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SOLVABLE 3-LIE ALGEBRAS WITH A MAXIMAL HYPO-NILPOTENT IDEAL N^*

RUI-PU BAI†, CAI-HONG SHEN†, AND YAO-ZHONG ZHANG‡

Abstract. This paper obtains all solvable 3-Lie algebras with the m-dimensional filiform 3-Lie algebra N ($m \geq 5$) as a maximal hypo-nilpotent ideal, and proves that the m-dimensional filiform 3-Lie algebra N can't be as the nilradical of solvable non-nilpotent 3-Lie algebras. By means of one dimensional extension of Lie algebras to the 3-Lie algebras, we get some classes of solvable Lie algebras directly.

Key words. 3-Lie algebra, hypo-nilpotent ideal, filiform n-Lie algebra.

AMS subject classifications. 17B05, 17D99.

1. Introduction. The concept of n-Lie algebras appeared in two different contexts [1, 2]. In [1], Nambu introduced n-ary multilinear operations in his description of simultaneous classical dynamics of n particles, and extended the Poisson bracket to the n-ary multilinear bracket. In [2], Filippov formulated a theory of n-Lie algebras based on his proposed $(2n-1)$-fold Jacobi type identity and gave a classification for n-Lie algebras of lower ($\leq n+1$) dimensions. The connection between the Nambu mechanics and the Filippov's theory of n-Lie algebras was established in 1994 by Takhtajan [3]. Recently n-Lie algebras have found important applications in string and membrane theories. For instance, in [4, 5] Bagger and Lambert proposed a supersymmetric field theory model for multiple M2-branes based on the metric 3-Lie algebras. More application of n-Lie algebras can be found in e.g., [6, 7, 8, 9, 10, 11, 12, 13].

In recent years, the structure of n-Lie algebras has been widely studied. Kasymov [14] developed the structure and representation theory of n-Lie algebras. Ling [15] proved that there is a unique $(n+1)$-dimensional simple n-Lie algebra for $n > 2$ over an algebraically closed field of characteristic zero. The first author of the current paper and her collaborators showed in [16] that there exist only $[\frac{n}{2}]+1$ classes of $(n+1)$-dimensional simple n-Lie algebras over a complete field of characteristic 2 and

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†College of Mathematics and Computer Science, Key Lab in Machine Learning and Computational Intelligence, Hebei University, Baoding (071002), China (bairp1@yahoo.com.cn, sch0925@163.com).
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‡School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia (yzz@maths.uq.edu.au). Supported by the Australian Research Council.

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gave a complete classification in [17] for six dimensional 4-Lie algebras. There are other results on structures and representations of n-Lie algebras.

The structure of n-Lie algebras is very different from that of Lie algebras, due to the n-ary multilinear operations involved. In particular, it turns out that the fundamental identity for an n-Lie algebra is much more restrictive than the Jocobi identity for a Lie algebra. One consequence is that higher, finite dimensional n-Lie algebras may be rare and are difficult to find. So it is very important to construct new examples of n-Lie algebras.

Filiform n-Lie algebras, i.e., nilpotent n-Lie algebras L satisfying $\dim L^i = \dim L - n - i$, are important class of nilpotent n-Lie algebras. In [18] we introduced the concept of hypo-nilpotent ideals of n-Lie algebras, and proved that an m-dimensional *simplest* filiform 3-Lie algebra N_0 can't be a nilradical of solvable non-nilpotent 3-Lie algebras. By m-dimensional *simplest* filiform 3-Lie algebra, we mean an m-dimensional filiform 3-Lie algebra with the following multiplication table in the basis $e_1, e_2, \ldots, e_m$,

$$[e_1, e_2, e_j] = e_{j-1}, 4 \leq j \leq m. \tag{1.1}$$

Moreover, it was shown that there are only four classes of $(m+1)$-dimensional and one class of $(m+2)$-dimensional solvable non-nilpotent 3-Lie algebras with N_0 as their maximal hypo-nilpotent ideal.

In this paper we generalize the results of [18]. Namely we consider a more complicated m-dimensional filiform 3-Lie algebra N $(m \geq 5)$ defined by the multiplication table (3.1) below (c.f. (1.1)). We obtain all solvable 3-Lie algebras with such an N as a maximal hypo-nilpotent ideal and prove that N can't be a nilradical of solvable non-nilpotent 3-Lie algebras.

The organization for the rest of this paper is as follows. Section 2 introduces some basic notions. Section 3 describes the structure of solvable 3-Lie algebras with the maximal hypo-nilpotent ideal N. Section 4 studies the solvable 3-Lie algebras with nilradical N. Section 5 gives an application of one dimensional extension of Lie algebras.

Throughout this paper we consider 3-Lie algebras over a field F of characteristic zero.

2. Fundamental notions. First we introduce some notions of n-Lie algebras (see [2, 14, 18]). A vector space A over a field F is an n-Lie algebra if there is an n-ary multilinear operation $[\ ,\cdots,\ ]$ satisfying the following identities

$$[x_1, \cdots, x_n] = (-1)^{\tau(\sigma)}[x_{\sigma(1)}, \cdots, x_{\sigma(n)}], \tag{2.1}$$

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and

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n], \tag{2.2}$$

where σ runs over the symmetric group S_n and the number $\tau(\sigma)$ is equal to 0 or 1 depending on the parity of the permutation σ.

