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THE SIGNLESS LAPLACIAN SEPARATOR OF GRAPHS*

ZHIFU YOU† AND BOLIAN LIU†

Abstract. The signless Laplacian separator of a graph is defined as the difference between the largest eigenvalue and the second largest eigenvalue of the associated signless Laplacian matrix. In this paper, we determine the maximum signless Laplacian separators of unicyclic, bicyclic and tricyclic graphs with given order.

Key words. Signless Laplacian, Separator, c-cyclic graphs.

AMS subject classifications. 15A18, 05C50.

1. Introduction. Let G be a simple graph with n vertices and m edges. A connected graph is called a *c-cyclic* graph if $c = m - n + 1$. A connected graph is unicyclic (resp., bicyclic, tricyclic) when $m = n$ (resp., $m = n + 1, m = n + 2$). Denote by $\delta(G)$ and $\Delta(G)$ the minimum and the maximum degree, respectively. Let A be the adjacency matrix of G. Since A is symmetric, its eigenvalues are real and can be written in descending order: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The Laplacian matrix of G is defined as $L = D - A$, where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ is the diagonal matrix of vertex degrees. The Laplacian spectrum of G (the spectrum of its Laplacian matrix) consists of the values $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$. The signless Laplacian matrix of G is defined as $Q = D + A$. Denote the eigenvalues of Q by $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0$.

The *separator* $S_A(G)$ of G is the difference between its largest and second largest eigenvalues, i.e., $S_A(G) = \lambda_1(G) - \lambda_2(G)$. In [1], Li et al. defined the *Laplacian separator* $S_L(G)$ as the difference between its largest and second largest Laplacian eigenvalues, i.e., $S_L(G) = \mu_1(G) - \mu_2(G)$.

The set of trees, unicyclic graphs and bicyclic graphs of order n are denoted by T_n, U_n and B_n, respectively.

Li et al. [1] obtained:

THEOREM 1.1. ([1]) *If $T \in T_n$ with $n \geq 4$, then $S_A(T) \leq \sqrt{n-1}$. Equality holds*

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†School of Mathematical Science, South China Normal University, Guangzhou, 510631, P.R. China (youzhf@hotmail.com, Zhifu You; liubl@scnu.edu.cn, Bolian Liu). This work was supported by the NNSF of China (No. 11071088).

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if and only if $T \cong K_{1,n-1}$.

THEOREM 1.2. ([1]) (1) *If* $T \in T_n$ *with* $n \geq 3$*, then* $S_L(T) \leq n-1$*, and equality holds if and only if* $T \cong K_{1,n-1}$.

(2) *For any* $T \in T_n$ *with* $n \geq 6$*, if* T *is neither* $K_{1,n-1}$ *nor* T_n^2*, then* $S_L(T) < n - \frac{5+\sqrt{5}}{2}$*, where* T_n^2 *is the tree obtained by joining two isolated vertices to two pendant vertices of* $K_{1,n-2}$.

THEOREM 1.3. ([1]) *If* $U \in U_n$ *with* $n \geq 6$*, then* $S_L(U) \leq \sqrt{n-3}$*, and equality holds if and only if* $U \cong U_n^1$*, where* U_n^1 *is the graph obtained by attaching* $n-3$ *pendant vertices to a common vertex of* C_3.

Similarly, the *signless Laplacian separator* $S_Q(G)$ is the difference between its largest and second largest signless Laplacian eigenvalues, i.e., $S_Q(G) = q_1(G) - q_2(G)$.

In fact, Das [2] has proved five conjectures among a series of 30 conjectures (see [3]) on Laplcian eigenvalues and signless Laplacian eigenvalues of G. And two conjectures are related to the *signless Laplacian separator*.

CONJECTURE 1.4. ([3]) *If* G *is a connected graph of order* $n \geq 4$*, then* $q_1 - q_2 \leq n$ *with equality holding if and only if* $G \cong K_n$.

CONJECTURE 1.5. ([3]) *If* T *is a tree of order* $n \geq 4$*, then* $q_1 - q_2 \leq n-1$ *with equality holding if and only if* $G \cong K_{1,n-1}$.

It is well known that the Laplacian matrix and signless Laplacian matrix of G have the same eigenvalues when G is bipartite (see [4]). Then Conjecture 1.5 can be directly deduced from Theorem 1.2.

