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GRAPHS DETERMINED BY THEIR (SIGNLESS) LAPLACIAN SPECTRA*

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Abstract. Let $S(n,c) = K_1 \vee (cK_2 \cup (n-2c-1)K_1)$, where $n \geq 2c+1$ and $c \geq 0$. In this paper, $S(n,c)$ and its complement are shown to be determined by their Laplacian spectra, respectively. Moreover, we also prove that $S(n,c)$ and its complement are determined by their signless Laplacian spectra, respectively.

Key words. Laplacian spectrum, Signless Laplacian spectrum, Complement graph.

AMS subject classifications. 05C50, 15A18, 15A36.

1. Introduction. In this paper, $G = (V,E)$ is an undirected simple graph. The neighbor set of a vertex u is denoted by $N(u)$. Let $d(u)$ be the degree of vertex u, namely, $d(u) = |N(u)|$. If $d(u) = 1$, then u is called a *pendant vertex* of G. Suppose the degree of vertex v_i equals d_i, for $i = 1, 2, \ldots, n$. Throughout this paper, we enumerate the degrees in non-increasing order, i.e., $d_1 \geq d_2 \geq \cdots \geq d_n$. Sometimes we write $d_i(G)$ in place of d_i, in order to indicate the dependence on G. By $v_1v_2 \in E(G)$, we mean an edge, of which the end vertices are v_1 and v_2. Let $G_1 \cup G_2$ denote the (disconnected) graph consisting of two components G_1 and G_2, and kG be the graph consisting of k (where $k \geq 0$ is an integer) copies of the graph G. The *join* $G_1 \vee G_2$ of two disjoint graphs G_1 and G_2 is the graph having vertex set $V(G_1 \vee G_2) = V(G_1 \cup G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. As usual, K_n, P_n and C_n denote the complete graph, path and cycle of order n, respectively. Specially, K_1 denotes an isolated vertex. A graph is a *cactus*, or a *treelike* graph, if any pair of its cycles has at most one common vertex [1, 20]. If all cycles of the cactus G have exactly one common vertex, then G is called a *bundle* [1]. Let $S(n,c)$ be the bundle with n vertices and c cycles of length 3 depicted in Figure 1.1, where $n \geq 2c+1$ and $c \geq 0$. By the definition, it follows that $S(n,c) = K_1 \vee (cK_2 \cup (n-2c-1)K_1)$.

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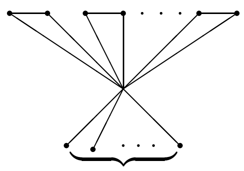

FIG. 1.1. *The bundle* $S(n,c)$.

The *adjacency matrix* $A(G)=[a_{ij}]$ of G is an $n\times n$ symmetric matrix of 0's and 1's with $a_{ij}=1$ if and only if $v_iv_j\in E(G)$. Let $D(G)$ be the diagonal matrix whose (i,i)-entry is d_i, where $1\leq i\leq n$. The *Laplacian matrix* of G is $L(G)=D(G)-A(G)$, and the *signless Laplacian matrix* of G is $Q(G)=D(G)+A(G)$. Sometimes, $Q(G)$ is also called the unoriented Laplacian matrix of G (see, e.g., [10, 22]).

It is well known that $L(G)$ is positive semidefinite so that its eigenvalues can be arranged as follows:

$$\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)=0.$$

Research on the signless Laplacian matrix has recently become popular [3, 5, 10, 22]. It is easy to see that $Q(G)$ is also positive semidefinite [5] and hence its eigenvalues can be arranged as:

$$\mu_1(G)\geq\mu_2(G)\geq\cdots\geq\mu_n(G)\geq 0.$$

If there is no confusion, sometimes we write $\lambda_i(G)$ as λ_i, and $\mu_i(G)$ as μ_i. In the following, let $SL(G)$ and $SQ(G)$ denote the spectra, i.e., eigenvalues of $L(G)$ and $Q(G)$, respectively.

A graph G is said to *be determined by its Laplacian spectrum* (resp. *adjacency spectrum, signless Laplacian spectrum*) if there does not exist a non-isomorphic graph H such that H and G share the same Laplacian spectrum (resp. adjacency spectrum, signless Laplacian spectrum). The question "which graphs are determined by their spectra?" is proposed by van Dam and Haemers in [6]. Up to now, only a few families of graphs are known to be determined by their spectra [6, 9]. For example, the path, the complement of a path, the complete graph, and the cycle were proved to be determined by their adjacency spectra [6, 9]. The path, the complete graph, the cycle, the star and some quasi-star graphs, together with their complement graphs were shown to be determined by their Laplacian spectra [6, 9, 15, 21]. Let K_n^m be the graph obtained by attaching m pendant vertices to a vertex of the complete graph K_{n-m}, and $U_{n,p}$ be the graph obtained by attaching $n-p$ pendant vertices to a vertex of C_p. Recently, Zhang and Zhang in [23] confirmed that K_n^m together with its

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complement are determined by their Laplacian and adjacency spectra, respectively, and $U_{n,p}$ is determined by its Laplacian spectrum. Moreover, they proved that $U_{n,p}$ is determined by its adjacency spectrum if p is odd. Very recently, the authors of [24] showed that $H_{n,p}$, which is obtained by appending a cycle C_p to a pendant vertex of a path P_{n-p}, is determined by its signless Laplacian spectrum.