A derivation of an n-Lie algebra A is a linear map $D : A \to A$, such that for any elements $x_1, \ldots, x_n$ of A

$$D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n].$$

The set of all derivations of A is a subalgebra of Lie algebra gl(A). This subalgebra is called the derivation algebra of A, and is denoted by Der$A$. The map $\mathrm{ad}(x_1, \cdots, x_{n-1}) : A \to A$ defined by $\mathrm{ad}(x_1, \cdots, x_{n-1})(x_n) = [x_1, \cdots, x_n]$ for $x_1, \ldots, x_n \in A$ is called a left multiplication. It follows from (2.2) that $\mathrm{ad}(x_1, \cdots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of Der$A$ and is denoted by ad(A). Every element in ad(A) is by definition an inner derivation, and for all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ of A,

$$\begin{aligned} &[\mathrm{ad}(x_1, \cdots, x_{n-1}), \mathrm{ad}(y_1, \cdots, y_{n-1})] \\ &= \mathrm{ad}(x_1, \cdots, x_{n-1})\mathrm{ad}(y_1, \cdots, y_{n-1}) - \mathrm{ad}(y_1, \cdots, y_{n-1})\mathrm{ad}(x_1, \cdots, x_{n-1}) \\ &= \sum_{i=1}^{n-1} \mathrm{ad}(y_1, \cdots, [x_1, \cdots, x_{n-1}, y_i], \cdots, y_{n-1}). \end{aligned} \tag{2.3}$$

Let $A_1, A_2, \ldots, A_n$ be subalgebras of n-Lie algebra A and let $[A_1, A_2, \cdots, A_n]$ denote the subspace of A generated by all vectors $[x_1, x_2, \cdots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The subalgebra $[A, A, \cdots, A]$ is called the derived algebra of A, and is denoted by A^1. If $A^1 = 0$, then A is called an abelian n-Lie algebra.

An ideal of an n-Lie algebra A is a subspace I such that $[I, A, \cdots, A] \subseteq I$. If $A^1 \neq 0$ and A has no ideals except for 0 and itself, then A is by definition a simple n-Lie algebra.

An ideal I of an n-Lie algebra A is called a solvable ideal, if $I^{(r)} = 0$ for some $r \geq 0$, where $I^{(0)} = I$ and $I^{(s)}$ is defined by induction,

$$I^{(s+1)} = [I^{(s)}, I^{(s)}, A, \cdots, A]$$

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for $s \geq 0$. When $A = I$, A is a solvable n-Lie algebra.

An ideal I of an n-Lie algebra A is called a nilpotent ideal, if I satisfies $I^r = 0$ for some $r \geq 0$, where $I^0 = I$ and I^r is defined by induction, $I^{r+1} = [I^r, I, A, \cdots, A]$ for $r \geq 0$. If $I = A$, A is called a nilpotent n-Lie algebra.

The sum of two nilpotent ideals of A is nilpotent, and the largest nilpotent ideal of A is called the nilradical of A, and is denoted by $NR(A)$.

Denote by A^* an associative algebra generated by all operators $\mathrm{ad}(x)$, where $x = (x_1, \cdots, x_{n-1}) \in A^{(n-1)}$. If I is an ideal of A, denote by I^*, $K(I)$ and $\mathrm{ad}(I, A)$ respectively the subalgebra of A^*, the ideal of A^* and the subalgebra of $\mathrm{ad}(A)$ generated by the operators of the form $\mathrm{ad}(c, x_1, \cdots, x_{n-2})$, $c \in I, x_i \in A, i = 1, \ldots, n-2$. It follows at once from (2.3) that $K(I) = I^* \cdot A^* = A^* \cdot I^*$, and $\mathrm{ad}(I, A)$ is an ideal of $\mathrm{ad}(A)$.

Lemma 2.1. *[14] An ideal I of an n-Lie algebra A is a nilpotent ideal if and only if $K(I)$ is a nilpotent ideal of the associative algebra A^*.*

An ideal I of an n-Lie algebra A may not be a nilpotent ideal although it is a nilpotent subalgebra. This property is different from that of Lie algebras. In the following, we concern such types of ideals of n-Lie algebras.

Definition 2.2. *Let A be an n-Lie algebra and I be an ideal of A. If I is a nilpotent subalgebra but is not a nilpotent ideal, then I is called a hypo-nilpotent ideal of A. If I is not properly contained in any hypo-nilpotent ideals, then I is called a maximal hypo-nilpotent ideal of A.*

From (2.2), a hypo-nilpotent ideal of A is a proper ideal, and the nilradical $NR(A)$ is properly contained in every maximal hypo-nilpotent ideals. But the sum of two hypo-nilpotent ideals of A may not be hypo-nilpotent.

In the following, any brackets of basis vectors not listed in the multiplication table of n-Lie algebras are assumed to be zero.

3. 3-Lie algebras with maximal hypo-nilpotent ideal N. In the following we suppose that N is an m-dimensional filiform 3-Lie algebra with the multiplication table

$$\begin{cases} [e_1, e_2, e_j] = e_{j-1}, & 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2}, & 5 \leq j \leq m-1, \end{cases} \tag{3.1}$$

where $e_1, \ldots, e_m$ is a basis of N.

Lemma 3.1. *Let N be an m-dimensional 3-Lie algebra with a basis $e_1, \ldots, e_m$ satisfying (3.1). Then the inner derivation algebra $ad(N)$ has a basis $ad(e_1, e_2)$,*