In this paper, we determine the maximum signless Laplacian separators of unicyclic, bicyclic and tricyclic graphs with given order.

2. Preliminaries.

We refer to the signless Laplacian eigenvalues as Q-eigenvalues.

In this section, we present some lemmas about Q-eigenvalues.

LEMMA 2.1. ([5]) *For* $e \notin E(G)$*, the* Q*-eigenvalues of* G *and* $G' = G+e$ *interlace, i.e.,*

$$q_1(G') \geq q_1(G) \geq q_2(G') \geq q_2(G) \geq \cdots \geq q_n(G') \geq q_n(G).$$

Denote by $d(v)$ and $N(v)$ the degree and set of neighbours of vertex v, respectively.

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LEMMA 2.2. ([6]) $q_1(G) \leq max\{d(v)+m(v) : v \in V\}$ *and* $m(v) = \sum_{u \in N(v)} \frac{d(u)}{d(v)}$.

LEMMA 2.3. ([7]) *Suppose* $c \geq 0$*, and* G *is a* c*-cyclic graph with* n *vertices with* $\Delta \leq n-2$*. If* $n \geq max\{c+5, 2c+3\}$*, then* $q_1(G) \leq n$.

In [2], Das obtained:

LEMMA 2.4. ([2]) *If* G *is a connected graph, then* $q_1 - q_2 \leq \Delta + 1$.

We will determine the maximum Q-separator among unicyclic, bicyclic, and tricyclic graphs according to the maximum degree Δ.

By Lemma 2.4, it is easy to see:

LEMMA 2.5. *If* G *is connected and* $\Delta(G) \leq n-4$*, then* $S_Q(G) \leq n-3$.

In the following, we first characterize the subgraphs which are contained in a c-cyclic graphs with $\Delta = n-3$ or $\Delta = n-2$.

LEMMA 2.6. *Let* G *be a connected* c*-cyclic graph with* $n \geq 7$ *vertices. If* $c \geq 1$ *and* $\Delta(G) = n-3$*, then* G *must have one of* H_i $(i = 1, 2, \ldots, 12)$ *as a subgraph, where* H_i *are depicted in Figure 1.*

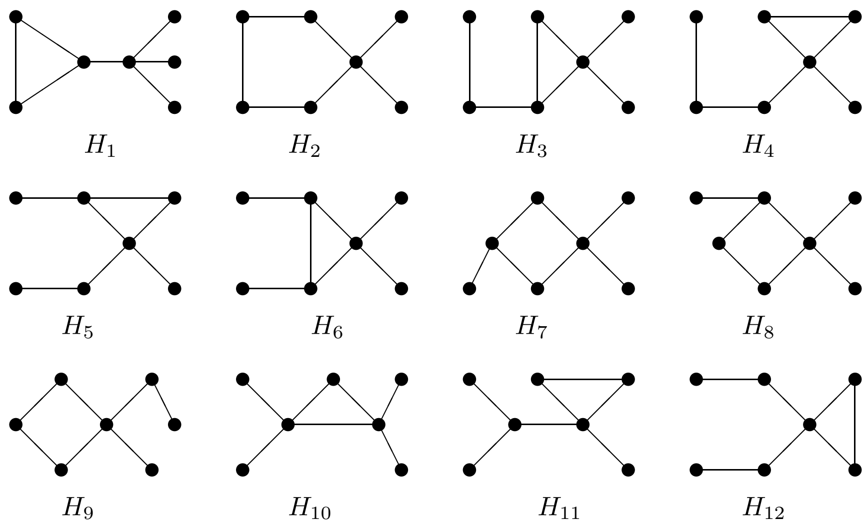

Figure 1. Subgraphs contained in c-cyclic graphs with $c \geq 1$ and $\Delta = n-3$.

Proof. Since $\Delta = n-3$, G has the subgraph $K_{1,n-3}$ and two vertices non-adjacent to the Δ-degree vertex. Note that $c \geq 1$. Then G has one of H_1, H_2, …, H_{12} as a subgraph. ∎

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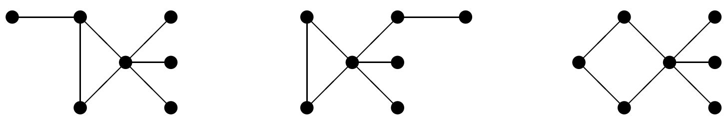
TABLE 2.1
The second largest Q-eigenvalues of H_i $(i = 1, 2, \dots, 12)$.