$S(n,c)$ is an extremal graph in some classes of graphs. For instance, $S(n,c)$ is the graph with the maximal spectral radius, the maximal Merrifield-Simmons index, the minimal Hosoya index, the minimal Wiener index, and the minimal Randić index in the set of all connected cacti on n vertices with c cycles [1, 14]. In this paper, by using a new method different from [6, 9, 15, 21, 23, 24], we show that $S(n,c)$ together with its complement are determined by their Laplacian spectra, and we also prove that $S(n,c)$ together with its complement are determined by their signless Laplacian spectra.

2. $S(n,c)$ and its complement are determined by their Laplacian spectra. The following lemmas are well-known:

LEMMA 2.1. [12, 18] *If G_1 and G_2 are two disjoint graphs on k and m vertices respectively, with Laplacian eigenvalues $0=\lambda_k(G_1)\le\lambda_{k-1}(G_1)\le\cdots\le\lambda_1(G_1)$ and $0=\lambda_m(G_2)\le\lambda_{m-1}(G_2)\le\cdots\le\lambda_1(G_2)$ respectively, then the Laplacian eigenvalues of $G_1\vee G_2$ are given by $0,\lambda_{k-1}(G_1)+m,\ldots,\lambda_1(G_1)+m,\lambda_{m-1}(G_2)+k,\ldots,\lambda_1(G_2)+k$, and $m+k$.*

LEMMA 2.2. [13] *If $G=(V,E)$ is a graph of order n, then $\lambda_1(G)\le n$. Moreover, $\lambda_1(G)=n\ge 2$ if and only if $G=G_1\vee G_2$, where each of G_1 and G_2 has at least one vertex.*

Let $G'=G+e$ be the graph obtained from G by inserting a new edge e into G, and $G-u$ be the graph obtained from G by deleting the vertex $u\in V(G)$ and all the edges adjacent to u. It follows by the Courant–Weyl inequalities [4, Theorem 2.1] that:

LEMMA 2.3. [7] *The Laplacian eigenvalues of G and $G'=G+e$ interlace, that is, $\lambda_1(G')\ge\lambda_1(G)\ge\lambda_2(G')\ge\lambda_2(G)\ge\cdots\ge\lambda_n(G')=\lambda_n(G)=0$.*

LEMMA 2.4. [17, 19] *If G is a graph with n vertices and at least one edge, then $\mu_1(G)\ge\lambda_1(G)\ge d_1(G)+1$. If G is connected, the first equality holds if and only if G is bipartite, the second equality holds if and only if $d_1(G)=n-1$.*

As usual, $K_{s,t}$ denotes the complete bipartite graph with s vertices in one part and t in the other. Specially, $K_{1,n-1}$ denotes the star of order n. By Lemmas 2.1–2.2, it is not difficult to prove that:

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LEMMA 2.5. [15, 21] *$K_{1,n-1}$ is determined by its Laplacian spectrum.*

THEOREM 2.6. *If $c \geq 0$, then $S(n,c)$ is determined by its Laplacian spectrum.*

Proof. If $c = 0$, then $S(n,c) \cong K_{1,n-1}$. By Lemma 2.5, the result follows. In the following, assume that $c \geq 1$. Since $n \geq 2c+1 \geq c+2$, $n = c+2$ if and only if $n = 3$ and $c = 1$. Thus, $n = c+2$ implies that $S(n,c) \cong C_3$, it can be readily checked that C_3 is determined by its Laplacian spectrum [6]. So, we may assume that $c \geq 1$ and $n > c+2$ in the sequel.

By Lemma 2.1 and $SL(K_2) = (2,0)$, we have

$$SL(S(n,c)) = (n, 3, \ldots, 3, 1, \ldots, 1, 0),$$

where the multiplicity of 3 is c, and the multiplicity of 1 is $n-c-2$. Now suppose there exists some graph G, such that $SL(G) = SL(S(n,c))$, then $\lambda_1(G) = n$. By Lemma 2.2, it follows that $G = G_1 \vee G_2$, where G_1 and G_2 are two disjoint graphs with $|V(G_1)| \geq |V(G_2)|$. Since $n > c+2$, we have $\lambda_{n-1}(G) = \lambda_{n-1}(S(n,c)) = 1$.

Next we shall prove that $|V(G_2)| = 1$. Otherwise, if $|V(G_2)| \geq 2$, by Lemmas 2.1 and 2.3, we can conclude that $\lambda_{n-1}(G) \geq \lambda_{n-1}(K_{|V(G_1)|,|V(G_2)|}) = |V(G_2)| \geq 2$, a contradiction. Thus, $|V(G_2)| = 1$ follows. Now suppose $V(G_2) = \{v_0\}$, then $G_1 = G - v_0$. By Lemma 2.1 and $SL(G) = SL(S(n,c))$, then $SL(G_1) = (2, 2, \ldots, 2, 0, 0, \ldots, 0)$, where the multiplicity of 2 is c, and the multiplicity of 0 is $n-c-1$. By Lemma 2.4, we can conclude that $d_1(G_1) = 1$, and hence $G_1 = cK_2 \cup (n-2c-1)K_1$. Therefore, $G \cong S(n,c)$. ∎

