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NEUTRAL SUBSPACES OF PAIRS OF SYMMETRIC/SKEWSYMMETRIC REAL MATRICES*

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Abstract. Let A and B be $n \times n$ real matrices with A symmetric and B skewsymmetric. Obviously, every simultaneously neutral subspace for the pair (A, B) is neutral for each Hermitian matrix X of the form $X = \mu A + i\lambda B$, where μ and λ are arbitrary real numbers. It is well-known that the dimension of each neutral subspace of X is at most $\mathrm{In}_+(X) + \mathrm{In}_0(X)$, and similarly, the dimension of each neutral subspace of X is at most $\mathrm{In}_-(X) + \mathrm{In}_0(X)$. These simple observations yield that the maximal possible dimension of an (A, B)-neutral subspace is no larger than

$$\min\{\min\{\mathrm{In}_+(\mu A + i\lambda B) + \mathrm{In}_0(\mu A + i\lambda B), \mathrm{In}_-(\mu A + i\lambda B) + \mathrm{In}_0(\mu A + i\lambda B)\}\},$$

where the outer minimum is taken over all pairs of real numbers (λ, μ). In this paper, it is proven that the maximal possible dimension of an (A, B)-neutral subspace actually coincides with the above expression.

Key words. Symmetric matrix, Skewsymmetric matrix, Hermitian matrix, Inertia, Neutral subspace.

AMS subject classifications. 15A21, 15A22, 15B57.

1. Introduction and main result.

Let F be the field of real numbers R, or the field of complex numbers C. Denote by $\mathsf{F}^{m\times n}$ the set of $m \times n$ matrices with entries in F, and let (x, y) be the standard inner product in F^n (short for $\mathsf{F}^{n\times 1}$).

Let $A, B \in \mathsf{R}^{n\times n}$, where A is symmetric and B is skewsymmetric. A subspace $\mathcal{M} \subseteq \mathsf{R}^n$ is called *simultaneously neutral* for A and B, or (A, B)-*neutral*, if

$$(Ax, y) = 0, \quad (Bx, y) = 0 \quad \text{for all } x, y \in \mathcal{M}.$$

Simultaneously neutral subspaces for a pair of real symmetric/skewsymmetric matrices, as well as those for a pair of complex hermitian matrices, play a key role in the theory of algebraic Riccati equations (see e.g. [7] and references therein), and in symmetric factorizations of matrix polynomials and rational matrix functions with

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certain symmetries [2, 3, 5, 6, 10]; in the latter application, the (A, B)-neutral subspaces have the additional property that they are $B^{-1}A$-invariant (and B is assumed to be invertible). In this paper, we focus on the following problem: *Find the maximal possible dimension of* (A, B)*-neutral subspaces for symmetric/skewsymmetric pairs of real matrices.* We will describe this dimension in terms of inertia of complex hermitian matrices; we denote by

$$\mathrm{In}(A) = (\mathrm{In}_+(A), \mathrm{In}_-(A), \mathrm{In}_0(A))$$

the inertia of a hermitian matrix $A \in \mathsf{C}^{n\times n}$. Thus, $\mathrm{In}_+(A)$, $\mathrm{In}_-(A)$, and $\mathrm{In}_0(A)$ stand for the number of positive, negative, and zero eigenvalues of A, respectively, counted with multiplicities.

The following observation will be useful:

LEMMA 1.1. *Let* $A, B \in \mathsf{R}^{n\times n}$, $A = A^T$, $B = -B^T$. *Then* $A + iB$ *and* $A - iB$ *are similar, and in particular*

$$\mathrm{In}\,(A + iB) = \mathrm{In}\,(A - iB). \tag{1.1}$$

Proof. Observe that $x + iy \in \mathsf{C}^n$, where $x, y \in \mathsf{R}^n$, is an eigenvector of $A + iB$ corresponding to the eigenvalue $t \in \mathsf{R}$ if and only if $y + ix$ is an eigenvector of $A - iB$ corresponding to the same eigenvalue t. Clearly, the set of vectors $x_1 + iy_1, \dots, x_p + iy_p$ is linearly independent if and only if the set $y_1 + ix_1, \dots, y_p + ix_p$ is linearly independent. Hence, $A + iB$ and $A - iB$ have the same eigenvalues with the same multiplicities. ∎

We now state our main result:

THEOREM 1.2. *Let* A *be symmetric,* B *skewsymmetric,* $A, B \in \mathsf{R}^{n\times n}$. *Then the maximal dimension of an* (A, B)*-neutral subspace* $\mathcal{M} \subseteq \mathsf{R}^n$ *coincides with*

$$\min\{\min\{\mathrm{In}_+(\mu A + i\lambda B) + \mathrm{In}_0(\mu A + i\lambda B), \mathrm{In}_-(\mu A + i\lambda B) + \mathrm{In}_0(\mu A + i\lambda B)\}\}, \tag{1.2}$$

where the outer minimum is taken over all pairs of real numbers (λ, μ).

Thus, the maximal dimension of an (A, B)-neutral subspace is described in terms of inertia of suitable combinations of A and B. Analogues of Theorem 1.2 in the context of pairs of complex or quaternionic hermitian matrices A and B, where $\mu A + i\lambda B$ of Theorem 1.2 is replaced by $\mu A + \lambda B$, have been obtained in [9, 11]. We mention in passing that an analogue of Theorem 1.2 for pairs of real symmetric matrices fails, see [11] for more details.

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Remark 1.3.

(1) Note that the inner minimum in (1.2) is attained at some nonzero (λ_0, μ_0); indeed, for $\lambda = \mu = 0$, (1.2) takes value n. Since

$$\mathrm{In}_+(tX) + \mathrm{In}_0(tX) = \mathrm{In}_+(X) + \mathrm{In}_0(X), \quad X \in \mathsf{C}^{n\times n}, \quad X = X^*, \quad t > 0,$$

and

$$\mathrm{In}_+(X) + \mathrm{In}_0(X) = \mathrm{In}_-(-X) + \mathrm{In}_0(-X), \quad X \in \mathsf{C}^{n\times n}, \quad X = X^*,$$

we have that (1.2) is equal to

$$\min_{0\le\alpha<2\pi}\{\mathrm{In}_+((\cos\alpha)A + i(\sin\alpha)B) + \mathrm{In}_0((\cos\alpha)A + i(\sin\alpha)B)\}. \tag{1.3}$$

(2) Note that (1.2) is also equal to

$$\min\left\{\begin{array}{l}\min_{t\in\mathsf{R}}\{\mathrm{In}_+(A + itB) + \mathrm{In}_0(A + itB)\} \\ \min_{t\in\mathsf{R}}\{\mathrm{In}_+(-A + itB) + \mathrm{In}_0(-A + itB)\}\end{array}\right\}; \tag{1.4}$$

as well as to the formula analogous to (1.4) with the roles of A and B interchanged. To verify that, one needs to observe that by the continuity of the spectrum there exists a real $M > 0$ such that

$$\begin{aligned}\mathrm{In}_+(iB) + \mathrm{In}_0(iB) &\ge \mathrm{In}_+\left(\frac{1}{t}A + iB\right) + \mathrm{In}_0\left(\frac{1}{t}A + iB\right) \\ &= \mathrm{In}_+(A + itB) + \mathrm{In}_0(A + itB)\end{aligned}$$

for all real numbers $t > M$.

