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THE MOORE-PENROSE INVERSE OF A LINEAR COMBINATION OF COMMUTING GENERALIZED AND HYPERGENERALIZED PROJECTORS*

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Abstract. In this paper, some representations for the Moore-Penrose inverse of a linear combination of generalized and hypergeneralized projectors are found. Also, the invertibility for some linear combinations of commuting generalized and hypergeneralized projectors is considered.

Key words. Idempotent, Projector, Generalized projector, Hypergeneralized projector, Moore-Penrose inverse.

AMS subject classifications. 15A09.

1. Introduction. Let $\mathbb{C}^{n\times m}$ denote the set of all $n \times m$ complex matrices. The symbols A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\mathrm{rank}(A)$ will denote the conjugate transpose, the range (column space), the null space, and the rank of a matrix A, respectively. By $\mathbb{C}_r^{n\times n}$ we will denote the set of all matrices from $\mathbb{C}^{n\times n}$ with a rank r. The Moore-Penrose inverse of A, is the unique matrix $A^\dagger$ satisfying the equations

$$(1)\ AA^\dagger A = A, \quad (2)\ A^\dagger AA^\dagger = A^\dagger, \quad (3)\ AA^\dagger = (AA^\dagger)^*, \quad (4)\ A^\dagger A = (A^\dagger A)^*.$$

For a square matrix A there exists a unique reflexive generalized inverse of A which commutes with A if and only if A is of the index 1, that is, $\mathrm{rank}(A) = \mathrm{rank}(A^2)$ ([4], Theorem 1). This generalized inverse is called the group inverse of A and is denoted by $A^\sharp$.

I_n will denote the identity matrix of order n while $0_{s,s}$ will denote the null-matrix of order s. We use the notations C_n^P, C_n^{OP}, C_n^{EP}, C_n^{GP}, and C_n^{HGP} for the subsets of $\mathbb{C}^{n\times n}$ consisting of projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices), EP (range-Hermitian) matrices, generalized, and

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hypergeneralized projectors, respectively, i.e.,

$$C_n^P = \{A \in \mathbb{C}^{n\times n} : A^2 = A\},$$
$$C_n^{OP} = \{A \in \mathbb{C}^{n\times n} : A^2 = A = A^*\},$$
$$C_n^{EP} = \{A \in \mathbb{C}^{n\times n} : \mathcal{R}(A) = \mathcal{R}(A^*)\} = \{A \in \mathbb{C}^{n\times n} : AA^\dagger = A^\dagger A\},$$
$$C_n^{GP} = \{A \in \mathbb{C}^{n\times n} : A^2 = A^*\},$$
$$C_n^{HGP} = \{A \in \mathbb{C}^{n\times n} : A^2 = A^\dagger\}.$$

The concepts of generalized and hypergeneralized projectors were introduced by Groß and Trenkler [9] who presented interesting properties of the classes of generalized and hypergeneralized projectors. Very interesting results concerning generalized and hypergeneralized projectors can be found in the papers of J.K. Baksalary, O.M. Baksalary, X. Liu, and G. Trenkler [2], O.M. Baksalary [1], J.K. Baksalary, O.M. Baksalary, and J. Groß [3], J. Benítez and N. Thome [6], and G.W. Stewart [11].

In this paper, we give the form for the Moore-Penrose inverse, i.e., the group inverse of a linear combination c_1A+c_2B of two commuting generalized or hypergeneralized projectors. Also, we studied the nonsingularity of c_1A+c_2B and $c_1A+c_2B+c_3C$, where A, B and C are commuting generalized or hypergeneralized projectors under various conditions.

2. The Moore-Penrose inverse and the invertibility of a linear combination of commuting generalized or hypergeneralized projections. J.K. Baksalary, O.M. Baksalary, X. Liu, and G. Trenkler [2], proved that any generalized projector $A \in \mathbb{C}_r^{n\times n}$ can be represented by

$$A = U \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where $U \in C^{n\times n}$ is unitary and $K \in \mathbb{C}^{r\times r}$ is such that $K^3 = I_r$ and $K^* = K^{-1}$. Any hypergeneralized projector $A \in \mathbb{C}_r^{n\times n}$ has a form

$$A = U \begin{bmatrix} \Sigma K & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

where $U \in C^{n\times n}$ is unitary, $\Sigma = \mathrm{diag}(\sigma_1 I_{r_1}, \dots, \sigma_t I_{r_t})$ is a diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$ and $K \in \mathbb{C}^{r\times r}$ satisfies $(\Sigma K)^3 = I_r$ and $KK^* = I_r$.

