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EXPONENTS AND DIAMETERS OF STRONG PRODUCTS OF DIGRAPHS*

BYEONG MOON KIM†, BYUNG CHUL SONG†, AND WOONJAE HWANG‡

Abstract. The exponent of the strong product of a digraph of order m and a digraph of order n is shown to be bounded above by $m+n-2$, with equality for $Z_m \boxtimes Z_n$. The exponent and diameter of the strong product of a graph and a digraph are also investigated.

Key words. Strong product of digraphs, Exponent, Diameter.

AMS subject classifications. 15A48, 05C50, 05C76.

1. Introduction. Let $D = (V, A)$ be a digraph on n vertices. Throughout this paper, we assume that D has no loops and no multiple arcs. A walk from u to v in D is a sequence $u = u_0, u_1, \ldots, u_k = v$ of vertices such that there is an arc from u_i to u_{i+1} in D for each i. We denote the walk by $u \to u_1 \to u_2 \to \cdots \to u_{k-1} \to v$ and its length is k. We use the notation $u \xrightarrow{k} v$ when there exits a walk in D of length k from u to v. The digraph is *primitive* if there is a k such that $u \xrightarrow{k} v$ for each pair of vertices u and v. Conventionally $u \xrightarrow{0} u$ is permitted. We say that the smallest such value of k is the *exponent* of D, which is denoted by $\exp(D)$. Wielandt [9] found that the maximum exponent of a primitive digraph on n vertices is $W_n = n^2 - 2n + 2$. See [1] for more details. Suppose that two digraphs $D = (V_D, A_D)$ and $E = (V_E, A_E)$ are given. Let $V = V_D \times V_E$. We define

$$\begin{aligned} A_1 = \{((u_1, u_2), (v_1, v_2)) \in V \times V | &((u_1, v_1) \in A_D \text{ and } u_2 = v_2) \\ &\text{or } ((u_2, v_2) \in A_E \text{ and } u_1 = v_1)\}, \end{aligned}$$

and

$$A_2 = \{((u_1, u_2), (v_1, v_2)) \in V \times V | (u_1, v_1) \in A_D \text{ and } (u_2, v_2) \in A_E\}.$$

The *strong product* $D \boxtimes E$ of D and E is the digraph $(V, A_1 \cup A_2)$. The *Cartesian product* $D \times E$ and the *direct product* $D \otimes E$ of D and E are defined by (V, A_1)

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†Department of Mathematics, Gangneung-Wonju National University, Gangneung 210-702, Korea (kbm@gwnu.ac.kr, bcsong@gwnu.ac.kr).

‡Department of Information and Mathematics, Korea University, Jochiwon 339-700, Korea (woonjae@korea.ac.kr).

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and (V, A_2) respectively. The strong product of graphs is used to define the concept of Shannon capacity which plays an important role in the information theory [8]. Definitions and related results on the product of graphs are provided in [2]. In 1979, Lamprey and Barnes [6] showed that $\exp(D \times E) \leq (n+m)^2 - 4(n+m) + 5$ for digraphs D and E on n and m vertices, respectively. They also showed $\exp(D \otimes E) = \max\{\exp(D), \exp(E)\}$ for primitive digraphs D and E. In 1987, Kwasnik [5] studied the exponent of other types of products such as the disjunction and lexicographic products of graphs. Recently, it has been proved in [3] that if D and E are digraphs on m and n vertices, respectively, and $D \times E$ is primitive, then $\exp(D \times E) \leq mn - 1$. In [3] it was also showed that $\exp(G \times D) = \exp(G) + \operatorname{diam}(D)$ for a primitive graph G and a strongly connected bipartite digraph D, and they computed the exponent of the Cartesian product of two cycles [4]. In this paper, we show

$$\exp(D \boxtimes E) \leq n + m - 2 \tag{1.1}$$

for strongly connected digraphs D and E on n and m vertices, respectively. Let Z_n and Z_m be the directed cycles of order n and m respectively. We also prove that

$$\exp(Z_n \boxtimes Z_m) = n + m - 2.$$

As a consequence, the bound in (1.1) is tight. A graph G is considered as a digraph by treating the edges of G as bidirectional. In particular, a cycle C_n of length n is considered as a digraph in the same manner. For a connected graph G and a strongly connected digraph E, we show $\exp(G \boxtimes E)$ is $\operatorname{diam}(G \boxtimes E)$ or $\operatorname{diam}(G \boxtimes E) + 1$, and we find the condition under which the latter case holds. As a consequence, we compute $\exp(C_n \boxtimes Z_m)$.