(3) It follows from (1.1) that (1.3) is actually equal to

$$\min_{0\le\alpha\le\pi}\{\mathrm{In}_+((\cos\alpha)A + i(\sin\alpha)B) + \mathrm{In}_0((\cos\alpha)A + i(\sin\alpha)B)\}.$$

The rest of the paper is devoted to the proof of Theorem 1.2. Preliminary results, including the canonical form for pairs of real symmetric/skewsymmetric matrices, are stated and sometimes proved in Sections 2 - 4. The proof of Theorem 1.2 itself is given in Sections 5 and 6.

We fix some notation. By $e_1, \ldots, e_n$ we denote the elements of the standard basis of F^n, and by $\mathrm{span}\,(x_1, \ldots, x_p)$ the linear span of vectors $x_1, \ldots, x_p$. The symbol $\#\mathcal{G}$ stands for the cardinality of the set $\mathcal{G}$. We denote by I_k and 0_k the $k \times k$ identity and zero matrices, respectively.

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2. Preliminaries on inertia of Hermitian matrices. If $X \in \mathsf{C}^{n\times n}$ is Hermitian, a subspace $\mathcal{M} \subseteq \mathsf{C}^n$ is said to be X-*neutral* if $(Xx, y) = 0$ for all $x, y \in \mathcal{M}$, or equivalently $(Xx, x) = 0$ for all $x \in \mathcal{M}$.

Proposition 2.1. *Let $X \in \mathsf{C}^{n\times n}$ be hermitian. Then an X-neutral subspace $\mathcal{M} \subseteq \mathsf{C}^n$ is maximal, in the sense that no subspace properly containing $\mathcal{M}$ is X-neutral, if and only if*

$$\dim(\mathcal{M}) = \min\{\mathrm{In}_+(X) + \mathrm{In}_0(X),\ \mathrm{In}_-(X) + \mathrm{In}_0(X)\}.$$

Proposition 2.1 is standard; see for example [4, Section 2.3], where it is proved under the additional assumption that X is invertible.

Lemma 2.2. *Let X be Hermitian matrix which is block partitioned as follows:*

$$X = \begin{bmatrix} 0_k & 0 & X_1 \\ 0 & X_0 & X_2 \\ X_1^* & X_2^* & X_3 \end{bmatrix}, \quad \text{or} \quad X = \begin{bmatrix} X_3 & X_2 & X_1 \\ X_2^* & X_0 & 0 \\ X_1^* & 0 & 0_k \end{bmatrix}, \tag{2.1}$$

where the block X_1 is $k \times k$ and invertible. Then

$$\mathrm{In}_0(X) = \mathrm{In}_0(X_0), \quad \mathrm{In}_\pm(X) = k + \mathrm{In}_\pm(X_0). \tag{2.2}$$

Proof. Say X is given by the first formula in (2.1). Replacing X with SXS^*, where

$$S = \begin{bmatrix} I_k & 0 & 0 \\ -X_2X_1^{-1} & I & 0 \\ -\frac{1}{2}X_3X_1^{-1} & 0 & I \end{bmatrix},$$

we may assume $X_2 = 0$, $X_3 = 0$. It is easy to see that

$$\mathrm{In}_\pm \begin{bmatrix} 0 & X_1 \\ X_1^* & 0 \end{bmatrix} = k.$$

Now (2.2) is obvious. ∎

3. Properties of $\Phi_\alpha(A, B)$. In this section, we let $A, B \in \mathsf{R}^{n\times n}$, where $A = A^T$, $B = -B^T$.

For convenience, denote

$$\Phi_\alpha(A, B) := \mathrm{In}_+((\cos\alpha)A + i(\sin\alpha)B) + \mathrm{In}_0((\cos\alpha)A + i(\sin\alpha)B), \quad 0 \le \alpha < 2\pi.$$

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We list some elementary properties of the quantity $\Phi_\alpha(A,B)$.

LEMMA 3.1. (a) *If* Q *is any finite subset of* $[0,2\pi)$*, then*

$$\min_{0\leq\alpha<2\pi}(\Phi_\alpha(A,B)) = \min_{0\leq\alpha<2\pi,\ \alpha\notin Q}(\Phi_\alpha(A,B)).$$

(b) *Assume*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

If

$$\Phi_\alpha(A_1,B_1) = \Phi_{\alpha'}(A_1,B_1) \tag{3.1}$$

for all $\alpha,\alpha'\in[0,2\pi)\setminus Q$*, where* Q *is a finite (or empty) set, then*

$$\min_{0\leq\alpha<2\pi}(\Phi_\alpha(A,B)) = \min_{0\leq\alpha<2\pi}(\Phi_\alpha(A_1,B_1)) + \min_{0\leq\alpha<2\pi}(\Phi_\alpha(A_2,B_2)). \tag{3.2}$$

Note that (3.2) is generally not valid without additional hypotheses on A_j and B_j (such as (3.1)).

Proof. Proof of (a). Let $\alpha_0\in[0,2\pi)$ be such that

$$\min_{0\leq\alpha<2\pi}(\Phi_\alpha(A,B)) = \Phi_{\alpha_0}(A,B). \tag{3.3}$$

Continuity of eigenvalues of a Hermitian matrix X (as functions of the entries of X; it is assumed that the eigenvalues are arranged in the nondecreasing order) implies that

$$\begin{aligned} &\mathrm{In}_+((\cos\alpha_0)A+i(\sin\alpha_0)B)+\mathrm{In}_0((\cos\alpha_0)A+i(\sin\alpha_0)B)\geq \\ &\qquad \mathrm{In}_+((\cos\beta)A+i(\sin\beta)B)+\mathrm{In}_0((\cos\beta)A+i(\sin\beta)B) \end{aligned} \tag{3.4}$$

for all values of $\beta\in[0,2\pi)$ sufficiently close to α_0. However, (3.3) implies that the strict inequality is impossible in (3.4). Thus,

$$\mathrm{In}_+((\cos\beta)A+i(\sin\beta)B)+\mathrm{In}_0((\cos\beta)A+i(\sin\beta)B) = \min_{0\leq\alpha<2\pi}(\Phi_\alpha(A,B))$$

for all β sufficiently close to α_0. We see that the minimum $\min_{0\leq\alpha<2\pi}(\Phi_\alpha(A,B))$ is attained on a set that contains a nondegenerate interval. The statement (a) is now clear.

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Proof of (b). We obviously have

$$\Phi_\alpha(A,B) = \Phi_\alpha(A_1,B_1) + \Phi_\alpha(A_2,B_2), \quad \forall \alpha \in [0,2\pi).$$

So (the first equality follows from part (a)):

$$\min_{0\le\alpha<2\pi} \Phi_\alpha(A,B) = \min_{0\le\alpha<2\pi,\ \alpha\notin Q} \Phi_\alpha(A,B) = \min_{0\le\alpha<2\pi,\ \alpha\notin Q} (\Phi_\alpha(A_1,B_1) + \Phi_\alpha(A_2,B_2))$$

which by (3.1) is equal to

$$\Phi_{\alpha'}(A_1,B_1) + \min_{0\le\alpha<2\pi,\ \alpha\notin Q} \Phi_\alpha(A_2,B_2),$$

where $\alpha' \in [0,2\pi) \setminus Q$ is fixed. By part (a) we have

$$\Phi_{\alpha'}(A_1,B_1) = \min_{\alpha\in[0,2\pi)} \Phi_\alpha(A_1,B_1),$$

$$\min_{0\le\alpha<2\pi,\ \alpha\notin Q} \Phi_\alpha(A_2,B_2) = \min_{0\le\alpha<2\pi} \Phi_\alpha(A_2,B_2),$$

and we are done. □

Remark 3.2. The result of Lemma 3.1 (with essentially the same proof) remains valid if the interval $[0,2\pi)$ is replaced by any nondegenerate subinterval, with or without one of both endpoints, of $[0,2\pi)$.

