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CHARACTERIZATION OF P-PROPERTY FOR SOME Z-TRANSFORMATIONS ON POSITIVE SEMIDEFINITE CONE*

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Abstract. The P-property of the following two **Z**-transformations with respect to the positive semidefinite cone is characterized:

(i) $I - S$, where $S : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ is a nilpotent linear transformation,
(ii) $I - L_A^{-1}$, where L_A is the Lyapunov transformation defined on $\mathbb{S}^{n\times n}$ by $L_A(X) = AX + XA^T$.

(Here $\mathbb{S}^{n\times n}$ denotes the space of all symmetric $n \times n$ matrices and I is the identity transformation.)

Key words. P-property, Stein-type transformations, Lyapunov transformations.

AMS subject classifications. 90C33, 17C55.

1. Introduction.

An $n \times n$ matrix is said to be a **Z**-matrix if all the off-diagonal entries are non-positive. Several interesting properties on **Z**-matrices can be found in [1]. For a square matrix of order n, by an easy verification, we find that the following are equivalent:

1. A is a **Z**-matrix.
2. If $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ then,

$$x \geq 0, \quad y \geq 0 \quad \text{(entrywise non-negative)}, \quad \text{and} \quad x^T y = 0 \implies y^T Ax \leq 0.$$

Motivated by the above fact, we consider **Z**-transformations with respect to positive semidefinite cone.

Let $\mathbb{S}^{n\times n}$ be the vector space of $n \times n$ symmetric matrices with real entries. A linear transformation $L : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ is called a **Z**-transformation with respect to the positive semidefinite cone if

$$X \succeq 0, \quad Y \succeq 0 \ \text{and} \quad XY = 0 \quad \implies \quad \langle L(X), Y\rangle := \text{trace}\,(L(X)Y) \leq 0.$$

(Here $X \succeq 0$ means X is symmetric and positive semidefinite.) Significances of **Z**-transformations (especially in mathematical programming) can be found in [2]. An important result on **Z**-transformations is the following:

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THEOREM 1.1 (Theorem 6 [2]). *Let* $L : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ *be a* **Z**-*transformation. Then the following are equivalent.*

1. *There exists a* $X \succ 0$ *such that* $L(X) \succ 0$.
2. *For every* $Q \succeq 0$*, there exists a unique* $X \succeq 0$ *such that* $L(X) = Q$.
3. *For every* $Q \in \mathbb{S}^{n\times n}$*, there exists a* $X \succeq 0$ *such that* $Y := L(X) + Q \succeq 0$ *and* $XY = 0$.

We will say that a transformation S (defined on $\mathbb{S}^{n\times n}$) has the property (c) if:

$$X \succeq 0 \Longrightarrow S(X) \succeq 0.$$

Transformations of the type $I-S$, where I is the identity transformation on $\mathbb{S}^{n\times n}$ and S is a linear transformation with property (c) are called *Stein-type* transformations. These transformations are important examples of **Z**-transformations. For a Stein-type transformation it is known that all the statements of Theorem 1.1 are equivalent to the condition $\rho(S) < 1$, where $\rho(S)$ is the spectral radius of S (see [3]).

A transformation $L : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ is said to have the P-property if the following condition is satisfied:

$$XL(X) = L(X)X \quad \text{and} \quad XL(X) \preceq 0 \quad \Longrightarrow \quad X = 0.$$

One of the unsolved problem on **Z**-transformations (see [2]) is to show that all the items in Theorem 1.1 are equivalent to the condition that L has the P-property. Even for the Stein-type transformations, the problem remains unsolved. More precisely, if $I - S$ is a Stein-type transformation such that $\rho(S) < 1$, then the problem of determining whether $I-S$ has the P-property has no answer. It is natural to consider the simplest case, when $\rho(S) = 0$. In other words, assuming S is nilpotent, we ask whether the Stein-type transformation $I - S$ has the P-property. First, we settle this question in this paper.