Let G^C be the *complement graph* of G. In particular, $S^C(n,c)$ denotes the complement graph of $S(n,c)$. For the relation between $SL(G)$ and $SL(G^C)$, it has been shown that:

LEMMA 2.7. [17] *Let G be a graph with n vertices. If $\lambda_i(G)$, $i = 1, 2, \ldots, n$ are the eigenvalues of $L(G)$, then the eigenvalues of $L(G^C)$ are $n - \lambda_i(G)$, $i = 1, 2, \ldots, n-1$ and 0.*

By Lemma 2.7 and Theorem 2.6, we have:

COROLLARY 2.8. *If $c \geq 0$, then $S^C(n,c)$ is determined by its Laplacian spectrum.*

3. $S(n,c)$ is determined by its signless Laplacian spectrum. In this section, we shall show that $S(n,c)$ is determined by its signless Laplacian spectrum. First we need some lemmas.

Suppose M and N are real symmetric matrices of order n and m with eigenvalues $\rho_1(M) \geq \cdots \geq \rho_m(M)$ and $\rho_1(N) \geq \cdots \geq \rho_n(N)$, respectively. It is well-known that:

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LEMMA 3.1. [11] *If M is a principal submatrix of N, then the eigenvalues of M interlace those of N, i.e., $\rho_i(N) \geq \rho_i(M) \geq \rho_{n-m+i}(N)$ for $i = 1, 2, \dots, m$.*

LEMMA 3.2. [8] *If G is a graph on n vertices with vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n$ and signless Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, then $\mu_2 \geq d_2 - 1$. Moreover, if $\mu_2 = d_2 - 1$, then $d_1 = d_2$, and the maximum and the second maximum degree vertices are adjacent.*

By Lemmas 2.4 and 3.2, it follows that $\mu_1 \geq d_1 + 1$ and $\mu_2 \geq d_2 - 1$. For the general case, we have:

THEOREM 3.3. *If G is a finite simple graph on n vertices with vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n$ and signless Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, then $\mu_m \geq d_m - m + 1$, where $m = 1, 2, \dots, n$.*

To prove Theorem 3.3, we need the next lemma.

LEMMA 3.4. [4] (Weyl) *Suppose A_n and B_n are two real symmetric matrices of order n, then $\rho_n(A) + \rho_n(B) \leq \rho_n(A+B)$, where $\rho_n(A)$, $\rho_n(B)$ and $\rho_n(A+B)$ denote the smallest eigenvalues of A, B and $A + B$, respectively.*

Proof of Theorem 3.3. Since $Q(G)$ is positive semidefinite, $\mu_m \geq 0$. If $d_m - m + 1 \leq 0$, the result already holds. So, we assume that $d_m > m - 1$ in the following.

Let $T = \{v_1, v_2, \dots, v_m\}$. Consider the principal submatrix Q_T of $Q(G)$ with rows and columns indexed by T. Let $Q(T)$ be the signless Laplacian matrix of the subgraph induced by T. Then, $Q_T = Q(T) + D'(T)$, where $D'(T)$ is the diagonal matrix and the (i, i)-entry of $D'(T)$ is the number of neighbors of v_i outside T. Since $Q(T)$ is positive semidefine, and $D'(T) \geq (d_m - m + 1)I_m$, by Lemma 3.4 we have $\rho_m(Q_T) \geq \rho_m(Q(T)) + \rho_m(D'(T)) \geq \rho_m(D'(T)) \geq d_m - m + 1$. Recall that Q_T is the principal submatrix of $Q(G)$, thus Lemma 3.1 implies that $\mu_m \geq \rho_m(Q_T) \geq d_m - m + 1$. We get the required inequality. □

REMARK 3.5. The main idea of the proof in Theorem 3.3 comes from Lemma 2 of [2]. In [2], it has been shown that "Let G be a finite simple graph on n vertices with vertex degree $d_1 \geq d_2 \geq \cdots \geq d_n$ and Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If $G \ncong K_m \cup (n - m)K_1$, then $\lambda_m \geq d_m - m + 2$, where $m = 1, 2, \dots, n$." Though $\mu_1 \geq \lambda_1 \geq d_1 + 1$ by Lemma 2.4, $\mu_m \geq d_m - m + 2$ does not hold for all connected graphs. For example, $\mu_2(K_n - e) = n - 2 < n - 1 = d_2(K_n - e)$, where $K_n - e$ is the graph obtained from K_n by deleting one edge and $n \geq 4$.

Let $\Phi(G, x) = \det(xI - Q(G))$ be the signless Laplacian characteristic polynomial of G.

LEMMA 3.6. *If $c \geq 1$, then $\mu_1(S(n, c)) > n$, $\mu_2(S(n, c)) \leq 3$ and $0 < \mu_n(S(n, c))$*

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≤ 1.

Proof. By a straightforward computation, we have

$$\Phi(S(n,c),x) = (x-1)^{n-c-2}(x-3)^{c-1}\varphi_1(x), \tag{3.1}$$

where $\varphi_1(x) = x^3 - (n+3)x^2 + 3nx - 4c$.

We consider the next two cases.

Case 1. $n \geq 2c+2$.

