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A. HORN'S RESULT ON MATRICES WITH PRESCRIBED SINGULAR VALUES AND EIGENVALUES*

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Abstract. We give a new proof of a classical result of A. Horn on the existence of a matrix with prescribed singular values and eigenvalues.

Key words. Eigenvalues, singular values.

AMS subject classifications. 15A45, 15A18.

Let $A \in \mathbb{C}_{n\times n}$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A arranged in the order $|\lambda_1| \geq \cdots \geq |\lambda_n|$. The singular values of A are the nonnegative square roots of the eigenvalues of the positive semi-definite matrix A^*A and are denoted by $s_1 \geq \cdots \geq s_n$. Weyl's inequalities [7] provide a very nice relation between the eigenvalues and singular values of A:

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} s_j, \quad k = 1, \ldots, n-1, \tag{1.1}$$

$$\prod_{j=1}^{n} |\lambda_j| = \prod_{j=1}^{n} s_j. \tag{1.2}$$

The equality follows from two ways of expressing the absolute value of the determinant of A. A. Horn [2] established the converse of Weyl's result.

THEOREM 1.1. (A. Horn) If $|\lambda_1| \geq \cdots \geq |\lambda_n|$ and $s_1 \geq \cdots \geq s_n$ satisfy (1.1) and (1.2), then there exists $A \in \mathbb{C}_{n\times n}$ such that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues and $s_1, \ldots, s_n$ are the singular values of A.

Horn's original proof is divided into two cases: (i) $s_n \neq 0$ (the nonsingular case) and (ii) $s_n = 0$ (the singular case. There is a typo: $C_{m,m+1} = \gamma$ and $C_{i,i+1} = \alpha_i$ should be $C_{m+1,m} = \gamma$ and $C_{i+1,i} = \alpha_i$ on [2, p.6]). In this note we provide a new proof of Horn's result. Our proof differs from Horn's proof in two ways that (i) our proof is divided into two cases according to $\lambda_1 = 0$ and $\lambda_1 \neq 0$, and (ii) our induction technique is different. It is very much like Chan and Li's technique [1] (the same

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technique is also used in [8]) for proving another result of Horn [3] (on the diagonal entries and eigenvalues of a Hermitian matrix): ours is multiplicative and Chan and Li's is additive. See [4, Section 3.6] for a proof of Theorem 1.1 using the result of Horn [3]. Also see [5] for an extension of Weyl-Horn's result and a numerically stable construction of A. A technique similar to that of Chan and Li can be found in Thompson's earlier work [6] on the diagonal entries and singular values of a square matrix.

Proof. We divide the proof into two cases: nilpotent or not.
Case i: $\lambda_1 = 0$. Then $s_n = 0$ by (1.2) and we choose

$$A := \begin{pmatrix} 0 & s_1 & & \\ & 0 & \ddots & \\ & & & s_{n-1} \\ & & & 0 \end{pmatrix}$$

Case ii: $\lambda_1 \neq 0$. We will use induction on n. When $n = 2$, the matrix

$$A = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_2 \end{pmatrix}$$

has singular values $s_1 \geq s_2$ if we set

$$\mu := (s_1^2 + s_2^2 - |\lambda_1|^2 - |\lambda_2|^2)^{1/2}.$$

Suppose that the statement of Theorem 1.1 is true for $\lambda_1 \neq 0$ when $n = m \geq 2$. Let $n = m + 1$ and let $j \geq 2$ be the largest index such that $s_{j-1} \geq |\lambda_1| \geq s_j$. Clearly $s_1 \geq \max\{|\lambda_1|, s_1 s_j/|\lambda_1|\} \geq \min\{|\lambda_1|, s_1 s_j/|\lambda_1|\}$. Then there exist 2×2 unitary matrices U_1 and V_1 such that

$$U_1 \begin{pmatrix} s_1 & \\ & s_j \end{pmatrix} V_1 = \begin{pmatrix} \lambda_1 & \mu' \\ 0 & s_1 s_j/|\lambda_1| \end{pmatrix},$$

where $\mu' = (s_1^2 + s_j^2 - |\lambda_1|^2 - s_1^2 s_j^2/|\lambda_1|^2)^{1/2}$. Set $U_2 := U_1 \oplus I_{m-1}, V_2 := V_1 \oplus I_{m-1}$. Then

$$\begin{aligned} A_1 &:= U_2 \operatorname{diag}(s_1, s_j, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{m+1}) V_2 \\ &= \begin{pmatrix} \lambda_1 & \mu' \\ 0 & s_1 s_j/|\lambda_1| \end{pmatrix} \oplus \operatorname{diag}(s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{m+1}). \end{aligned}$$

It suffices to show that $(s_1 s_j/|\lambda_1|, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{m+1})$ and $(\lambda_2, \ldots, \lambda_{m+1})$ satisfy (1.1) and (1.2). Since $s_{j-1} \geq |\lambda_1| \geq |\lambda_2|$,

$$|\lambda_2| \leq \max\{s_1 s_j/|\lambda_1|, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{m+1}\}.$$

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Moreover

$$\prod_{i=2}^{k} |\lambda_i| \le |\lambda_1|^{k-1} \le \prod_{i=2}^{k} s_i, \quad k = 2, \ldots, j-1,$$

$$\prod_{i=2}^{k} |\lambda_i| = \frac{1}{|\lambda_1|} \prod_{i=1}^{k} |\lambda_i| \le \frac{s_1 s_j}{|\lambda_1|} \prod_{i=2, i\neq j}^{k} s_i, \qquad k = j, \ldots, m, \qquad \text{by (1.1)}$$

$$\prod_{i=2}^{m+1} |\lambda_i| = \frac{s_1 s_j}{|\lambda_1|} \prod_{i=2, i\neq j}^{m+1} s_i \qquad \text{by (1.2)}.$$

We consider two cases: (a) $\lambda_2 = 0$ and apply Case i. (b) $\lambda_2 \neq 0$ and apply the induction hypothesis in Case ii. For both cases, there exist $m \times m$ unitary matrices U_3, V_3 such that

$$U_3 \operatorname{diag}\left(\frac{s_1 s_2}{|\lambda_1|}, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{m+1}\right) V_3$$

is upper triangular with diagonal $(\lambda_2, \ldots, \lambda_{m+1})$. Then $A = U_4 A_1 V_4$ is the desired matrix, where $U_4 := 1 \oplus U_3$, $V_4 := 1 \oplus V_3$. $\square$

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