If S is a **Z**-transformation satisfying any of the items in Theorem 1.1, we find that S^{-1} has property (c). We now ask whether $I - S^{-1}$ has the P-property if S is a **Z**-transformation with property (c) and such that $\rho(S^{-1}) < 1$. One of the well-studied **Z**-transformations is the Lyapunov transformation for which we know that all the items of Theorem 1.1 are equivalent to the fact that A is a positive stable matrix (See the definitions below for Lyapunov transformation and positive stable matrix). If $S = L_A^{-1}$, where L_A is the Lyapunov transformation corresponding to a positive stable matrix A with the property $\rho(L_A^{-1}) < 1$, then for the Stein-type transformation $I - L_A^{-1}$, we show that $I - L_A^{-1}$ has the P-property.

2. Preliminaries. All the matrices appearing here are assumed to be real. The following notations and definitions will be useful in the sequel.

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- Definition 2.1. Let A be a square matrix. Then A is said to be positive stable if every eigenvalue of A has a positive real part.
- Definition 2.2. For a square matrix A, the corresponding *Lyapunov* transformation $L_A : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ is defined by $L_A(X) := AX + XA^T$.
- If Q is an $n \times n$ matrix, and $\alpha = \{1, \dots, k\}$ $(k < n)$, $Q_{\alpha\alpha}$ will denote the $k \times k$ leading principal submatrix of Q.
- Definition 2.3. Let $L : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ be a linear transformation. For any $\alpha = \{1, \dots, k\}$, we define a linear transformation $L_{\alpha\alpha} : \mathbb{S}^{k\times k} \to \mathbb{S}^{k\times k}$ by

  $$L_{\alpha\alpha}(Z) := [L(X)]_{\alpha\alpha} \quad (Z \in \mathbb{S}^{k\times k}),$$

  where corresponding to $Z \in \mathbb{S}^{k\times k}$, $X \in \mathbb{S}^{n\times n}$ is the unique matrix such that

  $$X_{ij} = \begin{cases} Z_{ij} & (i,j) \in \alpha \times \alpha \\ 0 & \text{else.} \end{cases}$$

  We call $L_{\alpha\alpha}$ the principal subtransformation corresponding to α.
- If $\beta \in \mathbb{R}$, then we define $\beta^+ := \max(\beta, 0)$ and $\beta^- := \max(-\beta, 0)$. Suppose D is a diagonal matrix with diagonal entries d_i. Then D^+ will denote the diagonal matrix whose diagonal entries are d_i^+. Similarly, D^- will denote the diagonal matrix whose entries are d_i^-.
- If $X \in \mathbb{S}^{n\times n}$, then there exists an orthogonal matrix U such that $UXU^T = D$, where D is diagonal. Now we define $X^+ := UD^+U^T$ and $X^- := UD^-U^T$. It is easy to see that for every $X \in \mathbb{S}^{n\times n}$, $X = X^+ - X^-$; X^+ and X^- are positive semidefinite.
- We will use the fact that if T is a linear transformation on $\mathbb{S}^{n\times n}$ with property (c), then its spectral radius is an eigenvalue of T (see Theorem 0 in [4]).
- Let $T : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ be a linear transformation. Then T is a nilpotent transformation if there exists a positive integer m such that $T^m = 0$.

3. Results.

We prove our main results now.

3.1. Case 1.

We intend to show that $I - S$ has the P-property if S is nilpotent and has property (c). The result is trivial if $S = 0$ and so in the rest of the discussion, we assume S is nonzero. Let ν be the least positive integer satisfying

$$S^\nu = 0, \quad \text{and} \quad S^{\nu-1} \neq 0. \tag{3.1}$$

First we prove the following basic lemma.

Lemma 3.1. *Let S be a nilpotent transformation. Assume that S has property (c). Then the following are true:*

(a) *If $Q \succ 0$, then $Q \notin$* Image(S).

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(b) *If* $\operatorname{rank} S(X) = m$, *then there exists a* $P \succeq 0$ *such that* $\operatorname{rank} S(P) \geq m$. *In fact, if* $X \in \mathbb{S}^{n\times n}$, *then*

$$\text{rank } S(X) \leq \text{rank } S(|X|),$$

where $|X| := X^+ + X^-$.

