 0$, then $p(M) = aM + bI$ will have negative off-diagonal entries for any positive matrix M. Thus, $a \geq 0$. The converse is immediate. □

This completes our characterization of polynomial preserving exponentially nonnegative matrices. We next analyze the possible entire functions preserving these matrices having power series with infinite number of nonzero coefficient in its power series expansion.

THEOREM 5.7. *Let $f(z)$ be an entire function as in* (1.1) *such that $a_1 = 0$ and a_k is real for all nonnegative integers k. If the sequence $\{a_k\}_{k=2}^{\infty}$ has infinitely many nonzero terms, then f does not preserve the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$).*

Proof. Suppose to the contrary that the function f preserves the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$). Then there are three cases: case (1) the sequence $\{a_k\}_{k=2}^{\infty}$ has infinitely many positive terms and infinitely many negative terms, case (2) the sequence $\{a_k\}_{k=2}^{\infty}$ has infinitely many positive terms and finitely many negative terms, and case (3) the sequence $\{a_k\}_{k=2}^{\infty}$ has finitely many positive terms and infinitely many negative terms. Suppose with the hope of getting a contradiction that case (1) is true. Then, due to the absolute convergence of the power series of the entire function f, we may rearrange its terms to get the following:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + \sum_{k=r}^{\infty} b_k z^k + \sum_{k=m}^{\infty} c_k z^k,$$

where $b_k \leq 0$ ($k = r, r+1, r+2, \ldots$), $b_r < 0$, $c_k \geq 0$ ($k = m, m+1, m+2, \ldots$), and $c_m > 0$ for some distinct positive integers r and m. If $r < m$, then let t be a nonnegative scalar variable and consider the exponentially nonnegative matrix tM, where M is a positive matrix. If we consider the limit of the $(1, 2)$-entry of the matrix

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$t^{-r} f(tM)$ as $t \to 0^+$, then we see that $\lim_{t\to 0^+} t^{-r} e_1^T f(tM) e_2 = b_r e_1^T M^r e_2 < 0$, i.e., the $(1,2)$-entry, which is an off-diagonal entry, of the matrix $f(tM)$ is negative for all positive scalars t sufficiently small, a contradiction. Thus, $r > m$. Here we have two subcases: subcase (i) m is an even positive integer and subcase (ii) m is an odd positive integer. Suppose subcase (i) is true, i.e., m is even. Then let t be a nonnegative scalar variable and consider the exponentially nonnegative matrix tA, where A is the matrix given by (5.1). Note that $A^j = -A$ for all even positive integers j. If we consider the limit of the $(1,2)$-entry of the matrix $t^{-m} f(tA)$ as $t \to 0^+$ and use the hypothesis that m is even, then we get $\lim_{t\to 0^+} t^{-m} e_1^T f(tA) e_2 = c_m e_1^T A^m e_2 < 0$, i.e., the (1,2)-entry, which is an off-diagonal entry, of the matrix $f(tA)$ is negative for all positive scalars t sufficiently small, a contradiction. Thus, subcase (ii) must hold, i.e., m is odd. Then either $m = 1$ or m is an odd positive integer greater than 1. If $m = 1$, then $c_1 > 0$, i.e., the coefficient of z in the power series of f is positive. But the latter statement means that $a_1 = c_1 > 0$, a contradiction. Hence, m is an odd positive integer greater than 1. Then let t be a nonnegative scalar variable and consider the exponentially nonnegative matrix tB, where B is the matrix given by (5.2). We note here that in the proof of Theorem 5.5, we showed that the $(1,3)$-entry of the matrix B^j is negative for any odd positive integer $j > 1$. If we consider the limit of the $(1,3)$-entry of the matrix $t^{-m} f(tB)$ as $t \to 0^+$ and use the hypothesis that m is odd, then we see that $\lim_{t\to 0^+} t^{-m} e_1^T f(tB) e_2 = c_m e_1^T B^m e_2 < 0$, i.e., the $(1,3)$-entry, which is an off-diagonal entry, of the matrix $f(tB)$ is negative for all positive scalars t sufficiently small, a contradiction. Therefore, case (1) leads to a contradiction. Similarly, cases (2) and (3) lead to contradictions. Hence, for $n \geq 3$, f does not preserve the set of exponentially nonnegative matrices of order n. □

Remark 5.8. *The requirement that $a_1 = 0$ in Theorem 5.7 may not be dropped. Consider, e.g., the entire function*

$$f(z) = e^z = \sum_{k=0}^{\infty} a_k z^k, \quad \text{where} \quad a_k = \frac{1}{k!}$$

for all nonnegative integers k. If D is any exponentially nonnegative matrix of order n $(n \geq 1)$ then we can write $D = C - sI$ for some nonnegative matrix C and some real scalar $s > 0$. Using the fact that the matrices C and $-sI$ commute, we get $f(D) = e^D = e^{C-sI} = e^C e^{-sI} = e^{-s} e^C \geq 0$. Hence, $f(z) = e^z$ is an entire function that preserves the set of exponentially nonnegative matrices of order n $(n \geq 1)$ yet the sequence $\{a_k\}_{k=2}^{\infty}$ has infinitely many nonzero terms.

