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INERTIALLY ARBITRARY TREE SIGN PATTERNS OF ORDER 4[*]

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Abstract. An $n \times n$ sign pattern matrix A is an inertially arbitrary pattern if for every non-negative triple (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = n$, there is a real matrix in the sign pattern class of A having inertia (n_1, n_2, n_3). An $n \times n$ sign pattern matrix A is a spectrally arbitrary pattern if for any given real monic polynomial $r(x)$ of degree n, there is a real matrix in the sign pattern class of A with characteristic polynomial $r(x)$. In this paper, all 4×4 tree sign pattern matrices that are inertially arbitrary are characterized. As a result, in this paper, it is shown that a 4×4 tree sign pattern matrix is inertially arbitrary if and only if it is spectrally arbitrary.

1. Introduction. A *sign pattern matrix* (or a sign pattern, or a pattern) A is a matrix whose entries are in the set $\{+, -, 0\}$. Denote the set of all $n \times n$ sign patterns by Q_n. Associated with each $A = (a_{ij}) \in Q_n$ is a class of real matrices, called the *sign pattern class* of A, defined by

$$Q(A) = \{B = (b_{ij}) \mid B \text{ is an } n \times n \text{ real matrix, and } \operatorname{sign} b_{ij} = a_{ij} \text{ for all } i \text{ and } j\}.$$

For two sign patterns $A = (a_{ij})$ and $S = (s_{ij})$ in Q_n, if $a_{ij} = s_{ij}$ whenever $s_{ij} \neq 0$, then A is a *superpattern* of S, and S is a *subpattern* of A. Note that each sign pattern is a superpattern and a subpattern of itself.

A sign pattern $A \in Q_n$ is said to be *sign nonsingular* (or *sign singular*) if every matrix $B \in Q(A)$ is nonsingular (or singular).

A sign pattern matrix $P \in Q_n$ is called a *permutation pattern* if exactly one entry in each row and column is equal to $+$, and all other entries are 0. Two sign pattern matrices $A_1, A_2 \in Q_n$ are said to be *permutationally similar* if there is a permutation pattern P such that $A_2 = P^T A_1 P$.

A *signature pattern* is a diagonal sign pattern all of whose diagonal entries are

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nonzero. Two sign pattern matrices $A_1, A_2 \in Q_n$ are said to be *signature similar* if there is a signature pattern S such that $A_2 = SA_1S$.

A *combinatorially symmetric sign pattern matrix* is a square sign pattern $A = (a_{ij})$ where $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$. The *graph $G(A)$* of a combinatorially symmetric $n \times n$ sign pattern matrix $A = (a_{ij})$ is the graph with vertex set $\{1, 2, 3, \ldots, n\}$ where (i, j) is an edge if and only if $a_{ij} \neq 0$. A *tree sign pattern* (tsp) is a combinatorially symmetric sign pattern matrix whose graph is a tree (possibly with loops).

The *inertia* of a square real matrix B is the ordered triple

$$i(B) = (i_+(B), i_-(B), i_0(B)),$$

in which $i_+(B)$, $i_-(B)$ and $i_0(B)$ are the numbers of eigenvalues (counting multiplicities) of B with positive, negative and zero real parts, respectively. The *inertia set* of a square sign pattern A is the set of ordered triples $i(A) = \{i(B) \mid B \in Q(A)\}$. A sign pattern matrix $A \in Q_n$ is said to be an *inertially arbitrary pattern* (IAP) if for every nonnegative triple (n_1, n_2, n_3) with $n_1 + n_2 + n_3 = n$, there is a real matrix $B \in Q(A)$ such that $i(B) = (n_1, n_2, n_3)$.

An $n \times n$ matrix B is *stable* if $i(B) = (0, n, 0)$. An $n \times n$ sign pattern matrix A is *potentially stable* if $(0, n, 0) \in i(A)$. If there is a real matrix $B \in Q(A)$ having characteristic polynomial $f(x) = x^n$, then A is *potentially nilpotent*.

An $n \times n$ sign pattern matrix A is a *spectrally arbitrary pattern* (SAP) if, for any given real monic polynomial $r(x)$ of degree n, there is a matrix $B \in Q(A)$ with characteristic polynomial $r(x)$. That is, A is a SAP if for any possible spectrum of a real matrix (namely, any set of n complex numbers with nonreals occuring as conjugate pairs), there exists $B \in Q(A)$ with that spectrum.