2. Upper bound on the exponent of strong products of two digraphs.

LEMMA 1. *Let D and E be digraphs, $u, v \in V_D$ and $z, w \in V_E$. If $u \xrightarrow{t} v$ in D and $z \xrightarrow{s} w$ in E, then $(u, z) \xrightarrow{\alpha} (v, w)$ in $D \boxtimes E$ for all α with* $\max\{t, s\} \leq \alpha \leq t+s$.

Proof. We may assume that $t \leq s$. Let $u \to u_0 \to u_1 \to \cdots \to u_t = v$ in D and $z = z_0 \to z_1 \to \cdots \to z_s = w$ in E. If $i = \alpha - s$ for $0 \leq i \leq t$, then $(u, z) = (u_0, z_0) \to (u_1, z_1) \to \cdots \to (u_{t-i}, z_{t-i}) \to (u_{t-i}, z_{t-i+1}) \to \cdots \to (u_{t-i}, z_s) \to (u_{t-i+1}, z_s) \to \cdots \to (u_t, z_s) = (v, w)$ is a walk of length $t - i + s - (t-i) + t - (t-i) = s + i = \alpha$ in $D \boxtimes E$. ▯

LEMMA 2. *Let D and E be strongly connected digraphs, $u, v \in V_D$ and $z, w \in V_E$. If there are a cycle C passing through v of length k in D, $u \xrightarrow{t} v'$ in D for some vertex v' of C, and $z \xrightarrow{s} w$ in E for some $s \geq k-1$, then, for all α with* $\max\{t+k, s\} \leq \alpha$, *$(u, z) \xrightarrow{\alpha} (v, w)$ in $D \boxtimes E$.*

Proof. Since v and v' are vertices of C, $v' \xrightarrow{l} v$ in D for some l with $0 \leq l \leq k-1$. So $t + l \leq t + k$. Since $\alpha - t - l \geq \alpha - t - k \geq 0$, there is a $q \geq 0$ such that

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ELA
http://math.technion.ac.il/iic/ela

$kq \leq \alpha - t - l \leq kq + k - 1$. Since $u \xrightarrow{t} v' \xrightarrow{l} v \xrightarrow{kq} v$, $u \xrightarrow{kq+t+l} v$ in D. Since $z \xrightarrow{s} w$ in E and $\max\{kq+t+l, s\} \leq \alpha \leq kq+t+l+k-1 \leq kq+t+l+s$, by Lemma 1, $(u,z) \xrightarrow{\alpha} (v,w)$ in $D \boxtimes E$. $\square$

Theorem 1. *Let D and E be strongly connected digraphs on n and m vertices $(n, m \geq 2)$, respectively. Then $D \boxtimes E$ is primitive and*

$$\exp(D \boxtimes E) \leq n + m - 2.$$

Proof. It suffices to show that for each pair of vertices $(u,z), (v,w) \in D \boxtimes E$ and for each $\alpha \geq n+m-2$, we have $(u,z) \xrightarrow{\alpha} (v,w)$ in $D \boxtimes E$. Let k be the minimum length of the cycles in D passing through v, and C be one such cycle. Let t be the distance from u to C in D. Then $k + t \leq n$.