Since $\varphi_1(0) = -4c < 0$, $\varphi_1(1) = 2(n-2c-1) > 0$, $\varphi_1(3) = -4c < 0$, $\varphi_1(n) = -4c < 0$ and $\varphi_1(n+1) = n^2 - n - 2 - 4c \geq n^2 - n - 2 - 2n + 4 = n^2 - 3n + 2 > 0$. By Eq. (3.1), it follows that $\mu_1(S(n,c)) > n$, $\mu_2(S(n,c)) \leq 3$ and $0 < \mu_n(S(n,c)) < 1$.

Case 2. $n = 2c+1$.

If $c = 1$, then $n = 3$ and hence $S(n,c) = C_3$, it is easily checked the result follows. Thus, we may suppose that $n \geq 5$, i.e., $c \geq 2$ in the following. Then, Eq. (3.1) can be rewritten as

$$\Phi(S(n,c),x) = (x-1)^{n-c-1}(x-3)^{c-1}\varphi_2(x), \tag{3.2}$$

where $\varphi_2(x) = x^2 - (n+2)x + 4c$.

Since $\varphi_2(1) = 2c-2 > 0$, $\varphi_2(2) = -2 < 0$, $\varphi_2(n) = -2 < 0$ and $\varphi_2(n+1) = 2c-2 > 0$. By Eq. (3.2), it follows that $\mu_1(S(n,c)) > n$, $\mu_2(S(n,c)) = 3$ and $\mu_n(S(n,c)) = 1$.

By combining the above arguments, the result follows. □

LEMMA 3.7. [5] *Let $G = (V,E)$ be a graph on n vertices. Then, $\mu_1(G) \leq \max\{d(u)+d(v) : uv \in E\}$. For a connected graph G, equality holds if and only if G is regular or semi-regular bipartite.*

LEMMA 3.8. *For $c \geq 1$, if $SQ(G) = SQ(S(n,c))$, then G is connected with $d_2(G) \leq 4$. Moreover, $d_2(G) = 4$ implies that $d_1(G) = d_2(G)$.*

Proof. Since $SQ(G) = SQ(S(n,c))$, by Lemma 3.6 it follows that $\mu_1(G) = \mu_1(S(n,c)) > n$ and $\mu_2(G) = \mu_2(S(n,c)) \leq 3$. By Lemma 3.2, we can conclude that $d_2(G) \leq 4$, and $d_2(G) = 4$ implies that $d_1(G) = d_2(G)$.

Suppose to the contrary that G is disconnected. Let G_1 be the greatest connected component, i.e., the connected component with largest number of vertices, of G. Since $d_2(G) \leq 4$ and $\mu_1(G) > n$, we have $n-3 \leq d_1(G) \leq n-2$ by Lemma 3.7. We consider the next two cases.

Case 1. $d_1(G) = n-3$.

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Then, $|V(G_1)| \geq n-2$. If $|V(G_1)| = n-1$, then $G = G_1 \cup K_1$. This implies that $\mu_n(G) = 0$, a contradiction to $\mu_n(G) = \mu_n(S(n,c)) > 0$. If $|V(G_1)| = n-2$, then $G = G_1 \cup K_2$ or $G = G_1 \cup 2K_1$. This also implies that $\mu_n(G) = 0$, a contradiction.

Case 2. $d_1(G) = n-2$.

Then, $|V(G_1)| = n-1$, and hence $G = G_1 \cup K_1$. This also implies that $\mu_n(G) = 0$, a contradiction to $\mu_n(G) = \mu_n(S(n,c)) > 0$.

Thus, G is connected. □

Let $m(v)$ denote the average of the degree of the vertices adjacent to v, i.e., $m(v) = \sum_{u \in N(v)} d(u)/d(v)$.

LEMMA 3.9. [7] *Let G be a connected graph. Then $\mu_1(G) \leq \max\{d(v) + m(v) : v \in V\}$, and equality holds if and only if G is a regular graph or a semi-regular bipartite graph.*

LEMMA 3.10. *Let $G = (V,E)$ be a connected graph on $n \geq 2c+3$ vertices with $n+c-1$ edges. If $c \geq 1$ and $d_1(G) \leq n-2$, then $\mu_1(G) \leq n$.*

Proof. By Lemma 3.9, we only need to prove that $\max\{d(v)+m(v) : v \in V\} \leq n$. Suppose $\max\{d(v) + m(v) : v \in V\}$ occurs at the vertex u_0. Three cases arise: $d(u_0) = 1$, $d(u_0) = 2$, or $3 \leq d(u_0) \leq n-2$.

Case 1. $d(u_0) = 1$.

Suppose $v \in N(u_0)$. Since $d(v) \leq d_1(G) \leq n-2$, $d(u_0) + m(u_0) = d(u_0) + d(v) \leq n-1 < n$.

Case 2. $d(u_0) = 2$.

Suppose that $v,w \in N(u_0)$.

If $vw \in E$, since G is a connected graph with $n+c-1$ edges, it follows that $|N(v) \cap N(w)| \leq c$ and $|N(v) \cup N(w)| \leq n$. Therefore, $d(u_0) + m(u_0) = 2 + \frac{d(v)+d(w)}{2} \leq 2 + \frac{n+c}{2} \leq n$ by $n \geq 2c+3$.