Proof. Let S satisfy (3.1). Suppose $S(P) = Q$ for some $Q \succ 0$. If $X \succeq 0$, then there exists $\epsilon > 0$ such that $Q - \epsilon X \succ 0$. Since S has the property (c) and satisfies (3.1), we have:

$$S^{\nu-1}(Q - \epsilon X) + S^{\nu-1}(\epsilon X) = 0, \tag{3.2}$$

$$S^{\nu-1}(Q - \epsilon X) \succeq 0, \quad \text{and} \quad S^{\nu-1}(\epsilon X) \succeq 0. \tag{3.3}$$

In view of (3.2) and (3.3), $S^{\nu-1}(X) = 0$. Therefore for any $Y \in \mathbb{S}^{n\times n}$,

$$S^{\nu-1}(Y) = S^{\nu-1}(Y^+) - S^{\nu-1}(Y^-) = 0$$

and so $S^{\nu-1} = 0$ which is a contradiction to (3.1). This proves (a).

For any two positive semidefinite matrices U and V in $\mathbb{S}^{n\times n}$,

$$\text{rank}(U - V) \leq \text{rank}(U + V). \tag{3.4}$$

The above inequality can be proved as follows. Let $x \in \mathbb{R}^n$ be an element in the null space of $U + V$. This gives $Ux = -Vx$ and thus, $x^T Ux = -x^T V x$. Since U and V are symmetric and positive semidefinite, we get $Ux = 0 = Vx$ and thus,

$$\text{nullity } (U + V) \leq \text{nullity } (U - V).$$

By rank nullity theorem, (3.4) follows.

By setting $U = S(X^+)$ and $V = S(X^-)$ in (3.4), we find from the property (c) of S that the positive semidefinite matrix $P := X^+ + X^-$ satisfies $m \leq \operatorname{rank} S(P)$. This proves (b). □

We now prove the first main result.

THEOREM 3.2. *Suppose* $S : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ *is a nilpotent transformation with property* (c). *Then* $I - S$ *has the* P*-property.*

Proof. We prove the result by induction on n. If $n = 2$, the result is true (see Theorem 13 in [2]). For $k < n$, we will assume that the result holds and now we prove for $k = n$. Let $Q_0 \in \mathbb{S}^{n\times n}$ be such that

$$\operatorname{rank} S(Q_0) \geq \operatorname{rank} S(Q) \quad \text{for all } Q \in \mathbb{S}^{n\times n}.$$

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In view of Item (b) in Lemma 3.1, without any loss of generality, we assume $Q_0 \succeq 0$. If $\widehat{k} = \operatorname{rank} S(Q_0)$, then Item (a) of Lemma 3.1 implies $\widehat{k} < n$. There exists an orthogonal matrix U such that

$$US(Q_0)U^T = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix},$$

$D \in \mathbb{S}^{\widehat{k}\times\widehat{k}}$ being diagonal and nonsingular. Define $\widetilde{S} : \mathbb{S}^{n\times n} \to \mathbb{S}^{n\times n}$ by

$$\widetilde{S}(X) := US(U^T XU)U^T.$$

If $\widehat{Q}_0 = UQ_0U^T$, then

$$\widetilde{S}(\widehat{Q}_0) = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}.$$

By an easy verification, we find that $\widetilde{S}$ is nilpotent and has property (c). Further,

$$\operatorname{rank} \widetilde{S}(\widehat{Q_0}) \geq \operatorname{rank} \widetilde{S}(Q) \quad \text{for all } Q \in \mathbb{S}^{n\times n}. \tag{3.5}$$

We now claim that for any $X \in \mathbb{S}^{n\times n}$,

$$\widetilde{S}(X) = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{for some } E \in \mathbb{S}^{\widehat{k}\times\widehat{k}}. \tag{3.6}$$

Let $Q \succeq 0$ and $F := \widetilde{S}(Q)$. As $F = [f_{ij}] \succeq 0$, $f_{ii} = 0$ if and only if the ith column of F is zero. Suppose $f_{ii} > 0$ for some $i > \widehat{k}$. Then

$$\operatorname{rank} \widetilde{S}(\widehat{Q_0} + Q) = \operatorname{rank}(\widetilde{S}(\widehat{Q}_0) + \widetilde{S}(Q)) \geq \widehat{k} + 1 > \widehat{k}.$$