Lemma 3.3. *Assume that A and B have the following block form*

$$A = \begin{bmatrix} 0_k & 0 & A_1 \\ 0 & A_0 & A_2 \\ A_1^T & A_2^T & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0_k & 0 & B_1 \\ 0 & B_0 & B_2 \\ -B_1^T & -B_2^T & B_3 \end{bmatrix},$$

where the blocks A_1 and B_1 are $k\times k$. Assume furthermore that $(\cos\alpha)A_1 + i(\sin\alpha)B_1$ is invertible for all but finitely many values $\alpha \in [0,2\pi)$. Then

$$\min_{0\le\alpha<2\pi} \Phi_\alpha(A,B) = k + \min_{0\le\alpha<2\pi} \Phi_\alpha(A_0,B_0). \tag{3.5}$$

Proof. Let

$$Q = \{\alpha \in [0,2\pi) \,:\, (\cos\alpha)A_1 + i(\sin\alpha)B_1 \ \text{ is not invertible}\}.$$

By Lemma 3.1(a) we may replace the interval $[0,2\pi)$ with $[0,2\pi)\setminus Q$ in (3.5). By Lemma 2.2, $\Phi_\alpha(A,B) = k + \Phi_\alpha(A_0,B_0)$ for $\alpha \in [0,2\pi)\setminus Q$, and (3.5) follows. □

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4. Canonical form. We present here the known canonical form of real symmetric/skewsymmetric matrix pencils

$$A + \lambda B, \qquad A, B \in \mathsf{R}^{n\times n}, \quad A = A^T, \quad -B = B^T$$

under R-*congruence*:

$$A + \lambda B \quad \mapsto S^T AS + \lambda S^T BS, \qquad S \in \mathsf{R}^{n\times n} \ \text{ is invertible.}$$

(See e.g. [8] and references there.) The following notation will be used:

$$\Xi_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

F_q is the $q \times q$ real symmetric matrix with 1's in positions $(1, q), (2, q - 1), \ldots, (q, 1)$ and zeros elsewhere;

$$G_q = \begin{bmatrix} F_{q-1} & 0_{(q-1)\times 1} \\ 0_{1\times(q-1)} & 0_1 \end{bmatrix},$$

a $q \times q$ real symmetric matrix, and we take $G_1 = 0$; we denote by $J_{2m}(a \pm ib)$, where a and b are real and $b > 0$, the $2m \times 2m$ almost upper triangular real Jordan block of size $2m \times 2m$ having eigenvalues $a \pm ib$.

It will be convenient to list the elementary blocks first:

(sss0)

a square size zero matrix.

(sss1)

$$G_{2\varepsilon+1} + \lambda \begin{bmatrix} 0 & 0 & F_\varepsilon \\ 0 & 0_1 & 0 \\ -F_\varepsilon & 0 & 0 \end{bmatrix}.$$

(sss2)

$$F_k + \lambda \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{\frac{k-1}{2}} \\ 0 & -F_{\frac{k-1}{2}} & 0 \end{bmatrix}, \qquad k \ \text{ odd.}$$

(sss3)

$$F_k + \lambda \begin{bmatrix} 0_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_{\frac{k-2}{2}} \\ 0 & 0 & 0_1 & 0 \\ 0 & -F_{\frac{k-2}{2}} & 0 & 0 \end{bmatrix}, \qquad k \ \text{ even.}$$

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(sss4)

$$G_\ell + \lambda \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \qquad \ell \ \text{ even.}$$

(sss5)

$$\begin{bmatrix} 0 & G_{\ell/2} \\ G_{\ell/2} & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \qquad \ell \ \text{ even and } \ \ell/2 \ \text{ odd.}$$

(sss6)

$$\begin{bmatrix} 0 & \gamma F_{\ell/2} + G_{\ell/2} \\ \gamma F_{\ell/2} + G_{\ell/2} & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & F_{\ell/2} \\ -F_{\ell/2} & 0 \end{bmatrix}, \quad \ell \ \text{even}, \ \gamma \in \mathsf{R}\backslash\{0\}.$$

(sss7)

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \nu\Xi_2^{m+1} \\ 0 & 0 & \cdots & 0 & -\nu\Xi_2^{m+1} & -I_2 \\ 0 & 0 & \cdots & \nu\Xi_2^{m+1} & -I_2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ (-1)^{m-1}\nu\Xi_2^{m+1} & -I_2 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$+\lambda \begin{bmatrix} 0 & 0 & \cdots & 0 & \Xi_2^m \\ 0 & 0 & \cdots & -\Xi_2^m & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & (-1)^{m-2}\Xi_2^m & \cdots & 0 & 0 \\ (-1)^{m-1}\Xi_2^m & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad \nu > 0.$$

The pencil in (sss7) is $2m \times 2m$, where m is a positive integer. We denote the pencil in (sss7) by

$$\Omega_{2m}(\nu) + \lambda\widetilde{\Omega}_{2m}.$$

Note that the matrices $\Omega_{2m}(\nu)$ and $\widetilde{\Omega}_{2m}$ are symmetric and skewsymmetric, respectively, for every m (and every real ν).

(sss8)

$$\begin{bmatrix} 0 & J_{2m}(a \pm ib)^T \\ J_{2m}(a \pm ib) & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & I_{2m} \\ -I_{2m} & 0 \end{bmatrix},$$

where $a, b > 0$. The matrix pencil here is $4m \times 4m$.

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THEOREM 4.1. *Let $A + \lambda B$ be a real symmetric/skewsymmetric matrix pencil. Then $A + \lambda B$ is R-congruent to a real symmetric/skewsymmetric pencil of the form*

$$(A_0 + \lambda B_0) \oplus \bigoplus_{j=1}^{r} \delta_j \left(F_{k_j} + \lambda \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{\frac{k_j-1}{2}} \\ 0 & -F_{\frac{k_j-1}{2}} & 0 \end{bmatrix} \right) \tag{4.1}$$

$$\oplus \bigoplus_{t=1}^{p} \eta_t \left(G_{\ell_t} + \lambda \begin{bmatrix} 0 & F_{\ell_t/2} \\ -F_{\ell_t/2} & 0 \end{bmatrix} \right) \oplus \bigoplus_{u=1}^{q} \zeta_u (\Omega_{2m_u}(\nu_u) + \lambda \widetilde{\Omega}_{2m_u}). \tag{4.2}$$

Here, $A_0 + \lambda B_0$ is a direct sum of blocks of types (sss0), (sss1), (sss3), (sss5), (sss6), *and* (sss8) *in which several blocks of the same type and of different and/or the same sizes may be present, and the k_j's are odd positive integers, the ℓ_t's are even positive integers, the ν_u's are positive real numbers, $\delta_j, \eta_t, \zeta_u$ are signs ± 1, and the m_u's are positive integers.*

The blocks in (4.1) *and* (4.2) *are uniquely determined by $A + \lambda B$ up to a permutation of blocks.*

Theorem 4.1 is found in many sources; see, for example, [8] for a detailed proof.