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$ad(e_1,e_j)$, $ad(e_2,e_j)$, $j = 4,5, \ldots, m$. *And with respect to the basis* $e_1,\cdots,e_m$, $ad(e_k,e_l)$ *is represented by the following matrix form*

$$ad(e_1,e_2)=\sum_{j=4}^{m}E_{jj-1},\, ad(e_1,e_m)=\sum_{j=5}^{m-1}E_{jj-2}+E_{2m-1},$$

$$ad(e_1,e_i)=E_{2i-1}+E_{mi-2},\, ad(e_2,e_i)=E_{1i-1} \text{ for } 5\le i\le m-1,$$

$$ad(e_1,e_4)=E_{23},\, ad(e_2,e_4)=E_{13},\, ad(e_2,e_m)=E_{1m-1},$$

where E_{ij} *is the* $(m\times m)$ *matrix unit.*

Proof. The result follows from a direct computation. □

Let A be an $(m+1)$-dimensional 3-Lie algebra with the ideal N, and $x,e_1,\ldots,e_m$ be a basis of A. Then the multiplication table of A in the basis $x,e_1,\ldots,e_m$ is given by

$$\text{(3.2)}\qquad \begin{cases} [e_1,e_2,e_j]=e_{j-1}, & 4\le j\le m,\\ [e_1,e_j,e_m]=e_{j-2}, & 5\le j\le m-1,\\ [x,e_i,e_j]=\sum\limits_{k=1}^{m}a_{ij}^k e_k, & 1\le i,j\le m, \end{cases}$$

where $a_{ij}^k\in F, a_{ij}^k=-a_{ji}^k, 1\le i,j\le m$. Therefore, the following $(\frac{m(m-1)}{2}\times m)$ matrix M determines the structure of A

$$\text{(3.3)}\qquad M=\begin{pmatrix}
a_{12}^1 & a_{12}^2 & a_{12}^3 & a_{12}^4 & \cdot & a_{12}^{m-1} & a_{12}^m\\
a_{13}^1 & a_{13}^2 & a_{13}^3 & a_{13}^4 & \cdot & a_{13}^{m-1} & a_{13}^m\\
a_{14}^1 & a_{14}^2 & a_{14}^3 & a_{14}^4 & \cdot & a_{14}^{m-1} & a_{14}^m\\
a_{15}^1 & a_{15}^2 & a_{15}^3 & a_{15}^4 & \cdot & a_{15}^{m-1} & a_{15}^m\\
\vdots & \vdots & \vdots & \vdots & . & \vdots & \vdots\\
a_{1m-2}^1 & a_{1m-2}^2 & a_{1m-2}^3 & a_{1m-2}^4 & \cdot & a_{1m-2}^{m-1} & a_{1m-2}^m\\
a_{1m-1}^1 & a_{1m-1}^2 & a_{1m-1}^3 & a_{1m-1}^4 & \cdot & a_{1m-1}^{m-1} & a_{1m-1}^m\\
a_{1m}^1 & a_{1m}^2 & a_{1m}^3 & a_{1m}^4 & \cdot & a_{1m}^{m-1} & a_{1m}^m\\
a_{23}^1 & a_{23}^2 & a_{23}^3 & a_{23}^4 & \cdot & a_{23}^{m-1} & a_{23}^m\\
a_{24}^1 & a_{24}^2 & a_{24}^3 & a_{24}^4 & \cdot & a_{24}^{m-1} & a_{24}^m\\
a_{25}^1 & a_{25}^2 & a_{25}^3 & a_{25}^4 & \cdot & a_{25}^{m-1} & a_{25}^m\\
\vdots & \vdots & \vdots & \vdots & . & \vdots & \vdots\\
a_{2m-2}^1 & a_{2m-2}^2 & a_{2m-2}^3 & a_{2m-2}^4 & \cdot & a_{2m-2}^{m-1} & a_{2m-2}^m\\
a_{2m-1}^1 & a_{2m-1}^2 & a_{2m-1}^3 & a_{2m-1}^4 & \cdot & a_{2m-1}^{m-1} & a_{2m-1}^m\\
a_{2m}^1 & a_{2m}^2 & a_{2m}^3 & a_{2m}^4 & \cdot & a_{2m}^{m-1} & a_{2m}^m\\
a_{34}^1 & a_{34}^2 & a_{34}^3 & a_{34}^4 & \cdot & a_{34}^{m-1} & a_{34}^m\\
\vdots & \vdots & \vdots & \vdots & . & \vdots & \vdots\\
a_{m-1m}^1 & a_{m-1m}^2 & a_{m-1m}^3 & a_{m-1m}^4 & \cdot & a_{m-1m}^{m-1} & a_{m-1m}^m
\end{pmatrix}.$$

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The matrix M is called the structure matrix of A with respect to the basis x, e_1, $\dots$, e_m.

By the above notations we have the following result.

THEOREM 3.2. *Let A be an $(m+1)$-dimensional 3-Lie algebra with a maximal hypo-nilpotent ideal N. Then A is solvable, and up to isomorphism the following is the only possibility for the structural matrix M of A:*

$$
(3.4) \quad M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & m-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m-2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m-3 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Moreover, the multiplication table of A in a basis $x, e_1, \dots, e_m$ $(m \geq 5)$ is as follows

$$
(3.5) \quad \begin{cases}
[e_1, e_2, e_j] = e_{j-1} & \text{if } 4 \leq j \leq m, \\
[e_1, e_j, e_m] = e_{j-2} & \text{if } 5 \leq j \leq m-1, \\
[x, e_1, e_2] = e_2, & \\
[x, e_1, e_k] = (m-k+2)e_k & \text{if } 3 \leq k \leq m.
\end{cases}
$$

Proof. Since N is an ideal of A and $\dim A = m+1$, we have $A^1 = [A, A, A] = [A, A, N] \subseteq N$. Then the structural matrix M is of the form (3.3) with respect to a basis $x, e_1, \dots, e_m$.