Thus, we have $\operatorname{rank} \widetilde{S}(\widehat{Q}_0 + Q) > \operatorname{rank} \widetilde{S}(\widehat{Q}_0)$ which is a contradiction to (3.5). So, for any $Q \succeq 0$,

$$\widetilde{S}(Q) = \begin{bmatrix} E' & 0 \\ 0 & 0 \end{bmatrix}, \qquad E' \in \mathbb{S}^{\widehat{k}\times\widehat{k}}.$$

Since for any $X \in \mathbb{S}^{n\times n}$, $\widetilde{S}(X) = \widetilde{S}(X^+) - \widetilde{S}(X^-)$, using the c-property of $\widetilde{S}$, we see that (3.6) holds.

Let $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$ $(X_1 \in \mathbb{S}^{\widehat{k}\times\widehat{k}})$ be such that $X(X - \widetilde{S}(X)) \preceq 0$. If

$$\widetilde{S}(X) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}.$$

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Then from

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \left(\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} - \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \right) \preceq 0, \tag{3.7}$$

it follows that $X_2^T X_2 + X_3^2 \preceq 0$, and therefore, X_2 and X_3 are zero matrices. So, $F = \widetilde{S}_{\alpha\alpha}(X_1)$, where $\alpha = \{1, \ldots, \widehat{k}\}$. From (3.7) we now have

$$X_1(X_1 - \widetilde{S}_{\alpha\alpha}(X_1)) \preceq 0. \tag{3.8}$$

We next claim that $\widetilde{S}_{\alpha\alpha}$ has the property (c). Let $X_0 \in \mathbb{S}^{\widehat{k}\times\widehat{k}}$ be positive semidefinite and

$$Y_0 = \widetilde{S}\left(\begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Since $\widetilde{S}$ has property (c), Y_0 is a positive semidefinite matrix. Noticing that $\widetilde{S}_{\alpha\alpha}(X_0)$ is a leading principal submatrix of Y_0, we conclude $\widetilde{S}_{\alpha\alpha}(X_0)$ is positive semidefinite. This proves our claim.

Now we assert that $\widetilde{S}_{\alpha\alpha}$ is nilpotent. Since $\widetilde{S}_{\alpha\alpha}$ has property (c), $r := \rho(\widetilde{S}_{\alpha\alpha})$ is an eigenvalue of $\widetilde{S}_{\alpha\alpha}$. Let $X_0 \in$ be a nonzero matrix in $\mathbb{S}^{\widehat{k}\times\widehat{k}}$ such that

$$\widetilde{S}_{\alpha\alpha}(X_0) = rX_0.$$

In view of (3.6) and the definition of $\widetilde{S}_{\alpha\alpha}$,

$$\widetilde{S}\left(\begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} rX_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, r is an eigenvalue of $\widetilde{S}$. Since $\widetilde{S}$ is nilpotent, $r = 0$. Thus, $\widetilde{S}_{\alpha\alpha}$ is nilpotent.

By our induction assumption, $I - \widetilde{S}_{\alpha\alpha}$ must have P-property and hence from (3.8), $X_1 = 0$; thus, $X = 0$. This proves that $I - \widetilde{S}$ has the P-property. It is easy to see that $I - S$ has the P-property if and only if $I - \widetilde{S}$ has the P-property. The proof is now complete. ☐

Corollary 3.3. *Let $\{A_1, \ldots, A_\nu\}$ be a finite set of $n \times n$ nilpotent matrices. Assume that $A_iA_j = A_jA_i$ for all i and each A_i is nilpotent. Then the transformation $X - \sum_{i=1}^{\nu} A_iXA_i^T$ has the P-property.*

Proof. Let $M_{A_i}(X) = A_iXA_i^T$. Then, using $A_iA_j = A_jA_i$, we verify that $M_{A_i}M_{A_j} = M_{A_j}M_{A_i}$. Now it is easy to see that $\sum_{i=1}^{n} M_{A_i}$ is nilpotent, and hence, $X - \sum_{i=1}^{\nu} A_iXA_i^T$ has the P-property. ☐