5. Proof of Theorem 1.2: particular case. In this section, we prove the following particular case of Theorem 1.2:

THEOREM 5.1. *Let $A = A^T \in \mathsf{R}^{m\times m}$, $B = -B^T \in \mathsf{R}^{m\times m}$ be of the form*

$$A = \left(\oplus_{j=1}^{q} \kappa_j (-\nu_j I_2)\right) \oplus I_t, \quad B = \left(\oplus_{j=1}^{q} \kappa_j \Xi_2\right) \oplus 0_t,$$

where t is a nonnegative integer, ν_j are positive numbers, κ_j are signs ± 1, and if $\nu_{j_1} = \nu_{j_2}$ then $\kappa_{j_1} = \kappa_{j_2}$. Then there exists an (A, B)-neutral subspace of dimension $\min_{0\le\alpha<2\pi} \Phi_\alpha(A, B)$.

We will need preliminary results.

LEMMA 5.2. *Let*

$$A = \left(\oplus_{j=1}^{q} \kappa_j (-\nu_j I_2)\right) \oplus I_{t_1} \oplus -I_{t_2} \in \mathsf{R}^{m\times m}, \quad B = \left(\oplus_{j=1}^{q} \tau_j \Xi_2\right) \oplus 0_{t_1+t_2} \in \mathsf{R}^{m\times m}, \tag{5.1}$$

where t_1, t_2 are nonnegative integers, ν_j are positive numbers, κ_j and τ_j are signs ± 1, and if $\nu_{j_1} = \nu_{j_2}$ then $\kappa_{j_1} = \kappa_{j_2}$. Let

$$\rho_+(A, B) := \min_{v\in\mathsf{R}} \{\mathrm{In}_+(A + viB) + \mathrm{In}_0(A + viB)\}.$$

Then there exists an A-nonnegative B-neutral subspace $\mathcal{M}$ of $\mathsf{R}^{m\times m}$ of dimension $\rho_+(A, B)$.

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Recall that a subspace $\mathcal{M}$ is called *A-nonnegative* if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{M}$.

Before the proof of the lemma, it will be convenient to consider an example first.

Example 5.3. Let

$$A_0 = \nu' I_2 \oplus -\nu'' I_2, \qquad B_0 = \tau' \Xi_2 \oplus \tau'' \Xi_2,$$

where $\nu' > \nu'' > 0$ and $\tau', \tau'' = \pm 1$. It is easy to see that $\rho_+(A_0, B_0) = 2$. Then there exists an A_0-nonnegative B_0-neutral subspace of dimension two, for example,

$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \pm 1 \end{bmatrix}\right\}$$

where the sign ± 1 is taken $+1$ if $\tau' \neq \tau''$ and -1 if $\tau' = \tau''$.

Proof of Lemma 5.2. Without loss of generality, we assume that the ν_j are arranged in the nondecreasing order:

$$\nu_1 \leq \nu_2 \leq \cdots \leq \nu_q.$$

Let $\kappa = \kappa_1$, and separate the blocks in (5.1) according to the signs:

$$\kappa_j = \kappa \quad \text{for} \quad j = 1, 2, \ldots, p_1;$$

$$\kappa_j = -\kappa \quad \text{for} \quad j = p_1 + 1, p_1 + 2, \ldots, p_2;$$

$$\kappa_j = \kappa \quad \text{for} \quad j = p_2 + 1, p_2 + 2, \ldots, p_3;$$

and so on, and finally

$$\kappa_j = \pm\kappa \quad \text{for} \quad j = p_{s-1} + 1, p_{s-1} + 2, \ldots, p_s.$$

Here $1 \leq p_1 < p_2 < \cdots < p_s = q$. By the hypotheses of Lemma 5.2, $\nu_{p_\ell} < \nu_{p_\ell + 1}$ for $\ell = 1, 2, \ldots, s-1$.

In view of Lemma 1.1 and Remark 3.2, we have

$$\rho_+(A, B) = \min_{v \in \Omega} \{\text{In}_+(A + viB) + \text{In}_0(A + viB)\},$$

where

$$\Omega := \{v \,:\, v > 0 \quad \text{and} \quad v \notin \{\nu_1, \ldots, \nu_q\}\},$$

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and since $A + viB$ is invertible for $v \in \Omega$, we also have

$$\rho_+(A,B) = \min_{v\in\Omega}\{\mathrm{In}_+(A+viB)\}.$$

Letting

$$A' = \oplus_{j=1}^{q}\kappa_j(-\nu_j I_2), \quad B = \oplus_{j=1}^{q}\tau_j\Xi_2,$$

we clearly obtain

$$\rho_+(A',B') + t_1 = \rho_+(A,B).$$

On the other hand, if $\mathcal{M}'$ is an A'-nonnegative B'-neutral subspace of dimension $\rho_+(A',B')$, then

$$\begin{bmatrix} \mathcal{M} \\ \mathsf{R}^{t_1} \\ 0_{t_2} \end{bmatrix}$$

is an A-nonnegative B-neutral subspace of dimension $\rho_+(A',B') + t_1$. So, using induction on the size of matrices A and B, we may (and do) assume that $t_1 = t_2 = 0$.

Observe that for $\tau = \pm 1$ and $\nu > 0$, we have

$$\mathrm{In}_+(\tau(-\nu I_2) \pm iv\Xi_2) = \begin{cases} 0 & \text{if } 0 \le v < \nu \text{ and } \tau = 1, \\ 1 & \text{if } v > \nu \text{ and } \tau = \pm 1, \\ 2 & \text{if } 0 \le v < \nu \text{ and } \tau = -1. \end{cases} \tag{5.2}$$

Thus, for $v \in \Omega$ we have

$$\begin{aligned} \mathrm{In}_+(A+ivB) &= 2\#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ &\quad + \#\{j=1,2,\ldots,q : \nu_j < v\} \\ &= q + \#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ &\quad - \#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = 1\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \rho_+(A,B) = q + \min_{v\in\Omega}\{&\#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ &- \#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = 1\}\}. \end{aligned} \tag{5.3}$$

In particular, $\rho_+(A,B) \le q$. We now consider several cases.

Case (a): Assume $\rho_+(A,B) = q$. Then in view of (5.3),

$$\#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = -1\} \ge \#\{j=1,2,\ldots,q : \nu_j > v \ \text{ and } \ \kappa_j = 1\}\}$$

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for all $v \in \Omega$. So, rearranging blocks in A and B (this amounts to a simultaneous row and column permutation in A and B), we can bring A and B to the following form:

$$A'' = \oplus_{j=1}^{q'}(\nu_{j,1}I_2 \oplus -\nu_{j,2}I_2) \bigoplus \oplus_{j=1}^{q''}\mu_j I_2,$$
$$B'' = \oplus_{j=1}^{q'}(\tau_{j,1}\Xi_2 \oplus \tau_{j,2}\Xi_2) \bigoplus \oplus_{j=1}^{q''}\gamma_j \Xi_2,$$

where $\nu_{j,1} > \nu_{j,2} > 0$ for $j = 1, 2, \ldots, q'$; $\mu_j > 0$ for $j = 1, 2, \ldots, q''$; $\tau_{j,1}$, $\tau_{j,2}$ and γ_j are signs ± 1; $2q' + q'' = q$. Clearly, every pair $\mu_j I_2$, $\gamma_j \Xi_2$ produces a one-dimensional $\mu_j I_2$-nonnegative $\gamma_j \Xi_2$-neutral subspace, for example span $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and every pair $\nu_{j,1}I_2 \oplus -\nu_{j,2}I_2$, $\tau_{j,1}\Xi_2 \oplus \tau_{j,2}\Xi_2$ produces a two-dimensional $(\nu_{j,1}I_2 \oplus -\nu_{j,2}I_2)$-nonnegative $(\tau_{j,1}\Xi_2 \oplus \tau_{j,2}\Xi_2)$-neutral subspace in view of Example 5.3. Putting all these subspaces together we obtain an A-nonnegative B-neutral subspace of the requisite dimension q.