Firstly, imposing the Jacobi identities

$$[[x, e_1, e_2], e_1, e_j] = [[x, e_1, e_j], e_1, e_2] + [x, e_1, [e_2, e_1, e_j]] \text{ for } 3 \leq j \leq m-1,$$

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and

$$[[x, e_1, e_2], e_1, e_m] = [[x, e_1, e_m], e_1, e_2] + [x, e_1, [e_2, e_1, e_m]],$$

and using (3.2), we obtain

$$a^1_{1j} = a^2_{1j} = 0, a^{j+1}_{1j} = a^{j+2}_{1j} = \cdots = a^m_{1j} = 0 \text{ for } 3 \le j \le m-2;$$

$$a^{j-1}_{1j-1} = a^j_{1j} + a^2_{12}, 4 \le j \le m-1;$$

$$a^m_{12} = a^{j-1}_{1j} - a^{j-2}_{1j-1}, a^k_{1j} = a^{k-1}_{1j-1}, k \ne j-1, \text{ for } 4 \le k \le m, 5 \le j \le m-1,$$

and

$$a^2_{12} + a^m_{1m} - a^{m-1}_{1m-1} = 0, a^1_{1m-1} = a^2_{1m-1} = a^m_{1m-1} = 0,$$

$$a^{i+1}_{12} + a^i_{1m} - a^{i-1}_{1m-1} = 0, 4 \le i \le m-2, a^{m-1}_{1m} = a^{m-2}_{1m-1}$$

respectively.

Secondly, imposing the Jacobi identities on $\{[x, e_1, e_2], e_2, e_j\}$ for $3 \le j \le m$, we get

$$a^1_{2j} = a^2_{2j} = 0, a^{j+1}_{2j} = a^{j+2}_{2j} = \cdots = a^m_{2j} = 0 \text{ for } 3 \le j \le m-1;$$

$$a^{j-1}_{2j-1} = a^j_{2j} - a^1_{12}, 4 \le j \le m; a^k_{2j} = a^{k-1}_{2j-1}, \text{ for } 4 \le k < j \le m.$$

From

$$[[x, e_1, e_i], e_2, e_j] = [[x, e_2, e_j], e_1, e_i] + [x, [e_1, e_2, e_j], e_i] = 0, \text{ for } 3 \le i, j \le m-1,$$

$$[[x, e_1, e_4], e_1, e_m] = [[x, e_1, e_m], e_1, e_4],$$

and

$$[[x, e_1, e_i], e_1, e_m] = [[x, e_1, e_m], e_1, e_i] - [x, e_1, e_{i-2}] \text{ for } 5 \le i \le m-1,$$

we get

$$a^k_{ij} = 0, 3 \le i, j \le m-1, 1 \le k \le m, a^2_{1m} = 0,$$

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and

$$a_{1i}^{i} + a_{1m}^{m} - a_{1i-2}^{i-2} = 0, a_{1i}^{k} = a_{1i-2}^{k-2} \text{ for } k \neq i, 5 \leq k \leq m-1,$$

$$a_{1i-2}^{1} = a_{1i-2}^{2} = a_{1i-2}^{m-2} = a_{1i-2}^{m-1} = a_{1i-2}^{m} = 0.$$

Then we have $a_{1m}^{m} = 2a_{12}^{2}, a_{12}^{m} = 0$.

Again from

$$[[x, e_2, e_4], e_1, e_m] = [[x, e_1, e_m], e_2, e_4],$$

$$[[x, e_1, e_m], e_2, e_4] = [[x, e_2, e_4], e_1, e_m] + [x, e_3, e_m],$$

$$[[x, e_2, e_i], e_1, e_m] = [[x, e_1, e_m], e_2, e_i] - [x, e_2, e_{i-2}], 5 \leq i \leq m-1,$$

$$[[x, e_1, e_4], e_2, e_m] = [[x, e_2, e_m], e_1, e_4],$$

and

$$[[x, e_1, e_i], e_2, e_m] = [[x, e_2, e_m], e_1, e_i],$$

we get

$$a_{1m}^{1} = 0, a_{3m}^{k} = 0, 1 \leq k \leq m, a_{2i}^{k} = a_{2i-2}^{k-2} \text{ for } 5 \leq k \leq m-1,$$

$$a_{2i-2}^{1} = a_{2i-2}^{2} = a_{2i-2}^{m-2} = a_{2i-2}^{m-1} = a_{2i-2}^{m} = 0,\ a_{12}^{1} = 0,\ a_{2m}^{2} = 0 \text{ and } a_{2m}^{m} = 0.$$

By

$$[[x, e_2, e_m], e_1, e_i] = [[x, e_1, e_i], e_2, e_m] - [x, e_{i-1}, e_m] + [x, e_2, e_{i-2}], 5 \leq i \leq m-1,$$

$$[[x, e_2, e_4], e_2, e_m] = [[x, e_2, e_m], e_2, e_4],$$

and

$$[[x, e_1, e_m], e_2, e_m] = [[x, e_2, e_m], e_1, e_m] + [x, e_{m-1}, e_m],$$

we obtain

$$a_{i-1m}^{j} = a_{2i-2}^{j}, 1 \leq j \leq m, a_{2m}^{1} = 0, a_{2m}^{i} = a_{m-1m}^{i-2} \text{ for } 5 \leq i \leq m-1,$$

$$a_{m-1m}^{1} = a_{m-1m}^{2} = a_{m-1m}^{m-2} = a_{m-1m}^{m-1} = a_{m-1m}^{m} = 0.$$