If $vw \notin E$, since G is a connected graph with $n+c-1$ edges, it follows that $|N(v) \cap N(w)| \leq c+1$ and $|N(v) \cup N(w)| \leq n-2$. Therefore, $d(u_0) + m(u_0) = 2 + \frac{d(v)+d(w)}{2} \leq 2 + \frac{n+c-1}{2} < n$ by $n \geq 2c+3$.

Case 3. $3 \leq d(u_0) \leq n-2$.

Note that $3 \leq d(u_0) \leq n-2$ and the number of edges of G is $n+c-1$, then $d(u_0) + m(u_0) \leq d(u_0) + \frac{2(n+c-1)-d(u_0)-1}{d(u_0)} = d(u_0) - 1 + \frac{2n+2c-3}{d(u_0)}$. Next we shall prove that $d(u_0) - 1 + \frac{2n+2c-3}{d(u_0)} \leq n$, equivalently, $d(u_0)(n+1-d(u_0)) \geq 2n+2c-3$. Let

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$f(x) = (n+1-x)x$.

When $3 \le x \le \frac{n+1}{2}$, since $f'(x) = n+1-2x \ge 0$, we have $f(x) \ge f(3) = 3(n-2) \ge 2n+2c-3$ by $n \ge 2c+3$.

When $\frac{n+1}{2} \le x \le n-2$, since $f'(x) = n+1-2x \le 0$, we have $f(x) \ge f(n-2) = 3(n-2) \ge 2n+2c-3$ by $n \ge 2c+3$.

By combining the above arguments, the conclusion follows. □

Lemma 3.11. [5] *Let G be a graph with n vertices, m edges. We have $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} d_i = 2m$, and $\sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2$.*

Lemma 3.12. *For $c \ge 1$, if $n = 2c+2$ or $n = 2c+1$, then there does not exist any connected graph G on n vertices with $n+c-1$ edges and $d_1(G) \le n-2$ such that $SQ(G) = SQ(S(n,c))$.*

Proof. Here we only prove the case of $n = 2c+2$, because the proof of $n = 2c+1$ is analogous. When $3 \le n \le 7$, it is easily checked the result follows by the aid of computer. Thus, we may assume that $n \ge 8$ in the following. Suppose to the contrary, there exists some connected graph G on $n = 2c+2$ vertices with $n+c-1$ edges and $d_1(G) \le n-2$ such that $SQ(G) = SQ(S(n,c))$. By Lemmas 3.6–3.8, we can conclude that $d_2(G) \le 4$ and $n-3 \le d_1(G) \le n-2$ because $\mu_1(G) = \mu_1(S(n,c)) > n$. We divide the proof into the next two cases.

Case 1. $d_1(G) = n-3$.

If $d_2(G) \le 3$, then Lemma 3.7 implies that $\mu_1(G) \le n < \mu_1(S(n,c))$, a contradiction. Thus, $d_2(G) = 4$. So Lemma 3.8 implies that $d_1(G) = d_2(G)$, and hence $n = 7$, a contradiction to the fact that $n \ge 8$.

Case 2. $d_1(G) = n-2$.

If $d_2(G) \le 2$, then Lemma 3.7 implies that $\mu_1(G) \le n < \mu_1(S(n,c))$, a contradiction. If $d_2(G) = 4$, Lemma 3.8 implies that $d_1(G) = d_2(G)$, and hence $n = 6$, a contradiction. Thus, $d_2(G) = 3$. Suppose G has x vertices of degree 3, y vertices of degree 2. Then, G has $n-x-y-1$ pendant vertices. By Lemma 3.11, it follows that

$$(3.3) \qquad \begin{cases} n-2+3x+2y+n-x-y-1 = 2n+2c-2 \\ (n-2)^2+9x+4y+n-x-y-1 = (n-1)^2+8c+n-2c-1. \end{cases}$$

By Eqs. (3.3) and $n = 2c+2$, we have $x = n-3$ and $y = 5-n < 0$, a contradiction.

By combining the above arguments, this completes the proof of this result. □

Lemma 3.13. [5] *In any graph, the multiplicity of the eigenvalue 0 of the signless Laplacian is equal to the number of bipartite components. Moreover, the least eigen-*

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value of the signless Laplacian of a connected graph is equal to 0 *if and only if the graph is bipartite. In this case,* 0 *is a simple eigenvalue.*

LEMMA 3.14. *If* $n \neq 4$*, then* $K_{1,n-1}$ *is determined by its signless Laplacian spectrum.*

Proof. Suppose there exists some graph G such that $SQ(G) = SQ(K_{1,n-1})$. It is well-known that if G is bipartite graph, then $SQ(G) = SL(G)$ (see [5]). Thus, $SQ(K_{1,n-1}) = SL(K_{1,n-1}) = (n, 1, 1, \ldots, 1, 0)$, where the multiplicity of 1 is $n-2$. By Lemma 3.2, we have $d_2(G) - 1 \leq \mu_2(G) = \mu_2(K_{1,n-1}) = 1$. So, $d_2(G) \leq 2$.

If G is connected, since $\mu_n(G) = \mu_n(K_{1,n-1}) = 0$, by Lemma 3.13, G is connected bipartite, and hence $SL(G) = SQ(G) = SL(K_{1,n-1})$. By Lemma 2.5, it follows that $G \cong K_{1,n-1}$.