There are also some other very useful representations for generalized and hypergeneralized projectors. By using the fact that any generalized projector $A \in \mathbb{C}_r^{n\times n}$ is a normal matrix, by the spectral theorem we have that $A = U\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)U^*$,

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where U is a unitary matrix and λ_j, $j = \{1, \ldots, n\}$ are the eigenvalues of A. By [6, Theorem 2.1], we have that $\lambda_j \in \{0, 1, \omega, \overline{\omega}\}$, $j = \{1, \ldots, n\}$, where $\omega = \exp(2\pi i/3)$. Hence,

$$A \in C_n^{GP} \Leftrightarrow A = U\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U^*,$$

where $U^* = U^{-1}$ and $\lambda_j \in \{0, 1, \omega, \overline{\omega}\}$, $j = \{1, \ldots, n\}$, $\omega = \exp(2\pi i/3)$.

Similarly, for $A \in C_n^{HGP}$ using the fact that A is EP-matrix, by [7, Theorem 4.3.1] we can conclude that

$$A \in C_n^{HGP} \Leftrightarrow A = U(K \oplus 0)U^*,$$

where $U \in \mathbb{C}^{n\times n}$ is a unitary matrix and $K \in \mathbb{C}^{r\times r}$ is nonsingular such that $K^3 = I_r$, where $r = \mathrm{rank}(A)$.

From the above representations it is obvious that any generalized projector is a hypergeneralized projector.

The following fact will be used very often:

If $X, Y \in \mathbb{C}^{n\times n}$ and $c_1, c_2 \in \mathbb{C}$, then

$$\begin{aligned} &X^3 = Y^3 = I_n,\ XY = YX \Rightarrow \\ &(c_1X + c_2Y)(c_1^2X^2 - c_1c_2XY + c_2^2Y^2) = (c_1^3 + c_2^3)I_n. \end{aligned} \tag{2.1}$$

In this section, we first present the form for the Moore-Penrose inverse, i.e., the group inverse of $c_1A + c_2B$, where A, B are two commuting generalized or hypergeneralized projectors and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1^3 + c_2^3 \neq 0$.

THEOREM 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be commuting generalized or hypergeneralized projectors, and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $c_1^3 + c_2^3 \neq 0$. Then*

$$\begin{aligned} (c_1A + c_2B)^\dagger = &\frac{1}{c_1^3 + c_2^3}\Big(c_1^2A^2B^3 - c_1c_2AB + c_2^2A^3B^2\Big) + \frac{1}{c_1}A^2(I_n - B^3) \\ &+ \frac{1}{c_2}B^2(I_n - A^3). \end{aligned} \tag{2.2}$$

Furthermore, $c_1A + c_2B$ is nonsingular if and only if $n = \mathrm{rank}(A) + \mathrm{rank}(B) - \mathrm{rank}(AB)$ and in this case $(c_1A + c_2B)^{-1}$ is given by (2.2).

Proof. Since A and B are two commuting EP-matrices, by [5, Corollary 3.9], we have that

$$A = U(A_1 \oplus A_2 \oplus 0_{t,t} \oplus 0)U^*, \quad B = U(B_1 \oplus 0_{s,s} \oplus B_2 \oplus 0)U^*,$$

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where $A_1, B_1 \in \mathbb{C}^{r\times r}$, $A_2 \in \mathbb{C}^{s\times s}$, $B_2 \in \mathbb{C}^{t\times t}$ are nonsingular and $A_1B_1 = B_1A_1$. If in addition A and B are hypergeneralized projectors (the following reasoning works if A, B are generalized projectors), then $A_1^3 = B_1^3 = I_r$, $A_2^3 = I_s$, and $B_2^3 = I_t$. Since

$$c_1A + c_2B = U\Big((c_1A_1 + c_2B_1) \oplus c_1A_2 \oplus c_2B_2 \oplus 0\Big)U^*, \tag{2.3}$$

we can use (2.1) to get the expression for $(c_1A + c_2B)^\dagger$. Thus, by (2.1) we get that $c_1A_1 + c_2B_1$ is nonsingular and that

$$(c_1A_1 + c_2B_1)^{-1} = \frac{1}{c_1^3 + c_2^3}\Big(c_1^2A_1^2 - c_1c_2A_1B_1 + c_2^2B_1^2\Big).$$