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Therefore, we get

$$\operatorname{ad}(x,e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & a_{12}^3 & a_{12}^4 & a_{12}^5 & \cdot & a_{12}^{m-3} & a_{12}^{m-2} & a_{12}^{m-1} & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m}^{m-1} & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m-1}^{m-3} & a_{1m}^{m-1} & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{1m-1}^{m-4} & a_{1m-1}^{m-3} & a_{1m}^{m-1} & \cdot & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a_{1m-1}^4 & a_{1m-1}^5 & a_{1m-1}^6 & \cdot & a_{1m}^{m-1} & r_{m-4} a_{12}^2 & 0 & 0 \\ 0 & 0 & a_{1m-1}^3 & a_{1m-1}^4 & a_{1m-1}^5 & \cdot & a_{1m-1}^{m-3} & a_{1m}^{m-1} & r_{m-3} a_{12}^2 & 0 \\ 0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdot & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & r_{m-2} a_{12}^2 \end{pmatrix},$$

where $r_j = m - j$ for $1 \le j \le m-2$, $a_{1m-1}^{i-1} = a_{12}^{i+1} + a_{1m}^i$ for $4 \le i \le m-2$;

$$\operatorname{ad}(x,e_2)|_N = \begin{pmatrix} 0 & -a_{12}^2 & -a_{12}^3 & -a_{12}^4 & -a_{12}^5 & \cdots & -a_{12}^{m-3} & -a_{12}^{m-2} & -a_{12}^{m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & a_{2m}^{m-1} & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^4 & a_{2m}^5 & a_{2m}^6 & \cdots & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 & 0 \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 \end{pmatrix},$$

$$-\operatorname{ad}(x,e_m)|_N = \begin{pmatrix} 0 & 0 & a_{1m}^3 & a_{1m}^4 & a_{1m}^5 & \cdots & a_{1m}^{m-3} & a_{1m}^{m-2} & a_{1m}^{m-1} & a_{1m}^m \\ 0 & 0 & a_{2m}^3 & a_{2m}^4 & a_{2m}^5 & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & a_{2m}^{m-1} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^6 & a_{2m}^7 & a_{2m}^8 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^5 & a_{2m}^6 & a_{2m}^7 & \cdots & a_{2m}^{m-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we replace x by $x - a_{2m}^{m-1}e_1 + a_{1m}^{m-1}e_2 - a_{12}^3 e_4 - a_{12}^4 e_5 - \cdots - a_{12}^{m-1} e_m$, the above

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maps are reduced to

$$
\mathrm{ad}(x,e_1)|_N = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0\\
0 & a_{12}^2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0\\
0 & 0 & r_1a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0\\
0 & 0 & 0 & r_2a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0\\
0 & 0 & b_{1m}^{m-2} & 0 & r_3a_{12}^2 & \cdot & 0 & 0 & 0 & 0\\
0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdot & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & b_{1m}^{5} & b_{1m}^{6} & b_{1m}^{7} & \cdot & 0 & r_{m-4}a_{12}^2 & 0 & 0\\
0 & 0 & b_{1m}^{4} & b_{1m}^{5} & b_{1m}^{6} & \cdot & b_{1m}^{m-2} & 0 & r_{m-3}a_{12}^2 & 0\\
0 & 0 & b_{1m}^{3} & b_{1m}^{4} & b_{1m}^{5} & \cdot & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & r_{m-2}a_{12}^2
\end{pmatrix},
$$

where $r_j = m - j,$ for $1 \leq j \leq m-2$,

$$
\mathrm{ad}(x,e_2)|_N = \begin{pmatrix}
0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{4} & a_{2m}^{5} & a_{2m}^{6} & \cdots & a_{2m}^{m-2} & 0 & 0 & 0\\
0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0
\end{pmatrix},
$$

$$
-\mathrm{ad}(x,e_m)|_N = \begin{pmatrix}
0 & 0 & b_{1m}^{3} & b_{1m}^{4} & b_{1m}^{5} & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & 2a_{12}^2\\
0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & a_{2m}^{6} & a_{2m}^{7} & a_{2m}^{8} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Again by the Jacobi identities for vectors $\{[x, e_1, e_2], x, e_i\}$ for $3 \leq i \leq m$, we get

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$a_{12}^2 a_{2m}^i = 0$ for $i = 3, 4, \ldots, m-2$. Since $a_{12}^2 \neq 0$ (if $a_{12}^2 = 0$, then A is nilpotent), we get $a_{2m}^i = 0,$ for $i = 3, 4, \ldots, m-2$.

Therefore,

$$\mathrm{ad}(x, e_1)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-2} & 0 & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdot & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & b_{1m}^5 & b_{1m}^6 & b_{1m}^7 & \cdot & 0 & r_{m-4} a_{12}^2 & 0 & 0 \\ 0 & 0 & b_{1m}^4 & b_{1m}^5 & b_{1m}^6 & \cdot & b_{1m}^{m-2} & 0 & r_{m-3} a_{12}^2 & 0 \\ 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdot & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & r_{m-2} a_{12}^2 \end{pmatrix},$$

where $r_j = m - j$ for $1 \leq j \leq m-2$,

$$\mathrm{ad}(x, e_2)|_N = \begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-\mathrm{ad}(x, e_m)|_N = \begin{pmatrix} 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & 2a_{12}^2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix};$$

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that is

$$[x, e_1, e_2] = a_{12}^2 e_2, [x, e_1, e_3] = (m-1)a_{12}^2 e_3, [x, e_1, e_4] = (m-2)a_{12}^2 e_4,$$

$$[x, e_1, e_k] = \sum_{j=3}^{k-2} b_{2m}^{m-k+j} e_j + (m-k+2)a_{12}^2 e_k, \text{ for } k = 5, 6, \ldots, m,$$

and other brackets of the basis vectors are equal to zero.