It is easily seen that the class of $n \times n$ IAPs (SAPs) is closed under negation, transposition, permutation similarity, and signature similarity. We say that two sign patterns are *equivalent* if one can be obtained from the other by using a sequence of such operations.

The question of the existence of an inertially arbitrary sign pattern arose in [6]. The first inertially arbitrary sign pattern of order n for each $n \geq 2$ was provided in [7]. SAPs and IAPs are studied to some extent ([1]-[7], [9], [11]). In [1], all 4×4 tree sign patterns that are SAPs are characterized.

Clearly, if A is a SAP, then A is an IAP. In general the converse does not hold, as illustrated by an example with $n = 4$ in [3] (where the sign pattern is not a tsp).

QUESTION 1.1. Does there exist a tsp that is IAP but not SAP?

It was stated in reference [11] that the answer to this question is unknown in

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general. For star sign pattern, it was known that a star sign pattern is IAP if and only if it is also SAP ([11]).

In this paper, all 4×4 tree sign pattern matrices that are inertially arbitrary are characterized. By the result in this paper, for orders $2 \leq n \leq 4$, every tsp that is IAP is also SAP.

2. Preliminaries.

LEMMA 2.1. [6] *Up to equivalence, the sign pattern*

$$\mathcal{T}_2 = \left[\begin{array}{cc} - & + \\ - & + \end{array} \right]$$

is the only 2×2 pattern that is a SAP (IAP).

LEMMA 2.2. [1] *Up to equivalence,*

$$\mathcal{T}_3 = \left[\begin{array}{ccc} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{array} \right], \ \mathcal{U} = \left[\begin{array}{ccc} - & + & 0 \\ - & + & + \\ 0 & + & - \end{array} \right], \ \widetilde{\mathcal{T}}_3 = \left[\begin{array}{ccc} - & + & 0 \\ - & + & + \\ 0 & - & + \end{array} \right].$$

are the only 3×3 tree sign patterns which are SAPs.

LEMMA 2.3. [3] *For a sign pattern A of order 3, A is inertially arbitrary if and only if A is spectrally arbitrary.*

Up to equivalence, a 4×4 tsp A is a star sign pattern or a tridiagonal sign pattern. For $n \times n$ star sign patterns A, from [11], we know that A is a SAP if and only if A is an IAP. Now we consider 4×4 tridiagonal tree sign patterns.

Both the set of IAPs and the set of SAPs are subsets of the set of potentially stable sign patterns, and for tree sign patterns of order 4, the set of all potentially stable sign patterns is known (from [8] and [10] for paths and from [11] for stars). In [1], the authors precisely characterized all 4×4 tree sign patterns that are SAPs by determining which of these potentially stable sign patterns are SAPs.

Note that if A is a SAP, then A is an IAP; if A is sign nonsingular, then A is not an IAP; if A does not have a positive diagonal entry, then A is not an IAP. In order to determine which of these potentially stable sign patterns are IAPs, we only need to determine which of these sign patterns that are not potentially nilpotent are IAPs.

From [1], the sign patterns which are not potentially nilpotent are the following

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28 sign patterns:

$$\mathcal{A}_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{A}_2 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{A}_3 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

where patterns $\mathcal{A}_1 - \mathcal{A}_3$ appeared in the Proposition 3.1(d) of [1];

$$\mathcal{A}_4 = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix}, \ \mathcal{A}_5 = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \ \mathcal{A}_6 = \begin{bmatrix} - & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{A}_7 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \ \mathcal{A}_8 = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \ \mathcal{A}_9 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

$$\mathcal{A}_{10} = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_{11} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_{12} = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\mathcal{A}_{13} = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_{14} = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix},$$

where patterns $\mathcal{A}_4 - \mathcal{A}_{14}$ appeared in the Theorem 3.6 of [1];

$$\mathcal{A}_{15} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_{16} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_{17} = \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\mathcal{A}_{18} = \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}, \ \mathcal{A}_{19} = \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \ \mathcal{A}_{20} = \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