Case (b): Assume $\rho_+(A, B) < q$ and $\kappa_{p_s} = 1$. Let

$$A' = \oplus_{j=1}^{q-1}\kappa_j(-\nu_j I_2), \quad B = \oplus_{j=1}^{q-1}\tau_j \Xi_2.$$

Using formula analogous to (5.3) for the pair A', B', we have

$$\begin{aligned}\rho_+(A', B') = q - 1 + \min_{v\in\Omega}\{&\#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1\} \\ &- \#\{j = 1, 2, \ldots, q-1 : \nu_j > v \text{ and } \kappa_j = 1\}\},\end{aligned}$$

which is equal to

$$\begin{aligned}q - 1 + \min\{&\min_{v\in\Omega,\ v<\nu_q}\{\#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1\} \\ &- \#\{j = 1, 2, \ldots, q-1 : \nu_j > v \text{ and } \kappa_j = 1\}\}, \\ &\min_{v\in\Omega,\ v>\nu_q}\{\#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1\} \\ &- \#\{j = 1, 2, \ldots, q-1 : \nu_j > v \text{ and } \kappa_j = 1\}\}\} \\ &= q - 1 + \min\{\min_{v\in\Omega,\ v<\nu_q}\{\#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1\} \\ &- \#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1\} + 1\}, 0\} \\ &= q + \min\{\min_{v\in\Omega,\ v<\nu_q}\{\#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = -1\} \\ &- \#\{j = 1, 2, \ldots, q : \nu_j > v \text{ and } \kappa_j = 1\}\}, -1\}.\end{aligned}$$

In turn, this is equal to $\rho_+(A, B)$ in view of the formula (5.3) and our assumption $\rho_+(A, B) < q$. Using the induction hypothesis, we find A'-nonnegative B'-neutral subspace $\mathcal{M}'$ of dimension $\rho_+(A, B)$. Then

$$\begin{bmatrix} \mathcal{M}' \\ 0 \end{bmatrix} \subset \mathsf{R}^m$$

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is an A-nonnegative B-neutral subspace of dimension $\rho_+(A,B)$.

Case (c): Assume $\rho_+(A,B) < q$ and $\kappa_{p_s} = -1$.

Define the matrices A'_j, B'_j, $j = 1,2,\ldots,q$, as follows: A'_j is obtained from A by replacing the block $\kappa_j(-\nu_j I_2)$ with $\kappa_j(-\nu_j)$ (leaving all other blocks in A intact), and B'_j is obtained from B by replacing the block $\tau_j \Xi_2$ with zero (leaving all other blocks in B intact). Thus, $A'_j, B'_j \in \mathsf{R}^{(m-1)\times(m-1)}$. Since A'_j, resp. B'_j, is obtained from A, resp. B, by removing the $2(j-1)+1$th row and column, the interlacing inequalities for eigenvalues of principal submatrices of Hermitian matrices yield

$$\mathrm{In}_+(A+viB) - 1 \le \mathrm{In}_+(A'_j + ivB'_j) \le \mathrm{In}_+(A+viB), \qquad v \in \Omega, \qquad j = 1,2,\ldots,q,$$

and therefore

$$\rho_+(A,B) - 1 \le \rho(A'_j, B'_j) \le \rho_+(A,B), \quad j = 1,2,\ldots,q.$$

On the other hand, a computation using (5.2) shows that for $j_0 = 1,2,\ldots,q$, and for $v \in \Omega$:

$$\begin{aligned}\mathrm{In}_+(A'_{j_0} + ivB'_{j_0}) = \#\{&j = 1,2,\ldots,q \,:\, j \ne j_0, \quad \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ &+ q - 1 + \chi_{j_0} - \#\{j = 1,2,\ldots,q \,:\, j \ne j_0, \quad \nu_j > v \ \text{ and } \ \kappa_j = 1\},\end{aligned}$$

where $\chi_{j_0} = 1$ if $\kappa_{j_0} = -1$ and $\chi_{j_0} = 0$ if $\kappa_{j_0} = 1$. Thus,

$$\rho_+(A'_{j_0}, B'_{j_0}) = q$$

$$+ \min_{v\in\Omega}\{-1 + \chi_{j_0} + \#\{j = 1,2,\ldots,q \,:\, j \ne j_0, \quad \nu_j > v \ \text{ and } \ \kappa_j = -1\}$$

$$-\#\{j = 1,2,\ldots,q \,:\, j \ne j_0, \quad \nu_j > v \ \text{ and } \ \kappa_j = 1\}\}.$$

If there is j_0 such that

$$w := \rho_+(A'_{j_0}, B'_{j_0}) = \rho_+(A,B),$$

then we can use induction on the size $m \times m$ of the matrices A and B to show that there exists a w-dimensional A'_{j_0}-nonnegative B'_{j_0}-neutral subspace $\mathcal{M}_{j_0}$. Let $x_1,\ldots,x_w \in \mathsf{R}^{m-1}$ be a basis for $\mathcal{M}_{j_0}$, and write

$$x_\gamma = \begin{bmatrix} x_{\gamma,1} \\ x_{\gamma,2} \\ \vdots \\ x_{\gamma,m-1} \end{bmatrix}, \quad \gamma = 1,2,\ldots,w.$$

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Let

$$\widehat{x}_1, \ldots, \widehat{x}_w \in \mathsf{R}^m$$

be obtained from $x_1, \ldots, x_w$, respectively, by inserting a zero between $x_{\gamma,2(j_0-1)}$ and $x_{\gamma,2(j_0-1)+1}$, $\gamma = 1, 2, \ldots, w$. Then the subspace

$$\widehat{\mathcal{M}}_{j_0} := \operatorname{span}\{\widehat{x}_1, \ldots, \widehat{x}_w\}$$

is w-dimensional and A-nonnegative and B-neutral.

It remains therefore to consider the situation when

$$\rho_+(A'_{j_0}, B'_{j_0}) < \rho_+(A, B) \quad \forall \ \ j_0 = 1, 2, \ldots, q,$$

(in this case, necessarily

$$\rho_+(A'_{j_0}, B'_{j_0}) + 1 = \rho_+(A, B) \quad \forall \ \ j_0 = 1, 2, \ldots, q \quad),$$

in other words,

$$\begin{aligned} \min_{v\in\Omega}\{\#\{j = 1, 2, \ldots, q : \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ - \#\{j = 1, 2, \ldots, q : \nu_j > v \ \text{ and } \ \kappa_j = 1\}\} \\ = 1 + \min_{v\in\Omega}\{-1 + \chi_{j_0} + \#\{j = 1, 2, \ldots, q : j \neq j_0, \quad \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ -\#\{j = 1, 2, \ldots, q : j \neq j_0, \quad \nu_j > v \ \text{ and } \ \kappa_j = 1\}\} \end{aligned} \tag{5.4}$$

holds for $j_0 = 1, 2, \ldots, q$. Thus, we assume that (5.4) holds. As we will see, this leads to a contradiction.