For any l satisfying $3 \le l \le m-2$, we take a series of linear transformations defined by

$$\tilde{e}_k = e_k \text{ for } 1 \le k \le l+1 \text{ and } \tilde{e}_k = e_k - \frac{b_{1m}^{m-l+1}}{(l-1)a_{12}^2} e_{k-l+1} \text{ for } l+2 \le k \le m.$$

Then the basis vectors $\tilde{e}_1, \ldots, \tilde{e}_m$ satisfy (3.1). After replacing x by $\frac{x}{a_{12}^2}$, we get the structural matrix M of A with respect to the basis vectors $x, \tilde{e}_1, \ldots, \tilde{e}_m$ as follows

$$M = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & m-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m-2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & m-3 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Therefore, the multiplication table of A is

$$\begin{cases}
[e_1, e_2, e_j] = e_{j-1} & \text{for } 4 \le j \le m, \\
[e_1, e_j, e_m] = e_{j-2} & \text{for } 5 \le j \le m-1, \\
[x, e_1, e_2] = e_2, & \\
[x, e_1, e_k] = (m-k+2)e_k & \text{for } 3 \le k \le m. \qquad \square
\end{cases}$$

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THEOREM 3.3. *Let A be a solvable $(m+k)$-dimensional 3-Lie algebra with the maximal hypo-nilpotent ideal N. Then we have $k = 1$.*

Proof. If $k \geq 2$, let $x_1, \ldots, x_k, e_1, \ldots e_m$ be a basis of A. Thanks to the solvability of A, we have $[A, A, A] \subseteq N$. By the discussions of the proof of Theorem 3.2, we might as well suppose

$$\mathrm{ad}(x_1, e_1)|_N = \mathrm{diag}(0, 1, m-1, m-2, \ldots, 4, 3, 2),$$

$$\mathrm{ad}(x_1, e_2)|_N = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}(x_1, e_m)|_N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}(x_2, e_1)|_N$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a_{12}^2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 a_{12}^2 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_2 a_{12}^2 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-2} & 0 & r_3 a_{12}^2 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & \cdot & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & b_{1m}^5 & b_{1m}^6 & b_{1m}^7 & \cdot & 0 & r_{m-4} a_{12}^2 & 0 & 0 \\ 0 & 0 & b_{1m}^4 & b_{1m}^5 & b_{1m}^6 & \cdot & b_{1m}^{m-2} & 0 & r_{m-3} a_{12}^2 & 0 \\ 0 & 0 & b_{1m}^3 & b_{1m}^4 & b_{1m}^5 & \cdot & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & r_{m-2} a_{12}^2 \end{pmatrix},$$

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where $r_j = m - j$ for $1 \leq j \leq m-2$

$$\mathrm{ad}(x_2, e_2)|_N = \begin{pmatrix} 0 & -a_{12}^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{4} & a_{2m}^{5} & a_{2m}^{6} & \cdots & a_{2m}^{m-2} & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \end{pmatrix},$$

$$-\mathrm{ad}(x_2, e_m)|_N = \begin{pmatrix} 0 & 0 & b_{1m}^{3} & b_{1m}^{4} & b_{1m}^{5} & \cdots & b_{1m}^{m-3} & b_{1m}^{m-2} & 0 & 2a_{12}^2 \\ 0 & 0 & a_{2m}^{3} & a_{2m}^{4} & a_{2m}^{5} & \cdots & a_{2m}^{m-3} & a_{2m}^{m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{m-2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2m}^{6} & a_{2m}^{7} & a_{2m}^{8} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{2m}^{5} & a_{2m}^{6} & a_{2m}^{7} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\mathrm{ad}(x_2, e_1)|_N - a_{12}^2 \mathrm{ad}(x_1, e_1)|_N$, $\mathrm{ad}(x_2, e_2)|_N - a_{12}^2 \mathrm{ad}(x_1, e_2)|_N$ and

$$\mathrm{ad}(x_2, e_m)|_N - a_{12}^2 \mathrm{ad}(x_1, e_m)|_N$$

are nilpotent. It follows that $I = F(x_2 - a_{12}^2 x_1) + N$ is an $(m+1)$-dimensional hypo-nilpotent ideal of A. This is a contradiction. Therefore, we have $k = 1$. ☐

Corollary 3.4. *There are no $(m+k)$-dimensional solvable 3-Lie algebras with a maximal hypo-nilpotent ideal N when $k \geq 2$.*

4. 3-Lie algebras with nilradical N.

In this section we study the solvable 3-Lie algebras with the nilradical N.

Theorem 4.1. *There are no solvable non-nilpotent 3-Lie algebras with nilradical N.*

Proof. First let A be an $(m+k)$-dimensional 3-Lie algebra with the nilpotent ideal N, where $1 \leq k \leq 2$. We will prove that A is nilpotent.

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If $k = 1$, suppose $x, e_1, \dots, e_m$ is a basis of A. Then the associative algebra A^* is generated by left multiplications $\mathrm{ad}(x, e_i)$ and $\mathrm{ad}(e_i, e_j)$, where $1 \le i, j \le m$. Therefore, we have $A^* = K(N, A)$. It follows from Lemma 2.1 that A is nilpotent.

If $k = 2$, let $x_1, x_2, e_1, \dots, e_m$ be a basis of A. Set $B = Fx_1 + Fe_1 + \cdots + Fe_m$ and $C = Fx_2 + Fe_1 + \cdots + Fe_m$. Then B and C are $(m+1)$-dimensional subalgebras of A with the nilpotent ideal N. It follows from the result of the case $k = 1$, and Theorem 3.2 that the matrices of $\mathrm{ad}(x_i, e_j)|_N$ $(i = 1, 2, 1 \le j \le m)$ with respect to $e_1, \dots, e_m$ are of the form

$$S = \begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 & \cdots & a_{m-3} & a_{m-2} & 0 & 0 \\ 0 & 0 & b_3 & b_4 & b_5 & \cdots & b_{m-3} & b_{m-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_4 & c_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & c_{m-4} & c_{m-5} & c_{m-6} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{m-3} & c_{m-4} & c_{m-5} & \cdots & c_3 & 0 & 0 & 0 \\ 0 & 0 & c_{m-2} & c_{m-3} & c_{m-4} & \cdots & c_4 & c_3 & 0 & 0 \end{pmatrix},$$