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$$\mathcal{A}_{21} = \begin{bmatrix} - & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \; \mathcal{A}_{22} = \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \; \mathcal{A}_{23} = \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix},$$

where patterns $\mathcal{A}_{15} - \mathcal{A}_{23}$ appeared in the Theorem 3.7 of [1];

$$\mathcal{A}_{24} = \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix},$$

where pattern $\mathcal{A}_{24}$ appeared in Page 193 of [1], and it is a superpattern of $A_{4,8}$ (described in [10]);

$$\mathcal{A}_{25} = \begin{bmatrix} + & - & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & 0 \end{bmatrix}, \; \mathcal{A}_{26} = \begin{bmatrix} + & - & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & + \end{bmatrix},$$

where patterns $\mathcal{A}_{25} - \mathcal{A}_{26}$ appeared in Page 194 of [1], $\mathcal{A}_{25}$ is $A_{4,9}$ (described in [10]), and $\mathcal{A}_{26}$ is a superpattern of $A_{4,9}$; and

$$\mathcal{A}_{27} = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix}, \; \mathcal{A}_{28} = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix},$$

where patterns $\mathcal{A}_{27} - \mathcal{A}_{28}$ appeared in Page 194 of [1], $\mathcal{A}_{27}$ is $A_{4,10}$, and $\mathcal{A}_{28}$ is a superpattern of $A_{4,10}$.

3. Main results. In this section, we will prove that each of $\mathcal{A}_1 - \mathcal{A}_{28}$ in the above section is not an IAP.

LEMMA 3.1.   *Each of the sign patterns $\mathcal{A}_1 - \mathcal{A}_{28}$ is not an IAP.*

Proof. First, we note that the following facts.

(1) Sign patterns $\mathcal{A}_1$ and $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_{27}$, $\mathcal{A}_5$ and $\mathcal{A}_9$, $\mathcal{A}_6$ and $\mathcal{A}_{24}$ are equivalent, respectively, and $\mathcal{A}_5$ and $\mathcal{A}_{28}$ are the same.

(2) Sign patterns $\mathcal{A}_6$, $\mathcal{A}_8$ and $\mathcal{A}_{24}$ are sign nonsingular, and so they are not IAPs.

(3) Using a computer check, we verified that none of the patterns $-\mathcal{A}_4, -\mathcal{A}_{10},$ $-\mathcal{A}_{11}, \ldots, -\mathcal{A}_{23}, -\mathcal{A}_{25}, -\mathcal{A}_{26}$ are superpatterns of a combination of permutations and signature similarities of the tsp's listed in [8, Fig. 3 and Fig. 4] and [10, Section 5.2]

that are potentially stable. Therefore, each of $\mathcal{A}_4, \mathcal{A}_{10}, \mathcal{A}_{11}, \ldots, \mathcal{A}_{23}, \mathcal{A}_{25}, \mathcal{A}_{26}$ does not allow the inertia $(4, 0, 0)$, and hence, is not an IAP.

Then we now only need to prove that each of the sign patterns $\mathcal{A}_1$, $\mathcal{A}_3$, $\mathcal{A}_5$ and $\mathcal{A}_7$ is not an IAP.

For $\mathcal{A}_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}$ and any $B_1 \in Q(\mathcal{A}_1)$. By positive diagonal similarity, we may assume $B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a & 0 & 1 & 0 \\ 0 & -b & c & 1 \\ 0 & 0 & -d & -e \end{bmatrix}$, where $a, b, c, d, e > 0$. Then

$p_{B_1}(x) = \det(xI_4 - B_1) = x^4 + (e-c)x^3 + (a+b+d-ce)x^2 + (ae+be-ac)x + ad - ace$. Now suppose $i(B_1) = (0, 0, 4)$. Then $p_{B_1}(x)$ may be written as $p_{B_1}(x) = (x^2 + p)(x^2 + q) = x^4 + (p+q)x^2 + pq$, where $p \geq 0$ and $q \geq 0$. Thus, the coefficients of x^3 and x are 0, that is, $e - c = 0$ and $ae + be - ac = 0$. So $e = c$ and $be = 0$. It is a contradiction. Then $(0, 0, 4) \notin i(\mathcal{A}_1)$, and $\mathcal{A}_1$ is not an IAP.