Consider the function

$$\begin{aligned} f(v) = \#\{j = 1, 2, \ldots, q : \nu_j > v \ \text{ and } \ \kappa_j = -1\} \\ - \#\{j = 1, 2, \ldots, q : \nu_j > v \ \text{ and } \ \kappa_j = 1\}, \end{aligned}$$

where $v \in \Omega$. We have

$$\rho_+(A, B) = \min_{v\in\Omega} f(v) + q.$$

Select points $\lambda_0, \ldots, \lambda_s$ so that

$$0 < \lambda_0 < \nu_1, \quad \nu_{p_1} < \lambda_1 < \nu_{p_1+1}, \ldots, \nu_{p_{s-1}} < \lambda_{s-1} < \nu_{p_{s-1}+1}, \quad \nu_{p_s} < \lambda_s.$$

Clearly, at least one of the points λ_j, $j = 0, 1, \ldots, s$, is a point of (global) minimum for f. Since $f(\lambda_s) + q = q > \rho_+(A, B)$, the point λ_s is not a point of minimum. Also, it follows from our assumption $\kappa_q = -1$ that

$$f(\lambda_s) < f(\lambda_{s-1}), \quad f(\lambda_{s-1}) > f(\lambda_{s-2}), \quad f(\lambda_{s-2}) < f(\lambda_{s-3}),$$

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and so on. So, only the points $\lambda_{s-2}, \lambda_{s-4}, \lambda_{s-6}, \ldots$ can be points of (global) minimum of f.

Suppose s is odd; then $\kappa_1 = -1$, $\chi_1 = 1$, and λ_0 is not a point of minimum for f. The right hand side of (5.4) with $j_0 = 1$ takes the form

$$\begin{aligned} &1 + \min_{v\in\Omega}\{\#\{j = 1, 2, \ldots, q : j \neq 1, \quad \nu_j > v \text{ and } \kappa_j = -1\} \\ &\quad - \#\{j = 1, 2, \ldots, q : \quad \nu_j > v \text{ and } \kappa_j = 1\}\}. \end{aligned}$$

Clearly the minimum is achieved at one of the points $\lambda_0, \ldots, \lambda_s$. Thus,

$$\begin{aligned} &1 + \min_{v\in\Omega}\{\#\{j = 1, 2, \ldots, q : j \neq 1, \quad \nu_j > v \text{ and } \kappa_j = -1\} \\ &\quad - \#\{j = 1, 2, \ldots, q : \quad \nu_j > v \text{ and } \kappa_j = 1\}\} \\ &\quad = 1 + \min\{f(\lambda_0) - 1, f(\lambda_1), \ldots, f(\lambda_s)\} \\ &\quad = 1 + \min\{f(\lambda_0), f(\lambda_1), \ldots, f(\lambda_s)\} \end{aligned}$$

(because λ_0 is not a point of minimum for f), which is one more than the left hand side of (5.4), a contradiction with (5.4).

Thus, suppose s is even. Then $\kappa_1 = 1$. In this case, we select j_0 so that $\kappa_{j_0} = 1$, $\chi_{j_0} = 0$. The right hand side of (5.4) takes the form

$$\begin{aligned} &\min_{v\in\Omega}\{\#\{j = 1, 2, \ldots, q : \quad \nu_j > v \text{ and } \kappa_j = -1\} \\ &\quad -\#\{j = 1, 2, \ldots, q : j \neq j_0, \quad \nu_j > v \text{ and } \kappa_j = 1\}\}. \end{aligned} \tag{5.5}$$

Let λ_y be the point of (global) minimum of f having the largest index y; then we let $j_0 = j_{p_y} + 1$. (Note that we cannot have $y = s$ because λ_s is not a point of minimum of f.) Again, the minimal value of

$$\begin{aligned} &\#\{j = 1, 2, \ldots, q : \quad \nu_j > v \text{ and } \kappa_j = -1\} \\ &- \#\{j = 1, 2, \ldots, q : j \neq j_0, \quad \nu_j > v \text{ and } \kappa_j = 1\}, \end{aligned}$$

where $v \in \Omega$, is achieved at one of the points $\lambda_{s-2}, \lambda_{s-4}, \ldots$. So, (5.5) becomes

$$\begin{aligned} &\min_{z=s-2,s-4,\ldots}\{\#\{j = 1, 2, \ldots, q : \quad \nu_j > \lambda_z \text{ and } \kappa_j = -1\} \\ &\quad -\#\{j = 1, 2, \ldots, q : j \neq j_0, \quad \nu_j > \lambda_z \text{ and } \kappa_j = 1\}\}. \end{aligned} \tag{5.6}$$

By the choice of $j_0 = j_{p_y} + 1$, we see that (5.6) is equal to

$$\begin{aligned} &1 + \min_{z=s-2,s-4,\ldots}\{\#\{j = 1, 2, \ldots, q : \quad \nu_j > \lambda_z \text{ and } \kappa_j = -1\} \\ &\quad -\#\{j = 1, 2, \ldots, q : \quad \nu_j > \lambda_z \text{ and } \kappa_j = 1\}\}, \end{aligned} \tag{5.7}$$

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which is one more than the left hand side of (5.4), a contradiction again. □

The following result proved in [1] will be also needed for the proof of Theorem 5.1.

Proposition 5.4. *Let $A, B \in \mathsf{R}^{n\times n}$, $A = A^T$, $B = -B^T$. Assume that there exists a d-dimensional subspace $\mathcal{M} \subseteq \mathsf{R}^n$ which is simultaneously A-nonnegative, i.e., $(Ax, x) \geq 0$ for every $x \in \mathcal{M}$, and B-neutral, i.e., $(Bx, y) = 0$ for all $x, y \in \mathcal{M}$. Assume also that there exists a d-dimensional subspace $\mathcal{M}' \subseteq \mathsf{R}^n$ which is simultaneously A-nonpositive and B-neutral. Then there exists a d-dimensional (A, B)-neutral subspace.*

Proof of Theorem 5.1. By Lemma 5.2, there exists an A-nonnegative B-neutral subspace of dimension $\rho_+(A, B)$, and analogously there exists an A-nonpositive B-neutral subspace of dimension $\rho_+(-A, B)$. Since (cf. Remark 1.3 (1) and (2))

$$d := \min_{0\leq\alpha<2\pi} \Phi_\alpha(A, B) = \min\{\rho_+(A, B), \rho_+(-A, B)\},$$

it follows that there exist an A-nonnegative B-neutral subspace and an A-nonpositive B-neutral subspace of the same dimension d. Now Proposition 5.4 implies that there exists a d-dimensional (A, B)-neutral subspace. □

6. Proof of Theorem 1.2: general case. Since by Proposition 2.1 an (A, B)-neutral subspace cannot have dimension greater than (1.2), we only have to prove existence of an (A, B)-neutral subspace $\mathcal{M}$ having dimension $\min_{0\leq\alpha<2\pi} \Phi_\alpha(A, B)$.