Now, using that

$$I_n - A^3 = U(0 \oplus 0 \oplus I_t \oplus I_{n-(r+t+s)})U^*, \;\; A^3B^3 = U(I_r \oplus 0 \oplus 0 \oplus 0)U^*$$

and

$$I_n - B^3 = U(0 \oplus I_s \oplus 0 \oplus I_{n-(r+t+s)})U^*,$$

we have

$$\begin{aligned}(c_1A + c_2B)^\dagger &= U((c_1A_1 + c_2B_1)^{-1} \oplus \frac{1}{c_1}A_2^2 \oplus \frac{1}{c_2}B_2^2 \oplus 0)U^* \\ &= \frac{1}{c_1^3 + c_2^3}\Big(c_1^2A^2 - c_1c_2AB + c_2^2B^2\Big)A^3B^3 \\ &\quad + \frac{1}{c_1}A^2(I_n - B^3) + \frac{1}{c_2}B^2(I_n - A^3).\end{aligned}$$

Since $A^4 = A$ and $B^4 = B$, we get that (2.2) holds. Also, it is evident that $\operatorname{rank}(A) = r + s$, $\operatorname{rank}(B) = r + t$ and $\operatorname{rank}(AB) = r$. So, the last summand in the direct sum of (2.3) does not appear if and only if $n = \operatorname{rank}(A) + \operatorname{rank}(B) - \operatorname{rank}(AB)$, which is a necessary and sufficient condition for the invertibility of $c_1A + c_2B$. $\square$

As a corollary, we get that in the case when A is generalized or hypergeneralized projector and $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^3 + c_2^3 \neq 0$, a linear combination $c_1I_n + c_2A$ is always nonsingular.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be a generalized or hypergeneralized projector, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^3 + c_2^3 \neq 0$. Then $c_1I_n + c_2A$ is nonsingular and*

$$(c_1I_n + c_2A)^{-1} = \frac{1}{c_1^3 + c_2^3}\Big(c_1^2A^3 - c_1c_2A + c_2^2A^2\Big) + \frac{1}{c_1}(I_n - A^3).$$

Let $\mathcal{G} \subset C_n^{GP}$ denote a commuting family of generalized projectors and $\mathcal{H} \subset C_n^{HGP}$ denote a commuting family of hypergeneralized projectors, i.e., an infinite

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set of generalized projectors or hypergeneralized projectors in which each pair in the set commutes under multiplication. If we consider a finite commuting family $\{A_i\}_{i=1}^m$ where all of the members are generalized or hypergeneralized projectors, then $\prod_{i=1}^m A_i^{k_i}$, where $k_1, \ldots, k_m \in \mathbb{N}$, is also a generalized or hypergeneralized projector. Hence, we have the following result:

Proposition 2.3. *Let all of $A_i \in \mathbb{C}^{n\times n}$, $i = \{1, \ldots, m\}$ be commuting generalized or hypergeneralized projectors, $c_1, c_2 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^3 + c_2^3 \neq 0$ and $k_1, \ldots, k_m \in \mathbb{N}$. Then $c_1 I_n + c_2 \prod_{i=1}^m A_i^{k_i}$ is nonsingular.*

With the additional requirements of Theorem 2.1 it is possible to give a more precise form of Moore-Penrose inverse, i.e., the group inverse.

Corollary 2.4. *Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If A, B are commuting generalized or hypergeneralized projectors such that $AB = 0$, then*

$$(c_1 A + c_2 B)^\dagger = \frac{1}{c_1} A^2 + \frac{1}{c_2} B^2.$$

In the next result, we present the form of Moore-Penrose inverse, i.e., the group inverse of $c_1 A^m + c_2 A^k$, where $m, k \in \mathbb{N}$ and A is a generalized or hypergeneralized projector. It is a corollary of Theorem 2.1.