For $\mathcal{A}_3 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}$ and any $B_3 \in Q(\mathcal{A}_3)$. By positive diagonal similarity, we may assume $B_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a & 0 & 1 & 0 \\ 0 & b & -c & 1 \\ 0 & 0 & -d & e \end{bmatrix}$, where $a, b, c, d, e > 0$. Then $p_{B_3}(x) = x^4 + (c-e)x^3 + (a+d-b-ce)x^2 + (ac+be-ae)x + ad - ace$. Now suppose $i(B_3) = (0, 0, 4)$. Thus, the coefficients of x^3 and x are 0, that is, $c - e = 0, ac + be - ae = 0$. So $e = c$ and $be = 0$, a contradiction. Then $(0, 0, 4) \notin i(\mathcal{A}_3)$, and $\mathcal{A}_3$ is not an IAP.

For $\mathcal{A}_5 = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}$ and any $B_5 \in Q(\mathcal{A}_5)$. By positive diagonal similarity, we may assume $B_5 = \begin{bmatrix} -a & 1 & 0 & 0 \\ -b & c & 1 & 0 \\ 0 & d & -e & 1 \\ 0 & 0 & -f & 0 \end{bmatrix}$, where $a, b, c, d, e, f > 0$. Then $p_{B_5}(x) = x^4 + (a + e - c)x^3 + (b + f - d + ae - ac - ce)x^2 + (be + af - ad - cf - ace)x + bf - acf$. Now suppose $i(B_5) = (1, 0, 3)$. Then $p_{B_5}(x)$ may be written as

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$p_{B_5}(x) = x(x-p)(x^2+q) = x^4 - px^3 + qx^2 - pqx$, where $p > 0$ and $q \geq 0$. Thus,

$$(3.1) \quad \begin{cases} a + e - c < 0, \\ b + f - d + ae - ac - ce \geq 0, \\ be + af - ad - cf - ace \leq 0, \\ bf - acf = 0, \\ be + af - ad - cf - ace = (a + e - c)(b + f - d + ae - ac - ce). \end{cases}$$

By the fourth equation in (3.1), we have $b = ac$. By the last equation in (3.1), we have

$$\begin{aligned} & e(b + f - d + ae - ac - ce) \\ & = be + af - ad - cf - ace - (a - c)(b + f - d + ae - ac - ce) \\ & = -cd - e(a - c)^2 < 0. \end{aligned}$$

This contradicts the second equation in (3.1). Thus, $(1, 0, 3) \notin i(\mathcal{A}_5)$, and $\mathcal{A}_5$ is not an IAP.

For $\mathcal{A}_7 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}$ and any $B_7 \in Q(\mathcal{A}_7)$. By positive diagonal similarity, we may assume $B_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a & -b & 1 & 0 \\ 0 & -c & -d & 1 \\ 0 & 0 & -e & f \end{bmatrix}$, where $a, b, c, d, e, f > 0$. Then

$p_{B_7}(x) = x^4 + (b + d - f)x^3 + (a + c + e + bd - bf - df)x^2 + (ad + be - af - cf - bdf)x + ae - adf$. Now suppose $i(B_7) = (1, 0, 3)$. Then $p_{B_7}(x)$ may be written as $p_{B_7}(x) = x(x-p)(x^2+q) = x^4 - px^3 + qx^2 - pqx$, where $p > 0$ and $q \geq 0$. Thus,

$$(3.2) \quad \begin{cases} b + d - f < 0, \\ a + c + e + bd - bf - df \geq 0, \\ ad + be - af - cf - bdf \leq 0, \\ ae - adf = 0, \\ ad + be - af - cf - bdf = (b + d - f)(a + c + e + bd - bf - df). \end{cases}$$

By the fourth equation in (3.2), we have $e = df$. By the last equation in (3.2), we have

$$\begin{aligned} & b(a + c + e + bd - bf - df) \\ & = ad + be - af - cf - bdf - (d - f)(a + c + e + bd - bf - df) \\ & = -cd - b(d - f)^2 < 0. \end{aligned}$$

This contradicts the second equation in (3.2). Thus, $(1, 0, 3) \notin i(\mathcal{A}_7)$, and $\mathcal{A}_7$ is not an IAP. □

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Combining above discussions and Theorem 3.10 in [1], the following theorem is clear.

THEOREM 3.2. *If A is a tree sign pattern of order* 4*, then the following statements are equivalent:*

(1) *A is inertially arbitrary.*

(2) *A is spectrally arbitrary.*

(3) *A is potentially stable and potentially nilpotent.*

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