First, note that Lemma 3.1 leads to the following observation:

Proposition 6.1. *Under the hypotheses of* Lemma 3.1 *part (b), if there is an (A_j, B_j)-neutral subspace $\mathcal{M}_j$ of dimension* $\min_{0\leq\alpha<2\pi} \Phi_\alpha(A_j, B_j)$, *$j = 1, 2$, then there is an (A, B)-neutral subspace $\mathcal{M}$ of dimension* $\min_{0\leq\alpha<2\pi} \Phi_\alpha(A, B)$.

Proof. Let

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{M}_2 \end{bmatrix},$$

and take advantage of (3.2). □

Without loss of generality we may (and do) assume that $A+\lambda B$ is in the canonical form as presented in Theorem 4.1.

Let $v_0 \times v_0$ be the size of the zero block (if present) in $A_0 + \lambda B_0$, let $v_1 \times v_1$ be the total size of blocks of types (sss3), (sss5), (sss6), (sss8) (if present) in $A_0 + \lambda B_0$, and let

$$(2\varepsilon_1 + 1) \times (2\varepsilon_1 + 1), \ldots, (2\varepsilon_s + 1) \times (2\varepsilon_s + 1)$$

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be the sizes of blocks of type (sss1) (if present) in $A_0 + \lambda B_0$.

We shall calculate inertia of linear combinations of matrices in the blocks of types (sss0) - (sss8), and in each case show a neutral subspace of the requisite dimension. The calculations are straightforward.

(1) If $A' + \lambda B'$ is the block (sss0), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') = 0, \quad \forall\ \alpha \in [0, 2\pi).$$

Clearly, there exists an (A', B')-neutral subspace of dimension $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B')$.

(2) If $A' + \lambda B'$ is the block (sss1), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') = \epsilon, \quad \forall\ \alpha \in [0, 2\pi),$$

and span $(e_{\epsilon+1}, \ldots, e_{2\epsilon+1})$ is an (A', B')-neutral subspace of dimension equal to $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B') = \epsilon + 1$.

(3) If $A' + \lambda B'$ is the block (sss3), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') = \begin{cases} k/2 & \text{if } \cos\alpha \neq 0, \\ k/2 - 1 & \text{if } \cos\alpha = 0, \end{cases}$$

and span $(e_1, \ldots, e_{k/2})$ is an (A', B')-neutral subspace of dimension equal to $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B') = k/2$.

(4) If $A' + \lambda B'$ is the block (sss5), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') = \begin{cases} \ell/2 & \text{if } \sin\alpha \neq 0, \\ \ell/2 - 1 & \text{if } \sin\alpha = 0, \end{cases}$$

and span $(e_1, \ldots, e_{\ell/2})$ is an (A', B')-neutral subspace of dimension equal to $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B') = \ell/2$.

(5) If $A' + \lambda B'$ is the block (sss6), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') = \ell/2, \quad \forall\ \alpha \in [0, 2\pi),$$

and span $(e_1, \ldots, e_{\ell/2})$ is an (A', B')-neutral subspace of dimension equal to $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B') = \ell/2$.

(6) If $A' + \lambda B'$ is the block (sss8), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') = 2m, \quad \forall\ \alpha \in [0, 2\pi),$$

and span $(e_1, \ldots, e_{2m})$ is an (A', B')-neutral subspace of dimension equal to $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B') = 2m$.

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(7) If

$$A' + \lambda B' = \left(F_k + \lambda \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{\frac{k-1}{2}} \\ 0 & -F_{\frac{k-1}{2}} & 0 \end{bmatrix} \right)$$
$$\oplus - \left(F_{k'} + \lambda \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{\frac{k'-1}{2}} \\ 0 & -F_{\frac{k'-1}{2}} & 0 \end{bmatrix} \right),$$

where k, k' are odd, then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') \ = \ \frac{k+k'}{2},$$
$$\forall \ \alpha \in [0, 2\pi) \text{ such that } \cos\alpha \neq 0.$$

Thus, span $(e_1, \ldots, e_{(k-1)/2}, e_{(k+1)/2} + e_{k+(k'+1)/2}, e_{k+1}, \ldots, e_{k+(k'-1)/2})$ is an (A', B')-neutral subspace of dimension $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = \frac{k+k'}{2}$.

(8) If $A' + \lambda B'$ is the block (sss4), then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') \ = \ \ell/2,$$
$$\forall \ \alpha \in [0, 2\pi) \text{ such that } \sin\alpha \neq 0,$$

and span $(e_{\ell/2+1}, \ldots, e_\ell)$ is an (A', B')-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = \ell/2$.

(9) If $A' + \lambda B'$ is the block (sss7) (of size $2m \times 2m$), with m even, then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') = \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') \ = \ m,$$
$$\forall \ \alpha \in [0, 2\pi) \text{ such that } \tan\alpha \neq \pm\nu,$$

and span $(e_1, \ldots, e_m)$ is an (A', B')-neutral subspace of dimension equal to $\min_{0 \leq \alpha < 2\pi} \Phi_\alpha(A', B') = m$.

(10) Assume

$$A' + \lambda B' = \xi_1(\Omega_{2m_1}(\nu) + \lambda\widetilde{\Omega}_{2m_1}) \oplus \xi_2(\Omega_{2m_2}(\nu) + \lambda\widetilde{\Omega}_{2m_2}),$$

where $\nu > 0$, m_1, m_2 are odd, and

$$\xi_1(-1)^{\frac{m_1-1}{2}} = -\xi_2(-1)^{\frac{m_2-1}{2}}. \tag{6.1}$$

Then

$$\mathrm{In}_+((\cos\alpha)A' + i(\sin\alpha)B') \ = \ \mathrm{In}_-((\cos\alpha)A' + i(\sin\alpha)B') \ = \ m_1 + m_2 \tag{6.2}$$

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for all $\alpha \in [0, 2\pi)$ except those values for which $\tan\alpha = \pm\nu$. Indeed, a calculation shows that the direct sum of the middle 2×2 block in

$$\xi_1((\cos\alpha)\Omega_{2m_1}(\nu) + i(\sin\alpha)\widetilde{\Omega}_{2m_2})$$

and of the middle 2×2 block in

$$\xi_2((\cos\alpha)\Omega_{2m_2}(\nu) + i(\sin\alpha)\widetilde{\Omega}_{2m_2})$$

is

$$\xi_1((\cos\alpha)\nu\Xi_2^{m_1+1} + i(\sin\alpha)\Xi_2) \oplus \xi_2((\cos\alpha)\nu\Xi_2^{m_2+1} + i(\sin\alpha)\Xi_2). \tag{6.3}$$

Now (6.2) follows easily from (6.3). Also, the 4×4 matrix (6.3) has the following 2-dimensional neutral subspace $\mathcal{M}_0$ independent of α (the hypothesis (6.1) is essential here):

$$\mathcal{M}_0 = \begin{cases} \mathrm{span}\,(e_1 + e_3, e_2 + e_4) & \text{if } m_1 = 4k+3, \quad m_2 = 4\ell+3, \\ \mathrm{span}\,(e_1 + e_3, e_2 + e_4) & \text{if } m_1 = 4k+1, \quad m_2 = 4\ell+1, \\ \mathrm{span}\,(e_1 + e_4, e_2 + e_3) & \text{if } m_1 = 4k+3, \quad m_2 = 4\ell+1, \\ \mathrm{span}\,(e_1 + e_4, e_2 + e_3) & \text{if } m_1 = 4k+1, \quad m_2 = 4\ell+3, \end{cases} \tag{6.4}$$

where k and ℓ are nonnegative integers. Let

$$\mathcal{M} = \mathrm{span}\,(e_1, \ldots, e_{m_1-1}, e_{2m_1+1}, \ldots, e_{2m_1+m_2-1}, e_{m_1}+e_{2m_1+m_2}, e_{m_1+1}+e_{2m_1+m_2+1})$$

if $(m_1-1)/2$ and $(m_2-1)/2$ have the same parity, and

$$\mathcal{M} = \mathrm{span}\,(e_1, \ldots, e_{m_1-1}, e_{2m_1+1}, \ldots, e_{2m_1+m_2-1}, e_{m_1}+e_{2m_1+m_2+1}, e_{m_1+1}+e_{2m_1+m_2})$$

if $(m_1-1)/2$ and $(m_2-1)/2$ have different parity. It follows from (6.4) that $\mathcal{M}$ is an (A', B')-neutral subspace of dimension $\min_{0\le\alpha<2\pi} \Phi_\alpha(A', B') = m_1 + m_2$.