If G is disconnected, by Lemma 3.7, we have $d_1(G) = n - 2$ and $d_2(G) = 2$ by $\mu_1(G) = n$. Moreover, Lemma 3.2 implies that $n-2 = d_1(G) = d_2(G) = 2$, and hence $n = 4$, a contradiction. □

REMARK 3.15. It is easily checked that $SQ(K_{1,3}) = SQ(K_3 \cup K_1)$. Thus, $S(n, c)$ is not determined by its signless Laplacian spectrum when $c = 0$ and $n = 4$.

THEOREM 3.16. *Suppose* $c \geq 0$*, then* $S(n, c)$ *is determined by its signless Laplacian spectrum except for the case of* $c = 0$ *and* $n = 4$*.*

Proof. If $c = 0$, then $S(n, c) \cong K_{1,n-1}$. By Lemma 3.14 and Remark 3.15, the result follows. Next we assume that $c \geq 1$. Now suppose there exists some graph G such that $SQ(G) = SQ(S(n, c))$. Lemmas 3.8 and 3.11 imply that G is connected and $\sum_{i=1}^{n} d_i(G) = 2(n + c - 1)$. Thus, G has $n + c - 1$ edges. By Lemmas 3.8, 3.10 and 3.12, we can conclude that G is a connected graph with $d_1(G) = n - 1$ and $d_2(G) \leq 4$ because $\mu_1(G) = \mu_1(S(n, c)) > n$. Suppose G has x vertices of degree 4, y vertices of degree 3, z vertices of degree 2. Then, G has $n - x - y - z - 1$ pendant vertices. By Lemma 3.11, it follows that

$$(3.4) \begin{cases} n - 1 + 4x + 3y + 2z + n - x - y - z - 1 = 2n + 2c - 2 \\ (n-1)^2 + 16x + 9y + 4z + n - x - y - z - 1 = (n-1)^2 + 8c + n - 2c - 1. \end{cases}$$

By Eqs. (3.4), we have $6x + 2y = 0$. Thus, $x = y = 0$ and $z = 2c$. Note that $d_1(G) = n - 1$. Then, $G \cong S(n, c)$ follows. □

4. $S^C(n, c)$ **is determined by its signless Laplacian spectrum.** In this section, we shall show that $S^C(n, c)$ is determined by its signless Laplacian spectrum. We list more lemmas as follows.

LEMMA 4.1. [3] *The signless Laplacian eigenvalues of* G *and* $G' = G+e$ *interlace, that is,* $\mu_1(G') \geq \mu_1(G) \geq \mu_2(G') \geq \mu_2(G) \geq \cdots \geq \mu_n(G') \geq \mu_n(G) \geq 0$*.*

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LEMMA 4.2. [16] *Suppose G has n vertices and d_n is the minimum degree of vertices of G, then $\mu_n \le d_n$.*

LEMMA 4.3. *If $c \ge 1$ and $n \ge 7$, then $\mu_n(S^C(n,c)\,) = 0$, $\mu_{n-1}(S^C(n,c)\,) \ge n-5$, $\mu_2(S^C(n,c)\,) = n-3$ and $\mu_1(S^C(n,c)\,) \ge 2(n-3)$.*

Proof. By a straightforward computation, we have

$$\Phi(S^C(n,c),x) = x(x-n+5)^{c-1}(x-n+3)^{n-c-2}\varphi_3(x), \tag{4.1}$$

where $\varphi_3(x) = x^2 - 3(n-3)x + 2(n^2 - 7n + 10 + 2c)$.

It is easy to see that the roots of $\varphi_3(x) = 0$ are

$$\frac{3(n-3) \pm \sqrt{(n+1)^2 - 16c}}{2}.$$

Note that $n \ge 2c+1$. Then,

$$\mu_1 = \frac{3(n-3) + \sqrt{(n+1)^2 - 16c}}{2} \ge 2(n-3),$$

$$\text{and} \quad n-5 < \frac{3(n-3) - \sqrt{(n+1)^2 - 16c}}{2} \le n-3.$$

We divide the proof into the next two cases.

Case 1. $c = 1$.

By Eq. (4.1), it is easy to see that $\mu_n(S^C(n,c)\,) = 0$, $\mu_{n-1}(S^C(n,c)\,) > n-5$ and $\mu_2(S^C(n,c)\,) = n-3$.

Case 2. $c \ge 2$.

Since $n-c-2 > 0$, by Eq. (4.1) we can conclude that $\mu_n(S^C(n,c)\,) = 0$, $\mu_{n-1}(S^C(n,c)\,) = n-5$ and $\mu_2(S^C(n,c)\,) = n-3$. $\square$

LEMMA 4.4. *For $c \ge 1$ and $n \ge 8$, if there exists some graph $G = G^* \cup K_1$ such that G^* is connected and $SQ(G) = SQ(S^C(n,c)\,)$, then $d_{n-1}(G^*) = n-3$.*

Proof. By Lemmas 4.2 and 4.3, we can conclude that $n-5 \le \mu_{n-1}(G^*) \le d_{n-1}(G^*) \le n-2$. If $d_{n-1}(G^*) = n-2$, then $G^* \cong K_{n-1}$, and hence $SQ(G^*) = (2n-4, n-3, \ldots, n-3) \ne SQ(S^C(n,c)\,)$, a contradiction. We divide the proof into the next two cases.