Let l be the minimum length of the cycle in E passing through w and s be the distance from z to w. Then $l, s \leq m$. If $s \geq k-1$, then, by $\alpha \geq n \geq t+k$ and $\alpha \geq m \geq s$, Lemma 2 implies that $(u,z) \xrightarrow{\alpha} (v,w)$. If $s < k-1$, then there is a $q \geq 0$ such that $lq < k-s-1 \leq l(q+1)$. Then $l(q+1)+s = l+lq+s \leq l+k-2 \leq m+k-2 \leq n+m-2 \leq \alpha$. Since $z \xrightarrow{s} w \xrightarrow{l(q+1)} w$, $z \xrightarrow{l(q+1)+s} w$. By Lemma 2, $(u,z) \xrightarrow{\alpha} (v,w)$. $\square$

Theorem 2. *For $n, m \geq 2$,*

$$\exp(Z_n \boxtimes Z_m) = n + m - 2.$$

Proof. Let Z_n be a directed cycle $v_0 \to v_1 \to \cdots \to v_{n-1} \to v_0$ and Z_m be a directed cycle $w_0 \to w_1 \to \cdots \to w_{m-1} \to w_0$. We may assume $n \leq m$. Let r be the residue of $m-2$ modulo n. Suppose $(v_0, w_0) \xrightarrow{n+m-3} (v_r, w_{n-2})$. Let $(v_0, w_0) = (x_0, y_0) \to (x_1, y_1) \to \cdots \to (x_{n+m-3}, y_{n+m-3}) = (v_r, w_{n-2})$ be a path in $Z_n \boxtimes Z_m$ from (v_0, w_0) to (v_r, w_{n-2}). Then there are $i_0 < i_1 < \cdots < i_s$ and $j_0 < j_1 < \cdots < j_t$ such that $i_0 = j_0 = 0$, for all p, q with $0 \leq p \leq s-1$ and $0 \leq q \leq t-1$, $x_{i_p} = x_{i_p+1} = \cdots = x_{i_{p+1}-1} \neq x_{i_{p+1}}$, $y_{j_q} = y_{j_q+1} = \cdots = y_{j_{q+1}-1} \neq y_{j_{q+1}}$ and $x_{i_s} = x_{i_s+1} = \cdots = x_{n+m-3}$, $y_{j_t} = y_{j_t+1} = \cdots = y_{n+m-3}$. Then for all $i = 0, 1, \ldots, n+m-3$, $(x_i, y_i) = (x_{i_p}, y_{j_q})$ for some p and q. If $0 \leq i \leq n+m-4$ and $(x_i, y_i) = (x_{i_p}, y_{j_q})$, since $(x_{i+1}, y_{i+1}) \neq (x_i, y_i)$, $x_{i+1} = x_{i_{p+1}}$ or $y_{i+1} = y_{j_{q+1}}$. So if $(x_i, y_i) = (x_{i_p}, y_{j_q})$, we can show $i \leq p+q$, by induction. If $x_{i_p} = v_l$ and $l \neq n-1$, since $(x_{i_p}, x_{i_{p+1}}) \in A_{Z_n}$, $x_{i_{p+1}} = v_{l+1}$. If $x_{i_p} = v_{n-1}$, since $x_{i_p} = v_{n-1} \to x_{i_{p+1}}$, $x_{i_{p+1}} = v_0$. Since $x_{i_0} = v_0$, we can show by induction that if $x_{i_p} = v_l, p \equiv l \pmod{n}$. Similarly, we can show that if $y_{j_q} = w_k$, $q \equiv k \pmod{m}$. Since $y_{j_t} = w_{n-2}$, $t \equiv n-2 \pmod{m}$. Since $t \leq n+m-3$ and $n \leq m$, $t = n-2$. Since $x_{i_s} = v_r$, $s \equiv r \pmod{n}$. So $s \equiv r \equiv m-2 \pmod{n}$.

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ELA
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Since $s \leq n+m-3$, $s \leq m-2$. So $n+m-3 \leq s+t \leq (m-2)+(n-2) = n+m-4$. This is a contradiction. So $(v_0, w_0) \overset{n+m-3}{\nrightarrow} (v_r, w_{n-2})$. Thus, using Theorem 1, $\exp(Z_n \boxtimes Z_m) = n+m-2$. ☐

3. Exponents and diameters of strong products of digraphs. For any $u, v \in V_D$, the *distance* $\operatorname{dist}(u,v)$ from u to v is the smallest k such that there is a walk from u to v of length k. The *diameter* $\operatorname{diam}(D)$ of the strongly connected digraph D is the maximum of $\operatorname{dist}(u,v)$ for all $u, v \in V_D$.