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3.2. Case 2. Now we shall show that if a matrix A is positive stable and $\rho(L_A^{-1}) < 1$, then $I - L_A^{-1}$ has the P-property. Note that by Lyapunov theorem (cf. Theorem 6 [3]), L_A^{-1} will have the property (c). Hence, $I - L_A^{-1}$ is a Stein-type transformation and satisfy all the items in Theorem 1.1. Before proving the main result, we will prove some intermediate lemmas.

LEMMA 3.4. *Let A be a positive stable matrix of order n and $\rho(L_A^{-1}) < 1$. Then*

1. trace $A > \frac{n}{2}$.
2. *If there exist a nonsingular X and $Y := X - L_A^{-1}(X)$ such that $XY = YX$ and $XY \preceq 0$, then X must be indefinite.*

Proof. If λ is an eigenvalue of A, then it is straightforward to verify that $\lambda+\lambda^*$ is an eigenvalue of L_A. In other words, $2\,\mathrm{Re}(\lambda)$ is an eigenvalue of the linear transformation L_A. Our assumptions on A now imply that

$$0 < \frac{1}{2\,\mathrm{Re}(\lambda)} < 1,$$

and hence, $\mathrm{Re}(\lambda) > \frac{1}{2}$. As A is a real matrix, we now deduce that the sum of all the eigenvalues of A is greater than $\frac{n}{2}$. This proves 1.

Suppose $X \succeq 0$ is a nonsingular matrix such that $XY \preceq 0$. Because $XY = YX$, there exists an orthogonal matrix U such that $X = UDU^T$ and $Y = UEU^T$, where D and E are diagonal matrices and now $XY \preceq 0$ implies that

$$DE \preceq 0. \tag{3.9}$$

The matrix D must be positive definite as X is a nonsingular positive semidefinite matrix and by (3.9), we conclude $E \preceq 0$; hence,

$$Y \preceq 0.$$

This means that $X - L_A^{-1}(X) \preceq 0$. The matrix A is positive stable, and hence by Lyapunov theorem $I - L_A^{-1}$, is a **Z**-transformation. From the assumption $\rho(L_A^{-1}) < 1$, it follows from Item 2 of Theorem 1.1 that

$$(I - L_A^{-1})(X) \preceq 0 \quad \Longrightarrow \quad X \preceq 0.$$

Therefore, X cannot be positive semidefinite. This is a contradiction.

In a similar manner, it follows that X cannot be negative semidefinite. This proves 2. ∎

LEMMA 3.5. *If A is positive stable and $\rho(L_A^{-1}) < 1$, then*

1. *There does not exist a nonsingular matrix X commuting with $Y := X - L_A^{-1}(X)$, such that $XY \preceq 0$.*

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2. *If X is either positive semidefinite or negative semidefinite and if $Y := X - L_A^{-1}(X)$ is such that $XY = YX$, then*

$$XY \preceq 0 \Rightarrow X = 0.$$

Proof. Let X be a nonsingular matrix such that $XY = YX$ and $XY \preceq 0$, where $Y := X - L_A^{-1}(X)$. In view of previous lemma, X must be indefinite.

As $XY = YX$ and $XY \preceq 0$, there is an orthogonal matrix U such that

$$UXU^T = \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix}, \qquad UYU^T = \begin{bmatrix} -F & 0 \\ 0 & G \end{bmatrix},$$

where the matrices D and E are positive definite; F and G are positive semidefinite. Further D, E, F, and G are diagonal. Note that $X - Y = L_A^{-1}(X)$, and thus, $X = L_A(X - Y)$. We now have

$$(3.10)\qquad \left.\begin{aligned} \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix} = UXU^T &= UL_A(X-Y)U^T \\ &= UL_A(U^T(U(X-Y)U^T)U)U^T \\ &= UL_A(U^T \begin{bmatrix} D+F & 0 \\ 0 & -E-G \end{bmatrix} U)U^T. \\ &= L_{UAU^T}(\begin{bmatrix} D+F & 0 \\ 0 & -E-G \end{bmatrix}). \end{aligned}\right\}$$