H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}
4	3.3675	3.3880	3.2149	3.2470	3.2384	3.8360	3.4142	3.1129	3.7321
H_{11}	H_{12}								
3.5565	2.7935								

LEMMA 2.7. *Let G be a connected c-cyclic graph with $n \geq 7$ vertices. If $c \geq 1$ and $\Delta(G) = n - 2$, then G contains one of H_j $(j = 13, 14, 15)$ as a subgraph, where H_j are depicted in Figure 2.*

Proof. Since $\Delta = n - 2$, G has the subgraph $K_{1,n-2}$ and one vertex non-adjacent to the Δ-degree vertex. Note that $c \geq 1$. Then G has one of H_{13}, H_{14}, H_{15} as a subgraph. ☐



H_{13} H_{14} H_{15}

Figure 2. Subgraphs contained in c-cyclic graphs with $c \geq 1$ and $\Delta = n - 2$.

By computer calculations, we have $q_2(H_{13}) \doteq 3.3441$, $q_2(H_{14}) \doteq 2.7636$ and $q_2(H_{15}) \doteq 3.1404$.

3. The maximum Q-separator of unicyclic graphs.

LEMMA 3.1. *Let U be a unicyclic graph with n vertices. If $n \geq 7$ and $\Delta = n - 3$, then $S_Q(U) < n - 3$.*

Proof. Let U be a unicyclic graph with n $(n \geq 7)$ vertices. By Lemma 2.3, $q_1(U) \leq n$ holds. By Lemma 2.6, U contains one of H_i $(i = 1, 2, \dots, 12)$ as a subgraph. Then U has $H_i + N_{n-7}$ as a spanning subgraph, where N_{n-7} is the null graph of order $n - 7$. By Lemma 2.1 and Table 2.1, if U contains H_i $(i = 1, 2, \dots, 11)$, then $q_2(U) \geq q_2(H_i + N_{n-7}) = q_2(H_i) > 3$. We have $S_Q(U) < n - 3$.

Suppose U contains H_{12} as a subgraph. Note that $\Delta = n-3$. By Lemmas 2.1 and 2.2, then $q_1(U) \leq n-3+\frac{2n-(n-3)-1-1}{n-3} = n-2+\frac{4}{n-3}$ and $q_2(U) \geq q_2(H_{12}+N_{n-7}) = q_2(H_{12})$. Hence $S_Q(U) \leq n - 2 + \frac{4}{n-3} - 2.7935 < n - 3$ holds for $n \geq 7$. ☐

LEMMA 3.2. *Let U be a unicyclic graph with n vertices. If $n \geq 7$ and $\Delta = n - 2$, then $S_Q(U) < n - 3$.*

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Proof. Suppose $n \geq 7$. By Lemma 2.3, then $q_1(U) \leq n$. By Lemma 2.6, U must contain H_{13}, H_{14}, or H_{15} as a subgraph and $H_j + N_{n-7}$ a spanning subgraph. If U contains H_{13} or H_{15}, then $q_2(U) \geq q_2(H_j + N_{n-7}) = q_2(H_j) > 3$. Hence $S_Q(U) < n - 3$.

Suppose U contains H_{14} as a subgraph. Note that $\Delta = n-2$. By Lemma 2.2, then $q_1(U) \leq n-2+\frac{2n-(n-2)-1}{n-2} = n-1+\frac{3}{n-2}$. Hence $S_Q(U) \leq n-1+\frac{3}{n-2}-2.7636 < n-3$ holds for $n \geq 7$. ☐

Denote by U_n^1 the unicyclic graph obtained from C_3 by attaching $n-3$ pendant edges to a common vertex of C_3.