Case 1. $d_{n-1}(G^*) = n-5$.

Let H_1 be the graph obtained from K_{n-1} by deleting three edges, which are adjacent to the same vertex, from K_{n-1}. Clearly, $d_{n-1}(H_1) = n-5$ and G^* is a

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subgraph of H_1. By a straightforward computation, we have

$$\Phi(H_1, x) = (x - n + 4)^2 (x - n + 3)^{n-5} \varphi_4(x),$$

where $\varphi_4(x) = x^2 - (3n - 11)x + 2(n - 4)(n - 5)$.

It is easy to see that the roots of $\varphi_4(x) = 0$ are

$$\frac{3n - 11 \pm \sqrt{n^2 + 6n - 39}}{2}.$$

By Lemma 4.1, it follows that

$$\mu_{n-1}(G^*) \leq \mu_{n-1}(H_1) = \frac{3n - 11 - \sqrt{n^2 + 6n - 39}}{2} < n - 5.$$

On the other hand, $\mu_{n-1}(G^*) = \mu_{n-1}(G) = \mu_{n-1}(S^C(n, c) \geq n - 5$, a contradiction.

Case 2. $d_{n-1}(G^*) = n - 4$.

Let H_2 be the graph obtained from K_{n-1} by deleting two edges, which are adjacent to the same vertex, from K_{n-1}. Clearly, $d_{n-1}(H_2) = n-4$ and G^* is a subgraph of H_2. By a straightforward computation, we have

$$\Phi(H_2, x) = (x - n + 4)(x - n + 3)^{n-4} \varphi_5(x),$$

where $\varphi_5(x) = x^2 - (3n - 10)x + 2(n - 4)^2$.

It is easy to see that the roots of $\varphi_5(x) = 0$ are

$$\frac{3n - 10 \pm \sqrt{n^2 + 4n - 28}}{2}.$$

By Lemma 4.1, it follows that

$$\mu_{n-1}(G^*) \leq \mu_{n-1}(H_2) = \frac{3n - 10 - \sqrt{n^2 + 4n - 28}}{2} < n - 5.$$

On the other hand, $\mu_{n-1}(G^*) = \mu_{n-1}(G) = \mu_{n-1}(S^C(n, c) \geq n - 5$, a contradiction.

By combining the above arguments, we can conclude that $d_{n-1}(G^*) = n - 3$. ☐

LEMMA 4.5. *If $c = 0$ and $n \neq 4$, then $S^C(n, c)$ is determined by its signless Laplacian spectrum*

Proof. If $1 \leq n \leq 3$, it is easily checked the result follows. Thus, we may assume that $n \geq 5$ in the following. Suppose that there exists some graph G such that $SQ(G) = SQ(S^C(n, c))$. Note that $S^C(n, c) = K_{n-1} \cup K_1$. Then, $\mu_n(G) = \mu_n(K_{n-1} \cup K_1) = 0$ and $\mu_1(G) = \mu_1(K_{n-1} \cup K_1) = 2(n - 2)$.

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If G is connected, since $\mu_n(G) = 0$, by Lemma 3.13 it follows that G is bipartite. Lemma 2.2 implies that $\mu_1(G) = \lambda_1(G) \leq n < 2(n-2)$, a contradiction. Thus, G is disconnected and hence $d_1(G) \leq n-2$. Since $\mu_1(G) = 2(n-2)$, by Lemma 3.7 we can conclude that $G \cong K_{n-1} \cup K_1 = S^C(n,c)$. □

Remark 4.6. It is easily checked that $SQ(K_3 \cup K_1) = SQ(K_{1,3})$. Thus, $S^C(n,c)$ is not determined by its signless Laplacian spectrum when $c = 0$ and $n = 4$.

Theorem 4.7. *If $c \geq 0$, then $S^C(n,c)$ is determined by its signless Laplacian spectrum except for the case of $c = 0$ and $n = 4$.*

Proof. If $c = 0$, by Lemma 4.5 and Remark 4.6, the result follows. If $c \geq 1$ and $3 \leq n \leq 7$, it is easily checked the result follows by the aid of computer. Thus, we may assume that $n \geq 8$ and $c \geq 1$ in the sequel. Now suppose there exists some graph G such that $SQ(G) = SQ(S^C(n,c))$. We only need to prove the following facts:

Fact 1. $G = G^* \cup K_1$, where G^* is connected.

Proof of Fact 1. We first claim that G is disconnected. Suppose to the contrary, G is connected. By Lemma 4.3, we have $\mu_n(G) = \mu_n(S^C(n,c)) = 0$. Thus, G is bipartite by Lemma 3.13. So, $\mu_1(G) \leq n$ follows from Lemma 2.2. But $\mu_1(G) = \mu_1(S^C(n,c) \geq 2(n-3) > n$ by Lemma 4.3, a contradiction. Thus, G is disconnected.