Proposition 1. *If D and E are strongly connected digraphs, then*

$$\operatorname{diam}(D \boxtimes E) = \max\{\mathrm{diam(D)}, \mathrm{diam(E)}\}.$$

Proof. If $u, v \in V_D$ and $z, w \in V_E$, then by Lemma 1, we have

$$\operatorname{dist}((u,z),(v,w)) \leq \max\{\operatorname{dist}(u,v), \operatorname{dist}(z,w))\}.$$

Thus, $\operatorname{diam}(D \boxtimes E) \leq \max\{\operatorname{diam}(D), \operatorname{diam}(E))\}$.

Conversely, if $u, v \in V_D$, $z, w \in V_E$, and $\operatorname{dist}((u,z),(v,w)) = \alpha$, then $(u,z) = (u_0, z_0) \to (u_1, z_1) \to \cdots \to (u_\alpha, z_\alpha) = (v, w)$ for some $(u_i, z_i) \in V_{D\boxtimes E}$ where $i = 1, 2, \ldots, \alpha$. Thus, there are $0 = i_0 < i_1 < i_2 < \cdots < i_s \leq \alpha$ such that $u_{i_p} = u_{i_p+1} = \cdots = u_{i_{p+1}-1} \neq u_{ip+1}$ for all $p = 0, 1, \ldots, s-1$. Since $u = u_{i_0} \to u_{i_1} \to \cdots \to u_{i_s} = u_\alpha = v$, $\operatorname{dist}(u,v) \leq s \leq \alpha = \operatorname{dist}((u,z),(v,w))$. So $\operatorname{diam}(D) \leq \operatorname{diam}(D \boxtimes E)$. Similarly, $\operatorname{diam}(E) \leq \operatorname{diam}(D \boxtimes E)$. Thus, $\operatorname{diam}(D \boxtimes E) = \max\{\operatorname{diam}(D), \operatorname{diam}(E))\}$. ☐

Lemma 3. *Let G be a connected graph and D be a strongly connected digraph. If $(u,z), (v,w) \in V_{G\boxtimes D}$, $((u,z),(v,w)) \in A_{G\boxtimes D}$ and $z \neq w$, then $(u,z) \xrightarrow{k} (v,w)$ in $G \boxtimes D$ for all $k \geq 1$.*

Proof. Since $z \neq w$, $(z,w) \in A_D$. Since G is connected, there is $x \in V_G$ such that $\{u, x\} \in E_G$. Since $(u,z) \to (x,z) \to (u,z)$, $(u,z) \xrightarrow{2t} (u,z)$ for all $t \geq 0$. If $u = v$, since $(u,z) \xrightarrow{2t} (u,z) \xrightarrow{1} (u,w)$ and $(u,z) \xrightarrow{2t} (u,z) \xrightarrow{1} (x,z) \xrightarrow{1} (u,w)$, $(u,z) \xrightarrow{2t+1} (u,w) = (v,w)$ and $(u,z) \xrightarrow{2t+2} (u,w) = (v,w)$ for all $t \geq 0$. If $u \neq v$, $u \to v$. Since $(u,z) \xrightarrow{2t} (u,z) \xrightarrow{1} (v,w)$ and $(u,z) \xrightarrow{2t} (u,z) \xrightarrow{1} (u,w) \xrightarrow{1} (v,w)$, $(u,z) \xrightarrow{2t+1} (v,w)$ and $(u,z) \xrightarrow{2t+2} (v,w)$ for all $t \geq 0$. ☐