Repeatedly using Proposition 6.1, items (1) - (10) above, and Theorem 4.1, and replacing if necessary A and B by $-A$ and $-B$, respectively, we see that the proof of Theorem 1.2 is reduced to the consideration of the following case:

$$\begin{aligned} A + \lambda B = {} & \oplus_{j=1}^{q} \xi_j(\Omega_{2m_j}(\nu_j) + \lambda\widetilde{\Omega}_{2m_j}) \\ & \oplus \oplus_{i=1}^{s} \left(F_{k_i} + \lambda \begin{bmatrix} 0_1 & 0 & 0 \\ 0 & 0 & F_{\frac{k_i-1}{2}} \\ 0 & -F_{\frac{k_i-1}{2}} & 0 \end{bmatrix} \right), \end{aligned} \tag{6.5}$$

where $m_1, \ldots, m_q$ are odd and $k_1, \ldots, k_s$ are odd, and ξ_j are signs ±1 (the cases when $q = 0$, i.e, the first part of (6.5) is missing, or $s = 0$, i.e., the second part of (6.5)

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is missing, are not excluded); also, if $\nu_{j_1} = \nu_{j_2}$ then the signs of the corresponding blocks in (6.5) are the same.

Applying a suitable simultaneous permutation of rows and columns to $A + \lambda B$ in (6.5), we obtain $A' + \lambda B'$ in the following block form:

$$A' + \lambda B' = \begin{bmatrix} 0_k & 0 & A_1 + \lambda B_1 \\ 0 & A_0 + \lambda B_0 & * \\ A_1^T - \lambda B_1^T & * & * \end{bmatrix}, \tag{6.6}$$

where

$$k = \left(\sum_{j=1}^{q} (m_j - 1) \right) + \left(\sum_{i=1}^{s} \frac{k_i - 1}{2} \right).$$

In (6.6), $A_1 + \lambda B_1$ is a $k \times k$ block diagonal matrix pencil with the diagonal blocks of the forms

$$\begin{bmatrix} \cdots & 0 & 0 & \nu_j \Xi_2^{m_j+1} \\ \cdots & 0 & -\nu_j \Xi_2^{m_j+1} & -I_2 \\ \cdots & \nu_j \Xi_2^{m_j+1} & -I_2 & 0 \\ \iddots & \vdots & \vdots & \vdots \end{bmatrix} + \lambda \begin{bmatrix} \cdots & 0 & 0 & \Xi_2^{m_j} \\ \cdots & 0 & -\Xi_2^{m_j} & 0 \\ \cdots & \Xi_2^{m_j} & 0 & 0 \\ \iddots & \vdots & \vdots & \vdots \end{bmatrix},$$

where $j = 1, 2, \ldots, q$, and of the forms

$$F_{\frac{k_i-1}{2}} + \lambda G'_{\frac{k_i-1}{2}}, \quad i = 1, 2, \ldots, s,$$

where

$$G'_m = \begin{bmatrix} 0_1 & 0 \\ 0 & F_{m-1} \end{bmatrix} \in \mathsf{R}^{m \times m};$$

and

$$\begin{aligned} A_0 + \lambda B_0 &:= \left(\oplus_{j=1}^{q} \xi_j (-1)^{\frac{m_j-1}{2}} (\nu_j \Xi_2^{m_j+1} + \lambda \Xi_2^{m_j}) \right) \oplus I_s \\ &= \left(\oplus_{j=1}^{q} \xi_j (-\nu_j I_2 + \lambda \Xi_2) \right) \oplus I_s. \end{aligned}$$

Note that $(\cos\alpha)A_1 + i(\sin\alpha)B_1$ is invertible for all but finitely many values of $\alpha \in [0, 2\pi)$. By Lemma 3.3, we have

$$\min_{0 \le \alpha < 2\pi} \Phi_\alpha(A', B') = k + \min_{0 \le \alpha < 2\pi} \Phi_\alpha(A_0, B_0). \tag{6.7}$$

On the other hand, by Theorem 5.1, there exists an (A_0, B_0)-neutral subspace $\mathcal{M}_0$ of dimension $\min_{0 \le \alpha < 2\pi} \Phi_\alpha(A_0, B_0)$. Then clearly

$$\mathcal{M} := \begin{bmatrix} \mathsf{R}^k \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathcal{M}_0 \\ 0 \end{bmatrix}$$

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is an (A', B')-neutral subpace of dimension $k + \min_{0 \le \alpha < 2\pi} \Phi_\alpha(A_0, B_0)$. In view of (6.7), we have proved Theorem 1.2 for the pair (A, B). $\square$

REFERENCES

[1] A. Ben-Artzi, D. Ž. Djokovič, and L. Rodman. Totally isotropic subspaces for pairs of Hermitian forms and applications to Riccati equations. *Linear Algebra Appl.*, 159:121–128, 1991.

[2] I. Gohberg, P. Lancaster, and L. Rodman. Spectral analysis of selfadjoint matrix polynomials. *Ann. of Math.*, 112:33–71, 1980.

[3] I. Gohberg, P. Lancaster, and L. Rodman. *Matrices and Indefinite Scalar Products.* OT8, Birkhäuser, Basel, 1983.

[4] I. Gohberg, P. Lancaster, and L. Rodman. *Indefinite Linear Algebra and Applications.* Birkhäuser, Boston, 2006.

[5] P. Lancaster and L. Rodman. Invariant neutral subspaces for symmetric and skew real pairs. *Canad. J. Math.*, 46:602–618, 1994.

[6] P. Lancaster and L. Rodman. Minimal symmetric factorizations of symmetric real and complex rational matrix functions. *Linear Algebra Appl.*, 220:249–282, 1995.

[7] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Oxford University Press, New York, 1995.

[8] P. Lancaster and L. Rodman. Canonical forms for symmetric/skew-symmetric real matrix pairs under strict equivalence and congruence. *Linear Algebra Appl.*, 406:1–76, 2005.

[9] C.-K. Li and N.-S. Sze. Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations. *Proc. Amer. Math. Soc.*, 136:3013–3023, 2008.

[10] A.C.M. Ran. Minimal factorization of selfadjoint rational matrix functions. *Integral Equations Operator Theory*, 5:850–869, 1982.

[11] L. Rodman. Isotropic subspaces for pairs of Hermitian matrices. *Electron. J. Linear Algebra*, 17:508–517, 2008.