Corollary 2.5. *Let $A \in \mathbb{C}_r^{n\times n}$ be a generalized or hypergeneralized projector and let $c_1, c_2 \in \mathbb{C}$, $c_1^3 + c_2^3 \neq 0$ and $m, k \in \mathbb{N}$. Then*

$$(c_1 A^m + c_2 A^k)^\dagger = \frac{1}{c_1^3 + c_2^3}\big(c_1^2 A^{2m} - c_1 c_2 A^{m+k} + c_2 A^{2k}\big),$$

where $A^t = \begin{cases} A^3, & t \equiv_3 0, \\ A, & t \equiv_3 1 \\ A^2, & t \equiv_3 2 \end{cases}$ *. Furthermore, $c_1 A^m + c_2 A^k$ is nonsingular if and only if A is nonsingular and in this case the inverse of $c_1 A^m + c_2 A^k$ is given by*

$$(c_1 A^m + c_2 A^k)^{-1} = \frac{1}{c_1^3 + c_2^3}\big(c_1^2 A^p - c_1 c_2 A^q + c_2 A^r\big),$$

where $2m \equiv_3 p$, $m + k \equiv_3 q$ and $2k \equiv_3 r$.

Proof. It follows by Theorem 2.1 and the fact that $\operatorname{rank}(A^p) = \operatorname{rank}(A)$, for any $p \in \mathbb{N}$. ☐

As a corollary we get a result from [2].

Corollary 2.6. [2] *Let $A \in \mathbb{C}_r^{n\times n}$ be a generalized projector and let $c_1, c_2 \in \mathbb{C}$, $c_1^3 + c_2^3 \neq 0$. Then*

$$(c_1 A + c_2 A^*)^\dagger = \frac{1}{c_1^3 + c_2^3}(c_1^2 A^2 - c_1 c_2 A^3 + c_2^2 A).$$

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Let us recall that for the matrices $A, B \in \mathbb{C}^{n\times m}$, a matrix A is less than or equal to B with respect to the star partial ordering, denoted by $A \overset{*}{\leq} B$ [8], if

$$A^*A = A^*B \quad \text{and} \quad AA^* = BA^*.$$

If $A \in C_n^{EP}$, then for any $B \in \mathbb{C}^{n\times n}$,

$$A \overset{*}{\leq} B \Leftrightarrow AB = A^2 = BA.$$

In the next theorem, we present the form of Moore-Penrose inverse, i.e., the group inverse of $c_1A^m + c_2B^k$ under the condition that A, B are generalized projectors and $AB = BA = A^2$. Remark that the same result holds if we suppose that A, B are generalized projectors such that $B - A \in C_n^{GP}$; or $A \in C_n^{EP}$, $B \in C_n^{HGP}$ such that $A \overset{*}{\leq} B$.

Theorem 2.7. *Let $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, $c_1^3 + c_2^3 \neq 0$ and $m, k \in \mathbb{N}$. If $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ are hypergeneralized projectors such that $AB = BA = A^2$, then*

$$(c_1A^m + c_2B^k)^\dagger = \frac{1}{c_1^3 + c_2^3}\left(c_1^2A^{2m} - c_1c_2A^{m+k} + c_2^2A^{2k}\right) + \frac{1}{c_2}B^{2k}(I_n - A^3), \tag{2.4}$$

where $A^t = \begin{cases} A^3, & t \equiv_3 0 \\ A, & t \equiv_3 1 \\ A^2, & t \equiv_3 2 \end{cases}$ *and* $B^s = \begin{cases} B^3, & s \equiv_3 0 \\ B, & s \equiv_3 1 \\ B^2, & s \equiv_3 2 \end{cases}$.

Proof. By [5, Corollary 3.9] and the fact that $AB = BA = A^2$, we have that

$$A = U(A_1 \oplus 0_{t,t} \oplus 0)U^*, \quad B = U(B_1 \oplus B_2 \oplus 0)U^*,$$

where $A_1, B_1 \in \mathbb{C}^{r\times r}$, $B_2 \in \mathbb{C}^{t\times t}$ are nonsingular and $A_1B_1 = B_1A_1 = A_1^2$. Evidently $A_1 = B_1$. If in addition A and B are hypergeneralized projectors, then $A_1^3 = I_r$ and $B_2^3 = I_t$. Hence,

$$c_1A^m + c_2B^k = U\Big((c_1A_1^m + c_2A_1^k) \oplus c_2B_2^k \oplus 0\Big)U^*.$$

By (2.1) we get that $c_1A_1^m + c_2A_1^k$ is nonsingular and that

$$(c_1A_1^m + c_2A_1^k)^{-1} = \frac{1}{c_1^3 + c_2^3}\Big(c_1^2A_1^{2m} - c_1c_2A_1^{m+k} + c_2^2A_1^{2k}\Big).$$

Now, by using that

$$A^3 = U(I_r \oplus 0 \oplus 0)U^*, \quad B^3 - A^3 = U(0 \oplus I_t \oplus 0)U^*,$$

we have that (2.4) holds. ∎

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REMARK 2.8. If $A \in C_n^{EP}$, $B \in C_n^{HGP}$, and $A \overset{*}{\leq} B$, we can conclude that $B - A$ is a hypergeneralized projector. If A and B are hypergeneralized projectors, then $A \overset{*}{\leq} B$ or $AB = A^2 = BA$ is sufficient for $B - A$ to be a hypergeneralized projector [9].