where $a_l, b_l, c_l \in F, 3 \le l \le m-2$. Therefore, $\mathrm{ad}(x_i, e_j)$ are nilpotent maps of A for $i = 1, 2; j = 1, \dots, m$. Now suppose

$$[x_1, x_2, e_i] = \sum_{j=1}^{m} r_{ij} e_j, 1 \le i \le m.$$

With the help of the Jacobi identities for $\{[x_1, x_2, e_i], e_1, e_2\}$, $\{[x_1, x_2, e_i], e_1, e_4\}$, $i = 1, 2, \dots, m$; $\{[x_1, x_2, e_1], e_2, e_i\}$ for $4 \le i \le m$, we get that $\mathrm{ad}(x_1, x_2)|_N$ has the form

$$\begin{pmatrix} 0 & 0 & r_{13} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{23} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{m3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathrm{ad}(x_1, x_2)|_N$ is nilpotent, and $\mathrm{ad}(x_1, x_2)$ is also nilpotent to A. This proves that A is nilpotent when $k = 2$.

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Last we suppose that there is a solvable non-nilpotent $(m+k)$-dimensional 3-Lie algebra with the nilradical N for $k \geq 3$. Let $x_1, \ldots, x_k, e_1, \ldots, e_m$ be a basis of A. Then there exist x_i, x_j such that $\mathrm{ad}(x_i, x_j)|_N$ is not nilpotent. Set $T = Fx_i + Fx_j + Fe_1 + \cdots + Fe_m$, then N is a nilpotent ideal of $(m+2)$-dimensional subalgebra T. From the above discussions, T is a nilpotent subalgebra. Hence there exists an integer r such that $ad^r(x_i, x_j)(T) = 0$. Since A is solvable and N is the nilradical of A, we have $[A, \cdots, A] \subseteq N$. Therefore,

$$\mathrm{ad}^{r+1}(x_i, x_j)(A) \subseteq \mathrm{ad}^r(x_i, x_j)(N) \subseteq \mathrm{ad}^r(x_i, x_j)(T) = 0.$$

This is a contradiction. □

REMARK 4.2. *The solvable condition in Theorem 4.1 is necessary. See the following example. Let A be an $(m+4)$-dimensional 3-Lie algebra with the basis x_1, x_2, x_3, x_4, e_1, $\ldots$, e_m, and the multiplication table*

$$\begin{cases} [x_1, x_2, x_4] = x_3, \\ [x_1, x_3, x_4] = x_2, \\ [x_2, x_3, x_4] = x_1, \\ [x_4, e_1, e_2] = e_3, \\ [e_1, e_2, e_j] = e_{j-1} \text{ for } 4 \leq j \leq m, \\ [e_1, e_j, e_m] = e_{j-2} \text{ for } 5 \leq j \leq m-1. \end{cases}$$

By a direct computation we get that N is the nilradical of A, and

$$A^{(1)} = Fx_1 + Fx_2 + Fx_3 + Fe_3 + \cdots + Fe_{m-1},$$

$$A^{(s)} = Fx_1 + Fx_2 + Fx_3 \neq 0, s > 1.$$

It follows that A is an unsolvable 3-Lie algebra.

5. One dimensional extension of Lie algebras. In this section we describe the one dimensional extension of Lie algebras, first introduced in [6]. As an application of it we get all classes of solvable Lie algebras with the special nilradical given in [19, 20].

For any given s-dimensional Lie algebra g with a basis $y_1, \ldots, y_s$ and the multiplication table

$$[y_i, y_j] = \sum_{k=1}^{s} a_{ij}^k y_k, 1 \leq i, j \leq s$$

where a_{ij}^k are structure constants, we can define a corresponding 3-Lie algebra as follows. Let $y_0, y_1, \ldots, y_s$ be the basis of the $(s+1)$-dimensional vector space L_g. The

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3-ary multiplication table of L_g is defined by

$$\begin{cases} [y_0, y_i, y_j] = \sum_{k=1}^{s} a_{ij}^k y_k, 1 \leq i, j \leq s, \\ [y_t, y_i, y_j] = 0, 1 \leq t, i, j \leq s. \end{cases}$$

It is not difficult to check that L_g is a 3-Lie algebra. L_g is called the one dimensional extension of the Lie algebra g. Then we have following results.

THEOREM 5.1. *Let I be a subalgebra of g, then I is an ideal of Lie algebra g if and only if I is an ideal of 3-Lie algebra L_g, and I is a solvable (nilpotent) ideal of g if and only if I is a solvable (nilpotent) ideal of L_g.*

Proof. Since $[L_g, L_g, I] = [y_0, g, I] = [g, I] \subseteq I$, we get the first result. Denote the derived series (descending central series) of I in 3-Lie algebra L_g by $I_g^{(s)}$ (I_g^s), that is $I_g^{(s+1)} = [I_g^{(s)}, I_g^{(s)}, L_g]$, ($I_g^{s+1} = [I_g^s, I, L_g]$) for $s \geq 0$, $I_g^{(0)} = I = I^{(0)}$, ($I_g^0 = I = I^0$). By induction on s we get

$$I^{(s+1)} = [I^{(s)}, I^{(s)}] = [y_0, I^{(s)}, I^{(s)}] = [L_g, I_g^{(s)}, I_g^{(s)}] = I_g^{(s+1)}, s \geq 0,$$

$$I^{s+1} = [I^s, I] = [y_0, I^s, I] = [L_g, I_g^s, I] = I_g^{s+1}, s \geq 0.$$

It follows that $I^{(s+1)} = 0$ if and only if $I_g^{(s+1)} = 0$, and $I^{s+1} = 0$ if and only if $I_g^{s+1} = 0$. ◻