Theorem 3. *If G is a connected graph and D is a strongly connected digraph such that $|V_G| \geq 2$ and $|V_D| \geq 2$, then $\exp(G \boxtimes D)$ is $\mathrm{diam(G\boxtimes D)}$ or $\mathrm{diam(G\boxtimes D)}+1$. Moreover, $\exp(G \boxtimes D) = \mathrm{diam(G\boxtimes D)}+1$ if and only if G and D satisfy the following:*

1. $\mathrm{diam(D)} \geq \mathrm{diam(G)}$,

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ELA
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2. *there is* $v \in V_D$ *such that* $v \overset{l}{\nrightarrow} v$ *for all* $l = 1, 2, \ldots, \mathrm{diam(D)}$,
3. *either* G *is not primitive or* G *is primitive and* $\exp(G) > \mathrm{diam(D)}$.

Proof. Let $\mathrm{diam}(G) = m$ and $\mathrm{diam}(D) = n$. For all $u, v \in V_D$, there is $u' \in V_D$ such that $(u, u') \in A_D$. Since $\mathrm{diam}(D) = n$, $u' \xrightarrow{t} v$ for some $t \le n$. For all $x, y \in V_G$, $x \xrightarrow{s} y$ for some $s \le m$. If $\alpha = \max\{m, n\}$, by Lemma 1, $(x, u') \xrightarrow{l} (y, v)$ for some $l \le \alpha$. If $k \ge \alpha + 1$, by Lemma 3, $(x, u) \xrightarrow{k-l} (x, u')$. Since $(x, u) \xrightarrow{k-l} (x, u') \xrightarrow{l} (y, v)$, $(x, u) \xrightarrow{k} (y, v)$. So $\exp(G \boxtimes D) \le \alpha + 1 = \mathrm{diam}(G \boxtimes D) + 1$. Since $\mathrm{diam}(G \boxtimes D) \le \exp(G \boxtimes D)$, $\exp(G \boxtimes D)$ is $\mathrm{diam}(G \boxtimes D)$ or $\mathrm{diam}(G \boxtimes D) + 1$.

If $\exp(G\boxtimes D) = \alpha+1$, then there are $(x, u), (y, v) \in V_{G\boxtimes D}$ such that $(x, u) \overset{\alpha}{\nrightarrow} (y, v)$ in $G \boxtimes D$. If $u \xrightarrow{l} v$ for some l with $1 \le l \le \alpha$, then there is $u' \in V_D$ such that $(u, u') \in A_D$ and $u' \xrightarrow{l-1} v$. If $x \ne y$, then there is $x' \in V_G$ such that $\{x, x'\} \in E_G$ and $x' \xrightarrow{s-1} y$ where $s = \mathrm{dist}(x, y)$. If $\max\{s, l\} = p$, then $p \le \alpha$. By Lemmas 1 and 3, $(x', u') \xrightarrow{p-1} (y, v)$ and $(x, u) \xrightarrow{\alpha-p+1} (x', u')$. So $(x, u) \xrightarrow{\alpha} (y, v)$. This is a contradiction. If $x = y$, by Lemma 3, $(x, u) \xrightarrow{\alpha-l+1} (x, u')$. Since $(x, u) \xrightarrow{\alpha-l+1} (x, u') \xrightarrow{l-1} (x, v)$, $(x, u) \xrightarrow{\alpha} (x, v) = (y, v)$. This is a contradiction. So $u \overset{l}{\nrightarrow} v$ for all l such that $1 \le l \le \alpha$. If $u \ne v$, let $d = \mathrm{dist}(u, v)$. Then $1 \le d \le \alpha$ and $u \overset{d}{\nrightarrow} v$. This is a contradiction. So $u = v$ and G and D satisfy condition (2).

Since D is strongly connected, there is $\tilde{u} \in V_D$ such that $(\tilde{u}, u) \in A_D$. If $\mathrm{dist}(u, \tilde{u}) = r \le \alpha$, since $u \xrightarrow{r+1} u$, $r + 1 \ge \alpha + 1$. Since $r \le n \le \alpha$, $r = n = \alpha$. So $n \ge m$. Thus, G and D satisfy condition (1). Since $(x, u) \overset{\alpha}{\nrightarrow} (y, u)$, $x \overset{\alpha}{\nrightarrow} y$ in G. So G and D satisfy condition (3).