THEOREM 2.9. *Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be commuting hypergeneralized projectors. Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1^3 + c_2^3 \neq 0$ and $m, k, l \in \mathbb{N}$. Then*

$$\left[A^m(c_1A^k + c_2B^l)\right]^\dagger = \frac{1}{c_1^3 + c_2^3}\left(c_1^2A^{2(m+k)}B^3 - c_1c_2A^{(2m+k)}B^l + c_2^2A^{2m}B^{2l}\right) + \frac{1}{c_1}A^{2(m+k)}(I_n - B^3),$$

where $A^t = \begin{cases} A^3, & t \equiv_3 0 \\ A, & t \equiv_3 1 \\ A^2, & t \equiv_3 2 \end{cases}$ *and* $B^s = \begin{cases} B^3, & s \equiv_3 0 \\ B, & s \equiv_3 1 \\ B^2, & s \equiv_3 2 \end{cases}$.

Proof. The proof is similar to the proof of the Theorem 2.1. □

The following theorem presents a necessary and sufficient condition for the invertibility of $c_1A + c_2B + c_3C$ in the case when A, B, C are commuting hypergeneralized projectors such that $BC = 0$ and $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^3 + c_2^3 \neq 0$, $c_1^3 + c_3^3 \neq 0$. Remark that the same result holds if we suppose that $A, B, C \in \mathcal{G}$ such that $B + C \in C_n^{GP}$, or when $A, B, C \in \mathcal{H}$ such that $B \overset{*}{\perp} C$ with the same conditions for the scalars c_1, c_2, c_3.

The notion of star-orthogonality was introduced by Hestenes [10]. Let us recall that matrices $A, B \in \mathbb{C}^{n\times m}$ are star-orthogonal, denoted by $A \overset{*}{\perp} B$, if $AB^* = 0$ and $A^*B = 0$. It is well-known that for $A, B \in C_n^{EP}$,

$$A \overset{*}{\perp} B \Leftrightarrow AB = 0 \Leftrightarrow BA = 0.$$

If A,B are hypergeneralized projectors, then $A \overset{*}{\perp} B$ or $AB = BA = 0$ is sufficient for $A + B$ to be a hypergeneralized projector (see [9]).

THEOREM 2.10. *Let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^3 + c_2^3 \neq 0$, $c_1^3 + c_3^3 \neq 0$. If $A, B, C \in \mathbb{C}^{n\times n}$ are commuting hypergeneralized projectors such that $BC = 0$, then the following conditions are equivalent:*

(*i*) $c_1A + c_2B + c_3C$ *is nonsingular,*

(*ii*) $B^3 + C^3 + A(I_n - B^3 - C^3)$ *is nonsingular,*

(*iii*) $\operatorname{rank}(A(I_n - B^3 - C^3)) = n - (\operatorname{rank}(B) + \operatorname{rank}(C))$.

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Proof. By [5, Corollary 3.9], we have that

$$B = U(B_1 \oplus 0_{s,s} \oplus 0)U^*, \ C = U(0_{r,r} \oplus C_1 \oplus 0)U^*,$$

where $B_1 \in \mathbb{C}^{r\times r}$, $C_1 \in \mathbb{C}^{s\times s}$ are nonsingular and U is unitary. Since $B^2 = B^\dagger$ and $C^2 = C^\dagger$, we get that $B_1^3 = I_r$ and $C_1^3 = I_s$. Also, since A commutes with B and C, it follows that $A = U(A_1 \oplus A_2 \oplus A_3)U^*$ where A_1, A_2, A_3 are hypergeneralized projectors and $A_1B_1 = B_1A_1$, $A_2C_1 = C_1A_2$.