Lemma 3.3. *If $n \geq 7$, then $n - 3 < S_Q(U_n^1) = r_1 - r_2 < n$, where r_1 and r_2 are the first two largest solutions of the equation $x^3 - (3+n)x^2 + 3nx - 4 = 0$.*

Proof. Consider the characteristic polynomial of $Q(U_1)$ which is

$$\Phi(Q(U_n^1); x) = (x-1)^{n-3}[x^3 - (3+n)x^2 + 3nx - 4].$$

Suppose $f(x) = x^3 - (3+n)x^2 + 3nx - 4$. Note that $n \geq 7$. By direct calculations, we have

$$f(n+1) = n^2 - n - 6 > 0, \quad f(n) = -4 < 0, \quad f(3) = -4 < 0,$$

$$f(1) = 2n - 6 > 0 \quad \text{and} \quad f(0) = -4 < 0.$$

The roots of the equation $f(x) = 0$ lie in $(0,1)$, $(1,3)$ and $(n, n+1)$. Then $n < q_1(U_n^1) < n+1$ and $1 < q_2(U_n^1) < 3$. Hence $n - 3 < S_Q(U_n^1) = r_1 - r_2 < n$. ☐

Lemma 3.4. *If $3 \leq n \leq 6$, then $S_Q(U) \leq S_Q(U_n^1) = r_1 - r_2 < n$ for $n = 3, 4, 5, 6$, respectively, where r_1 and r_2 are the first two largest solutions of the equation $x^3 - (3+n)x^2 + 3nx - 4 = 0$.*

Proof. The unique unicyclic graph is $C_3 \cong U_3^1$ if $n = 3$. For $n = 4$, there are two unicyclic graphs and $S_Q(U) \leq S_Q(U_4^1) \doteq 4.5616 - 2$. For $n = 5$, there are 5 unicyclic graphs and $S_Q(U) \leq S_Q(U_5^1) \doteq 5.3234 - 2.3579$.

By the table of graphs on six vertices [8] and direct calculations among the 13 unicyclic graphs, $S_Q(U) \leq S_Q(U_6^1) \doteq 6.2015 - 2.5451$ holds. ☐

Combining Lemmas 2.5 and 3.1–3.4, we arrive at the main result of this section:

Theorem 3.5. *Let U be a unicyclic graph with n vertices. If $n \geq 3$, then $S_Q(U) \leq S_Q(U_n^1) = r_1 - r_2 < n-1$, where r_1 and r_2 are the first two largest solutions of the equation $x^3-(3+n)x^2+3nx-4 = 0$, and equality holds if and only if $U \cong U_n^1$.*

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4. The maximum Q-separator of bicyclic graphs. By Lemma 2.5, we only need to consider bicyclic graphs with $\Delta \geq n-3$.

LEMMA 4.1. *Let B be a bicyclic graph with n vertices. If $n \geq 7$ and $\Delta = n-3$, then $S_Q(B) < n-3$.*

Proof. Similar to Lemma 3.1, if B contains H_i $(i = 1, 2, \ldots, 11)$ as a subgraph, then $S_Q(B) < n-3$.

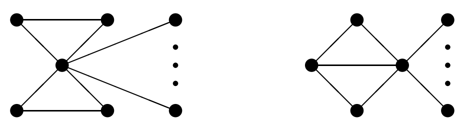
Suppose B contains H_{12} as a subgraph. By Lemma 2.2, then $q_1(B) \leq n-3+\frac{2(n+1)-(n-3)-1-1}{n-3} = n-2+\frac{6}{n-3}$. Hence $S_Q(B) \leq n-2+\frac{6}{n-3} - 2.7935 < n-3$ holds for $n \geq 7$. ∎

LEMMA 4.2. *Let B be a bicyclic graph with n vertices. If $n \geq 7$ and $\Delta = n-2$, then $S_Q(B) \leq n-3$.*

Proof. Similar to Lemma 3.2, if B contains H_{13} or H_{15}, then $S_Q(B) < n-3$.

Suppose B contains H_{14} as a subgraph. Note that $\Delta = n-2$ and B is a bicyclic graph. Then B contains a subgraph obtained from H_{14} by adding an edge. There are six such subgraphs and the minimal q_2-value is 3. Hence $S_Q(B) \leq n-3$. ∎

The bicyclic graphs with n vertices and $\Delta = n-1$ are B_n^1 and B_n^2 (Figure 3).



B_n^1 B_n^2

Figure 3. Bicyclic graphs with $\Delta = n-1$ and n vertices.

In [7], by computer calculations, one of the present authors has obtained that the characteristic polynomials of $Q(B_n^1)$ and $Q(B_n^2)$ are $(x-1)^{n-4}(x-3)[x^3-(n+3)x^2+3nx-8]$ and $(x-1)^{n-4}(x-2)[x^3-(n+4)x^2+4nx-8]$, respectively.