Let G_1 be the greatest connected component, i.e., the connected component with largest number of vertices, of G. Since $\mu_n(G) = 0$ and $\mu_{n-1}(G) = \mu_{n-1}(S^C(n,c)) \geq n-5 > 0$, by Lemmas 3.13 and 4.2 we can conclude that G has exactly one bipartite component and $|V(G_1)| \geq n-4$. Moreover, Lemma 4.3 implies that $\mu_1(G) = \mu_1(S^C(n,c) \geq 2(n-3)$, thus $|V(G_1)| \geq n-2$ by Lemma 3.7.

If $|V(G_1)| = n-2$, since G has exactly one bipartite component, we can deduce that $G = G_1 \cup K_2$. Then G has 2 as its signless Laplacian eigenvalue. On the other hand, Lemma 4.3 implies that $\mu_{n-1}(G) = \mu_{n-1}(S^C(n,c) \geq n-5 > 2$, a contradiction. Thus, $|V(G_1)| = n-1$ and hence Fact 1 follows.

Fact 2. $G \cong S^C(n,c)$.

Proof of Fact 2. By Fact 1 and Lemma 4.4, it follows that $G = G^* \cup K_1$, where G^* is connected with $d_{n-1}(G^*) = n-3$. By Lemma 3.11, it follows that G^* has $n-2c-1$ vertices of degree $n-2$ and $2c$ vertices of degree $n-3$, then $G \cong S^C(n,c)$ follows.

This completes the proof of this result. □

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REFERENCES

[1] B. Borovićanin and M. Petrović. On the index of cactuses with n vertices. *Publ. Inst. Math. (Beograd)*, 79(93):13–18, 2006.

[2] A.E. Brouwer and W.H. Haemers. A lower bound for the Laplacian eigenvalues of a graph–Proof of a conjecture by Guo. *Linear Algebra Appl.*, 429(8):2131–2135, 2008.

[3] D.M. Cardoso, D. Cvetković, P. Rowlinson, and S.K. Simić. A sharp lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph. *Linear Algebra Appl.*, 429:2770–2780, 2008.

[4] D.M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs. Theory and Application*, second edition. VEB Deutscher Verlag der Wissenschaften, Berlin, 1982.

[5] D. Cvetković, P. Rowlinson, and S. K. Simić. Signless Laplacians of finite graphs. *Linear Algebra Appl.*, 423:155–171, 2007.

[6] E.R. van Dam and W.H. Haemers. Which graphs are determined by their spectrum? *Linear Algebra Appl.*, 373:241–272, 2003.

[7] K.C. Das. The Laplacian spectrum of a graph. *Comput. Math. Appl.*, 48:715–724, 2004.

[8] K.C. Das. On conjectures involving second largest signless Laplacian eigenvalue of graphs. *Linear Algebra Appl.*, 432:3018–3029, 2010.

[9] M. Doob and W.H. Haemers. The complement of the path is determined by its spectrum. *Linear Algebra Appl.*, 356:57–65, 2002.

[10] Y.Z. Fan, B.S. Tam, and J. Zhou. Maximizing spectral radius of unoriented Laplacian matrix over bicyclic graphs of a given order. *Linear Multilinear Algebra*, 56(4):381–397, 2008.

[11] W.H. Haemers. Interlacing eigenvalues and graphs. *Linear Algebra Appl.*, 226/228:593–616, 1995.

[12] S. Kirkland. Completion of Laplacian integral graphs via edge addition. *Discrete Math.*, 295:75–90, 2005.

[13] B.L. Liu, Z.B. Chen, and M.H. Liu. On graphs with the largest Laplacian index. *Czechoslovak Math. J.*, 58:949–960, 2008.

[14] H.Q. Liu and M. Lu. A unified approach to extremal cacti for different indices. *MATCH Commun. Math. Comput. Chem.*, 58:183–194, 2007.

[15] M.H. Liu and B.L. Liu. Some results on the Laplacian spectrum. *Comput. Math. Appl.*, 59:3612–3616, 2010.

[16] M.H. Liu and B.L. Liu. The signless Laplacian spread. *Linear Algebra Appl.*, 432(2-3):505–514, 2010.

[17] R. Merris. Laplacian matrices of graphs: a survey. *Linear Algebra Appl.*, 197/198:143–176, 1994.

[18] R. Merris. Laplacian graph eigenvectors. *Linear Algebra Appl.*, 278:221–236, 1998.

[19] Y.L. Pan. Sharp upper bounds for the Laplacian graph eigenvalues. *Linear Algebra Appl.*, 355:287–295, 2002.

[20] Z. Radosavljević and M. Rašajski. A class of reflexive cactuses with four cycles. *Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat.*, 14:64–85, 2003.

[21] X.L. Shen and Y.P. Zhang. The star and all starlike trees with largest degree 3 are determined by their spectra. *J. Nat. Sci. Hunan Norm. Univ.*, 28:17–20, 2005.

[22] B.S. Tam, Y.Z. Fan, and J. Zhou. Unoriented Laplacian maximizing graphs are degree maximal. *Linear Algebra Appl.*, 429:735–758, 2008.

[23] X.L. Zhang and H.P. Zhang. Some graphs determined by their spectra. *Linear Algebra Appl.*, 431:1443–1454, 2009.

[24] Y.P. Zhang, X.G. Liu, B.Y. Zhang, and X.R. Yong. The lollipop graph is determined by its Q-spectrum. *Discrete Math.*, 309:3364–3369, 2009.