Now,

$$c_1A + c_2B + c_3C = U\Big((c_1A_1 + c_2B_1) \oplus (c_1A_2 + c_3C_1) \oplus c_1A_3\Big)U^*,$$

so $c_1A + c_2B + c_3C$ is nonsingular if and only if $c_1A_1 + c_2B_1$, $c_1A_2 + c_3C_1$, and A_3 are nonsingular. By Proposition 2.3 we get that $c_1A_1B_1^2 + c_2I$ is nonsingular. Now, by $c_1A_1 + c_2B_1 = (c_1A_1B_1^2 + c_2I)B_1$ it follows that $c_1A_1 + c_2B_1$ is nonsingular. Similarly, we get that $c_1A_2 + c_3C_1$ is nonsingular. Thus, $c_1A + c_2B + c_3C$ is nonsingular if and only if A_3 is nonsingular i.e., $B^3 + C^3 + A(I_n - B^3 - C^3)$ is nonsingular. Hence, $(i) \Leftrightarrow (ii)$. Also, we have that A_3 is nonsingular if and only if $\text{rank}(A_3) = n - (r + s)$ which is equivalent with the fact that $\text{rank}(A(I_n - B^3 - C^3)) = n - (r + s) = n - (\text{rank}(B) + \text{rank}(C))$. So, $(i) \Leftrightarrow (iii)$. ∎

Remark that from the proof of Theorem 2.10 and Theorem 2.1, if one of the conditions $(i) - (iii)$ holds, we get the formula for the inverse of $c_1A + c_2B + c_3C$:

$$\begin{aligned}(c_1A + c_2B + c_3C)^{-1} = {} & \frac{1}{c_1^3 + c_2^3}\Big(c_1^2A^2B^3 - c_1c_2AB + c_2^2A^3B^2\Big) + \frac{1}{c_2}B^2(I_n - A^3) \\ & + \frac{1}{c_1^3 + c_3^3}\Big(c_1^2A^2C^3 - c_1c_3AC + c_2^2A^3C^2\Big) + \frac{1}{c_3}C^2(I_n - A^3) \\ & + \frac{1}{c_1}\Big(B^3 + C^3 + A(I_n - B^3 - C^3)\Big)^{-1}(I_n - B^3 - C^3),\end{aligned} \tag{2.5}$$

which will be useful later in Theorem 2.11.

In the following theorem, under the assumption that $c_1, c_2, c_3 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^3 + c_2^3 \neq 0$, $c_1^3 + c_3^3 \neq 0$, we show that $c_1I_n + c_2A + c_3B$ is nonsingular, in the case when A, B are commuting hypergeneralized projectors such that $AB = 0$. Remark that the same theorem holds if we suppose that A, B are generalized projectors such that $A + B \in C_n^{GP}$ or when A, B are hypergeneralized projectors such that $A \overset{*}{\perp} B$.

Theorem 2.11. *Let $c_1, c_2, c_3 \in \mathbb{C}$, $c_1 \neq 0$, $c_1^3 + c_2^3 \neq 0$, $c_1^3 + c_3^3 \neq 0$. If A, $B \in \mathbb{C}^{n\times n}$ are commuting hypergeneralized projectors such that $AB = 0$, then $c_1I_n + c_2A + c_3B$ is nonsingular and*

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$$(c_1 I_n + c_2 A + c_3 B)^{-1} = \frac{1}{c_1^3 + c_2^3}\left(c_1^2 A^3 - c_1 c_2 A + c_2^2 A^2\right)$$
$$+\frac{1}{c_1^3 + c_3^3}\left(c_1^2 B^3 - c_1 c_3 B + c_3^2 B^2\right) + \frac{1}{c_1}(I_n - A^3 - B^3).$$

Proof. The proof follows by Theorem 2.10 and (2.5). □

Corollary 2.12. *Let $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$, $c_1^3 + c_2^3 \neq 0$, $c_1^3 + c_3^3 \neq 0$. If $A, B, C \in \mathbb{C}^{n\times n}$ are commuting hypergeneralized projectors such that $BC = 0$, then the invertibility of $c_1A+c_2B+c_3C$ is independent of the choice of the scalars c_1, c_2, c_3.*

Corollary 2.13. *Let $c_1, c_2, c_3 \in \mathbb{C}\setminus\{0\}$, $c_1^3+c_2^3 \neq 0$, $c_1^3+c_3^3 \neq 0$. If $A, B, C \in \mathcal{G}$ such that $B + C \in C_n^{GP}$ or $A, B, C \in \mathcal{H}$ such that $B \overset{*}{\perp} C$, then the invertibility of $c_1A + c_2B + c_3C$ is independent of the choice of the scalars c_1, c_2, c_3.*

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