Let d_i, e_i, f_i and g_i be the diagonal entries of D, E, F and G, respectively. Assume that the order of D and F is ν. If $\alpha_{11}, \alpha_{22}, \ldots, \alpha_{nn}$ are the diagonal entries of UAU^T, then we find from the above equations that

$$\alpha_{kk} = \begin{cases} \dfrac{d_k}{2(d_k + f_k)} & \text{if } k = 1, \ldots, \nu \\ \dfrac{e_k}{2(e_k + g_k)} & \text{if } k = \nu + 1, \ldots, n. \end{cases}$$

Thus, $\operatorname{trace} A = \operatorname{trace}(UAU^T) \leq \frac{n}{2}$. This contradicts Lemma 3.4. Therefore item 1 is proved.

The proof of item 2 follows easily by replacing $E = 0$ in the above. ◻

THEOREM 3.6. *Let A be an $n \times n$ positive stable matrix with real entries. If L_A is the corresponding Lyapunov transformation then the following are equivalent:*

(i) $\rho(L_A^{-1}) < 1$.
(ii) $I - L_A^{-1}$ *has the P-property.*

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Proof. Since $I - L_A^{-1}$ is a Stein-type-transformation, (ii) $\Rightarrow$ (i) follows immediately from the fact that $\rho(L_A^{-1})$ is an eigenvalue of L_A^{-1}. We now prove (i) $\Rightarrow$ (ii).

Let X be such that

$$X(X - L_A^{-1}(X) \preceq 0.$$

Put $Y := X - L_A^{-1}(X)$. In view of Lemma 3.4 and Lemma 3.5, we see that X must be indefinite and X is singular. Since X and Y commute and $XY \preceq 0$, there is an orthogonal matrix U such that

$$UXU^T = \begin{bmatrix} D & 0 & 0 \\ 0 & -E & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad UYU^T = \begin{bmatrix} -F & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & L \end{bmatrix},$$

where the matrices D and E are positive definite; F and G are positive semidefinite. Further, D, E, F, G and L are diagonal. Assume that D and E are of order ν_1 and ν_2, respectively.

Now working similarly as in (3.10) of previous lemma, it is easy to show that

$$\begin{bmatrix} D & 0 & 0 \\ 0 & -E & 0 \\ 0 & 0 & 0 \end{bmatrix} = L_{UAU^T}\left(\begin{bmatrix} D+F & 0 & 0 \\ 0 & -E-G & 0 \\ 0 & 0 & -L \end{bmatrix}\right). \tag{3.11}$$

Put $\widetilde{A} = UAU^T$. It is straightforward to verify that $\rho(L_{\widetilde{A}}) = \rho(L_A)$. First we consider the case $L = 0$. We now define two diagonal matrices of order $\nu_1 + \nu_2$ viz.

$$\widetilde{D} := \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix}, \qquad \widetilde{E} := \begin{bmatrix} D+F & 0 \\ 0 & -E-G \end{bmatrix}.$$

It is easy to note that $\widetilde{D}$ and $\widetilde{E}$ are nonsingular.

Let $\widetilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where A_1 is of order $\nu_1 + \nu_2$. Since $L = 0$, from (3.11), we have

$$\begin{bmatrix} \widetilde{D} & 0 \\ 0 & 0 \end{bmatrix} = L_{\widetilde{A}}\left(\begin{bmatrix} \widetilde{E} & 0 \\ 0 & 0 \end{bmatrix}\right).$$

From the above equation, we have

$$\begin{bmatrix} \widetilde{D} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1\widetilde{E} + \widetilde{E}A_1^T & \widetilde{E}A_3^T \\ A_3\widetilde{E} & 0 \end{bmatrix}; \tag{3.12}$$

hence, $A_3\widetilde{E} = 0$. The matrix $\widetilde{E}$ must be nonsingular and therefore $A_3 = 0$. Thus, every eigenvalue of A_1 must be an eigenvalue of A and so A_1 is positive stable. We