THEOREM 5.2. *Let I be an ideal of Lie algebra g. Then $J = I + Fy_0$ is an ideal of 3-Lie algebra L_g, and I is a solvable ideal of g if and only if J is a solvable ideal of L_g.*

Proof. It is evident that J is an ideal of L_g if I is an ideal of L. Since

$$J^{(1)} = [J, J, L_g] \subseteq I, J^{(2)} = [J^{(1)}, J^{(1)}, L_g] \subseteq [I, I, L_g] = [I, I] = I^{(1)},$$

by induction on s, we get

$$J^{(s+1)} = [J^{(s)}, J^{(s)}, L_g] \subseteq [I^{(s-1)}, I^{(s-1)}] = I^{(s)}.$$

Conversely,

$$I^{(1)} = [I, I] = [I, I, y_0] \subseteq [J, J, L_g] = J^{(1)},$$

by induction on s, we get

$$I^{(s+1)} = [I^{(s)}, I^{(s)}] = [I^{(s)}, I^{(s)}, y_0] \subseteq [J^{(s)}, J^{(s)}, L_g] = J^{(s+1)}.$$

Therefore, I is a solvable ideal of L if and only if J is a solvable ideal of L_g. ◻

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REMARK 5.3. *If I is a nilpotent ideal of g, then $J = I + Fy_0$ is a nilpotent subalgebra of L_g, but J may not be a nilpotent ideal of L_g. See the following example.*

EXAMPLE 5.4. *Suppose $g = Fx + Fe_1 + Fe_2 + Fe_3 + Fe_4$ is a 5-dimensional Lie algebra and the multiplication table of g in the basis x, e_1, e_2, e_3, e_4 is*

$$\begin{cases} [e_1, e_3] = e_2, \\ [e_1, e_4] = e_3, \\ [x, e_1] = e_1, \\ [x, e_2] = 2e_2, \\ [x, e_3] = e_3. \end{cases}$$

$I = Fe_1 + Fe_2 + Fe_3 + Fe_4$ is a nilpotent ideal of g since $I^3 = 0$. Let $L_g = Fy_0 + g$ is the one dimensional extension of g. Then the multiplication table of L_g in the basis $y_0, x, e_1, e_2, e_3, e_4$ is as follows

$$\begin{cases} [y_0, e_1, e_3] = e_2, \\ [y_0, e_1, e_4] = e_3, \\ [y_0, x, e_1] = e_1, \\ [y_0, x, e_2] = 2e_2, \\ [y_0, x, e_3] = e_3. \end{cases}$$

Then $J = Fy_0 + I = Fy_0 + Fe_1 + Fe_2 + Fe_3 + Fe_4$ is an ideal of L_g. Since $J^s = J^1 = Fe_1 + Fe_2 + Fe_3 \neq 0$ for $s \geq 1$, J is not a nilpotent ideal of L_g.

Suppose $N_1 = N_0 = Fe_1 + \cdots + Fe_m$ (as vector spaces), with $m \geq 4$, the multiplication table of Lie algebra N_0 in the basis $e_1, \ldots, e_m$ is as follows

$$[e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m,$$

and the multiplication table of Lie algebra N_1 in the basis $e_1, \ldots, e_m$ is

$$\begin{cases} [e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m, \\ [e_j, e_m] = e_{j-2} \text{ for } 4 \leq j \leq m-1. \end{cases}$$

In [19, 20], authors constructed all solvable Lie algebras with the nilradical N_0 and N_1 respectively. By Theorem 3.1, and Theorem 4.1 and Theorem 4.2 in [18], we have

(1). Let A be an $(m+k)$-dimensional solvable Lie algebra with the nilradical N_1 $(k \geq 1)$. Then we have $k = 1$, and up to isomorphism the following is the only possibility:

$$\begin{cases} [e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m, \\ [e_j, e_m] = e_{j-2} \text{ for } 4 \leq j \leq m-1, \\ [x, e_1] = e_1, \\ [x, e_k] = (m-k+2)e_k, \text{ for } 2 \leq k \leq m. \end{cases}$$

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(2). Let A be an $(m+k)$-dimensional solvable Lie algebra with the nilradical N_0 $(k \geq 1, m \geq 4)$. Then we have $k=1$ or $k=2$. And in the case of $k=1$, up to isomorphisms one and only one of the following possibilities holds:

$$(M_1). \begin{cases} [e_1, e_j] = e_{j-1}, \\ [x, e_2] = e_2, \\ [x, e_3] = e_3, \\ [x, e_r] = \sum_{k=2}^{r-2} b_{r-k+1} e_k + e_r; \end{cases} \qquad (M_2) \begin{cases} [e_1, e_j] = e_{j-1}, \\ [x, e_1] = e_1, \\ [x, e_t] = (m-t)e_t; \end{cases}$$

$$(M_3) \begin{cases} [e_1, e_j] = e_{j-1}, \\ [x, e_1] = e_1, \\ [x, e_t] = (m-t+\alpha)e_t; \end{cases} \qquad (M_4) \begin{cases} [e_1, e_j] = e_{j-1}, \\ [x, e_1] = e_1 - e_m, \\ [x, e_t] = (m-t+1)e_t; \end{cases}$$

where $3 \leq j \leq m$, $2 \leq t \leq m$, $4 \leq r \leq m$, $b_{ij}, \alpha \in F$, and $\alpha \neq 0$.

In the case of $k=2$, up to isomorphism the only possibility is the following:

$$\begin{cases} [e_1, e_j] = e_{j-1} \text{ for } 3 \leq j \leq m, \\ [x_1, e_2] = e_2, \\ [x_1, e_i] = (m-i)e_i \text{ for } 2 \leq i \leq m, \\ [x_2, e_i] = e_i \text{ for } 2 \leq i \leq m. \end{cases}$$

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