Corollary 5.9. *Let f be an entire function such that $f'(0) = 0$. Then f preserves the set of $n \times n$ exponentially nonnegative matrices $(n \geq 3)$ if and only if f is a constant real-valued function.*

Proof. Let $f(z)$ be an entire function as in (1.1) such that $a_1 = f'(0) = 0$. If

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f preserves the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$), then by Theorem 5.7 the sequence $\{a_k\}_{k=2}^{\infty}$ has only finitely many nonzero terms. Thus, f is a polynomial and by Theorem 5.6 the entire function f is linear with real coefficients. Hence, $f(z) = a_0 + a_1 z = a_0 \in \mathbb{R}$ for all $z \in \mathbb{C}$. The converse is immediate. □

The technique of proof for the following theorem is similar to that of Theorem 5.7 and thus the proof will be omitted.

THEOREM 5.10. *Let $f(z)$ be an entire function as in* (1.1) *such that a_k is real for all nonnegative integers k. If $a_1 < 0$ or $a_k < 0$ for all but finitely many nonnegative integers k, then f does not preserve the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$).*

Combining the results from Theorem 5.7 through Theorem 5.10, we obtain the following necessary condition on entire functions that preserve the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$).

THEOREM 5.11. *Let $f(z)$ be an entire function as in* (1.1). *If f preserves the set of $n \times n$ exponentially nonnegative matrices ($n \geq 3$), then either*

(1) $a_1 = 0$ in which case $f(z) = a_0 \in \mathbb{R}$ for all $z \in \mathbb{C}$,
or
(2) $a_1 > 0$ in which case
   (i) $f(z) = a_0 + a_1 z$ for some $a_0 \in \mathbb{R}$,
   or
   (ii) $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where all the a_k's are real, finitely many a_k's are negative, and infinitely many of them are positive.

6. Conclusions and outlook. We were able to completely characterize matrix functions preserving PFn and WPFn, and we gave separate necessary and sufficient conditions for matrix functions to preserve eventually nonnegative matrices and exponentially nonnegative matrices. There are of course other questions related to matrix functions preserving generalized nonnegative matrices worth considering. As already mentioned, the case of matrix functions preserving 2×2 exponentially nonnegative matrices was not covered here. Another aspect of interest would consider matrix functions that preserve generalized M-matrices such as GM-matrices (see [5]) or M_V-matrices (see [18]). In particular, it would be interesting to find analogues in the case of generalized M-matrices and generalized nonnegative matrices to the results by Varga, Bapat, Catral, and Neumann [1], [20] on matrix functions taking M-matrices to nonnegative matrices or taking inverse M-matrices to inverse M-matrices or nonnegative matrices.

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REFERENCES

[1] R. Bapat, M. Catral, and M. Neumann. On functions that preserve M-matrices and inverse M-matrices. *Linear and Multilinear Algebra*, 53:193–201, 2005.

[2] A. Berman and R. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1994.

[3] G. Bharali and O. Holtz. Functions preserving nonnegativity of matrices. *SIAM Journal on Matrix Analysis and Applications*, 30:84–101, 2008.

[4] A. Brauer. Limits for characteristics roots of a matrix. IV: Applications to stochastic matrices. *Duke Mathematics Journal*, 19:75–91, 1952.

[5] A. Elhashash and D.B. Szyld. Generalizations of M-matrices which may not have a nonnegative inverse. *Linear Algebra and its Applications*, 429:2435–2450, 2008.

[6] A. Elhashash and D.B. Szyld. On general matrices having the Perron-Frobenius property. *Electronic Journal of Linear Algebra*, 17:389–413, 2008.

[7] G.H. Golub and C.F. Van Loan. *Matrix Computations.* Third Edition, The Johns Hopkins University Press, Baltimore, Maryland 1996.

[8] N.J. Higham. *Functions of Matrices, Theory and Computations.* SIAM, Philadelphia, 2008.

[9] R. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

[10] R. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[11] A. Jakimovski, A. Sharma, and J. Szabados. *Walsh equiconvergence of complex interpolating polynomials.* Springer, New York, 2006.

[12] C.R. Johnson and P. Tarazaga. On matrices with Perron-Frobenius properties and some negative entries. *Positivity*, 8:327–338, 2004.

[13] C.R. Johnson and P. Tarazaga. A characterization of positive matrices. *Positivity*, 9:149–152, 2005.

[14] P. Lancaster and M. Tismenetsky. *The theory of matrices: with applications.* Academic Press, London, 1985.

[15] C.A. Micchelli and R.A. Willoughby. On functions which preserve the class of Stieltjes matrices. *Linear Algebra and its Applications*, 23:141–156, 1979.

[16] D. Noutsos. On Perron-Frobenius property of matrices having some negative entries. *Linear Algebra and its Applications*, 412:132–153, 2006.

[17] D. Noutsos and M.J. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM Journal on Matrix Analysis and Applications*, 30:700–712, 2008.

[18] D.D. Olesky, M.J. Tsatsomeros, and P. van den Driessche. $M_\vee$-matrices: A generalization of M-matrices based on eventually nonnegative matrices. *Electronic Journal of Linear Algebra*, 18:339–351, 2009.

[19] P. Tarazaga, M. Raydan, and A. Hurman. Perron-Frobenius theorem for matrices with some negative entries. *Linear Algebra and its Applications*, 328:57–68, 2001.

[20] R.S. Varga. Nonnegatively posed problems and completely monotonic functions. *Linear Algebra and its Applications*, 1:329–347, 1968.

[21] R.S. Varga. *Matrix Iterative Analysis.* Second edition, Springer-Verlag, Berlin, 2000.