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claim that $r := \rho(L_{A_1}^{-1}) < 1$. Since A_1 is positive stable, $L_{A_1}^{-1}$ will have the property (c) (by Lyapunov theorem) and so r is an eigenvalue of $L_{A_1}^{-1}$. Let V be such that $L_{A_1}^{-1}(V) = rV$. Let $\widetilde{V}$ be the $n \times n$ matrix defined by

$$\widetilde{V} = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that $L_{\widetilde{A}}^{-1}(\widetilde{V}) = r\widetilde{V}$ and since $\rho(L_A^{-1}) = \rho(L_{\widetilde{A}}^{-1}) < 1$, we deduce $r < 1$.

From (3.12), we have $\widetilde{D} = A_1\widetilde{E} + \widetilde{E}A_1^T$, and thus, $L_{A_1}^{-1}(\widetilde{D}) = \widetilde{E}$. Now we have

$$\begin{aligned} \widetilde{D}(\widetilde{D} - L_{A_1}^{-1}(\widetilde{D})) &= \widetilde{D}(\widetilde{D} - \widetilde{E}) \\ &= \begin{bmatrix} D & 0 \\ 0 & -E \end{bmatrix} \begin{bmatrix} -F & 0 \\ 0 & G \end{bmatrix} \\ &\preceq 0. \end{aligned}$$

Thus, $\widetilde{D}$ is a nonsingular matrix such that $\widetilde{D}$ and $\widetilde{D} - L_{A_1}^{-1}(\widetilde{D})$ commute and $\widetilde{D}(\widetilde{D} - L_{A_1}^{-1}(\widetilde{D})) \preceq 0$. This contradicts the previous lemma.

We now consider the case where L is nonzero. First assume L is nonsingular. Since L is a diagonal matrix, the diagonal entries of L must be nonzero now. In this case using (3.11), we compute the diagonal entries α_{kk} of $\widetilde{A}$:

$$\alpha_{kk} = \begin{cases} \dfrac{d_k}{2(d_k + f_k)} & \text{if } k = 1, \ldots, \nu_1 \\ \dfrac{e_k}{2(e_k + g_k)} & \text{if } k = \nu_1 + 1, \ldots, \nu_1 + \nu_2 \\ 0 & \text{if } k > \nu_1 + \nu_2. \end{cases}$$

Now it is easy to see that trace $\widetilde{A} \leq \frac{1}{2}(\nu_1 + \nu_2) < \frac{n}{2}$ which contradicts Lemma 3.4.

Finally, we consider the case L is singular but nonzero. In this case, we can write UXU^T and UYU^T as follows:

$$UXU^T = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad UYU^T = \begin{bmatrix} -F & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & L_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the matrix L_1 is nonsingular. Suppose the order of L_1 is ν_3. Let the matrix $\widetilde{A}$ be partitioned conformally (as above in UXU^T and UYU^T) into

$$\widetilde{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

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Following the same arguments as above, we see that A_{41}, A_{42}, and A_{43} are zero matrices. Further if $\widehat{A}$ is the $(\nu_1+\nu_2+\nu_3)\times(\nu_1+\nu_2+\nu_3)$ leading principal submatrix of A, then we see that

$$\begin{bmatrix} D & 0 & 0 \\ 0 & -E & 0 \\ 0 & 0 & 0 \end{bmatrix} = L_{\widehat{A}}\left(\begin{bmatrix} -F & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & L_1 \end{bmatrix}\right), \tag{3.13}$$

$\widehat{A}$ is positive stable and $\rho(L_{\widehat{A}}^{-1}) < 1$. Invoking Lemma 3.4, we find that

$$\operatorname{trace}\widehat{A} > \frac{1}{2}(\nu_1+\nu_2+\nu_3).$$

However, calculating the trace of $\widehat{A}$ by finding the sum of all the diagonal entries of $\widehat{A}$ from (3.13), we see that

$$\operatorname{trace}\widehat{A} \leq \frac{1}{2}(\nu_1+\nu_2).$$

This is a contradiction. The proof is now complete. □

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