Conversely, if G and D satisfy conditions (1), (2) and (3), then there is $u \in V_D$ such that $u \overset{l}{\nrightarrow} u$ for all l such that $1 \le l \le n = \alpha$, and there are $x, y \in V_G$ such that $x \overset{\alpha}{\nrightarrow} y$ in G. If $(x, u) \xrightarrow{\alpha} (y, u)$, then $(x, u) = (x_0, u_0) \to (x_1, y_1) \to \cdots \to (x_\alpha, y_\alpha) = (y, u)$ for some $x_0, x_1, \ldots, x_\alpha \in V_G$ and $u_0, u_1, \ldots, u_\alpha \in V_D$. If $u_0 = u_1 = \cdots = u_\alpha = u$, since $x = x_0 \to x_1 \to \cdots \to x_\alpha = y$, $x \xrightarrow{\alpha} y$. This is a contradiction. If $u_i \ne u$ for some i, there are $0 = i_0 < i_1 < \cdots < i_s \le \alpha$ such that $u_{i_p} = u_{i_p+1} = \cdots = u_{i_{p+1}-1} \ne u_{i_{p+1}}$ for all $p = 0, 1, \ldots, s-1$ and $u_{i_s} = u_{i_s+1} = \cdots = u_\alpha$. Since $u = u_{i_0} \to u_{i_1} \to \cdots \to u_{i_s} \to u_\alpha = u$, $u \xrightarrow{s} u$. Since $u_i \ne u$ for some i, $1 \le s \le \alpha$. This is a contradiction. So $(x, u) \overset{\alpha}{\nrightarrow} (y, u)$. Thus, $\exp(G \boxtimes D) = \alpha + 1$. ◻

Corollary 1. *If* G *and* H *are connected graphs, then*

$$\exp(G \boxtimes H) = \mathrm{diam(G \boxtimes H)}$$

except when both G *and* H *are complete graphs.*

Proof. If $\exp(G \boxtimes H) = \mathrm{diam}(G \boxtimes H) + 1$, since $v \xrightarrow{2} v$ for all $v \in V_H$, by Theorem 3, $1 \le \mathrm{diam}(G) \le \mathrm{diam}(H) = 1$. So G and H are complete graphs. ◻

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Note that the strong product of two complete graphs is also a complete graph, whose exponent is 2.

Corollary 2.

$$\exp(C_n \boxtimes Z_m) = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \textit{if } n \geq 2m \\ m-1, & \textit{if } n \textit{ is odd and } n \leq m \\ m, & \textit{if } n \textit{ is even and } n \leq 2m-2, \textit{ or } n \textit{ is odd} \\ & \textit{and } m+1 \leq n \leq 2m-1. \end{cases}$$

Proof. If $\exp(C_n \boxtimes Z_m) = \operatorname{diam}(C_n \boxtimes Z_m) + 1$, $\operatorname{diam}(C_n) = \lfloor \frac{n}{2} \rfloor \leq \operatorname{diam}(Z_m) = m-1$. So $n \leq 2m-1$. Moreover, C_n is not primitive, or C_n is primitive and $\exp(C_n) = n-1 > \operatorname{diam}(Z_m) = m-1$. So n is even, or n is odd and $n \geq m+1$. Thus, if n is even, $n \leq 2m-2$. And if n is odd, $m+1 \leq n \leq 2m-1$. In this case, $\exp(C_n \boxtimes Z_m) = (m-1)+1 = m$. Otherwise, $\exp(C_n \boxtimes Z_m) = \operatorname{diam}(C_n \boxtimes Z_m) = \max\{\lfloor \frac{n}{2} \rfloor, m-1\} = \begin{cases} \lfloor \frac{n}{2} \rfloor, & n \geq 2m \\ m-1, & \text{if } n \leq 2m-1. \end{cases}$ ◻

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