LEMMA 4.3. *If $n \geq 7$, then $S_Q(B_n^2) < n-3 < S_Q(B_n^1) = r_1 - 3$, where r_1 is the largest solution of the equation $x^3-(n+3)x^2+3nx-8 = 0$.*

Proof. Let $f(x) = x^3-(n+3)x^2+3nx-8$. We have

$$f(n+1) = n^3-n-10 > 0, \quad f(n) = -8 < 0, \quad f(3) = -8 < 0,$$

$$f(2) = 2(n-6) > 0 \quad \text{and} \quad f(0) = -8 < 0.$$

The roots of the equation $f(x) = 0$ lie in $(0,2)$, $(2,3)$ and $(n, n+1)$. Then

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$n < q_1(B_n^1) < n+1$ and $q_2(B_n^1) = 3$. Hence $n-3 < S_Q(B_n^1) = r_1 - 3$, where r_1 is the largest eigenvalue of the equation $x^3 - (n+3)x^2 + 3nx - 8 = 0$.

Let $g(x) = x^3 - (n+4)x^2 + 4nx - 8$. Note that $n \geq 7$. We have

$$g\left(n+\frac{1}{3}\right) = \frac{1}{27}(9n^2 - 30n - 227) > 0, \quad g(n) = -8 < 0,$$

$$g(4) = -8 < 0, \quad g\left(\frac{10}{3}\right) = \frac{4}{27}(15n - 104) > 0,$$

$$g(1) = 3n - 11 > 0 \text{ and } g(0) = -8 < 0.$$

The roots of the equation $g(x) = 0$ lie in $(0,1)$, $(\frac{10}{3}, 4)$ and $(n, n+\frac{1}{3})$. Then $n < q_1(B_n^2) < n + \frac{1}{3}$ and $\frac{10}{3} < q_2(B_n^1) < 4$. Hence $S_Q(B_n^2) < n + \frac{1}{3} - \frac{10}{3} = n - 3$. ∎

Lemma 4.4. *Let B ba a bicyclic graph with n vertices. If $n = 4$, 5, or 6, then $S_Q(B) \leq S_Q(B_n^2)$.*

Proof. There is only one bicyclic graph B_4^2 for $n = 4$. For $n = 5$, there are five bicyclic graphs. By direct calculations, we have $S_Q(B) \leq S_Q(B_5^2) \doteq 5.7785 - 2.7108 = 3.0677$.

By the table of graphs on six vertices [8], there are 19 bicyclic graphs. By direct calculations, then $S_Q(B) \leq S_Q(B_6^2) \doteq 6.4940 - 3.1099 = 3.3841$. ∎

By Lemmas 2.5 and 4.1–4.4, we have:

Theorem 4.5. *Let B be a bicyclic graph with n vertices.*

(1) *If $n \geq 7$, then $S_Q(B) \leq S_Q(B_n^1) = r_1 - 3$, where r_1 is the largest solution of the equation $x^3 - (n+3)x^2 + 3nx - 8 = 0$, and equality holds if and only if $U \cong B_n^1$.*

(2) *If $n = 4$, 5, or 6, then $S_Q(B) \leq S_Q(B_n^2)$.*

5. The maximum Q-separator of tricyclic graphs.

Lemma 5.1. *Let F be a tricyclic graph with n vertices. If $n \geq 9$ and $\Delta = n - 3$, then $S_Q(F) < n - 3$.*

Proof. As in Lemmas 3.1 and 4.1, if F contains H_i $(i = 1, 2, \ldots, 11)$ as a subgraph, then $S_Q(F) < n - 3$.

Suppose F contains H_{12} as a subgraph. By Lemma 2.2, then $q_1(F) \leq n - 3 + \frac{2(n+2)-(n-3)-1-1}{n-3} = n - 2 + \frac{8}{n-3}$. Hence $S_Q(F) \leq n - 2 + \frac{8}{n-3} - 2.7935 < n - 3$ holds for $n \geq 9$. ∎

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By the proof of Lemma 4.2, we similarly obtain:

Lemma 5.2. *Let F be a tricyclic graph with n vertices. If $n \geq 9$ and $\Delta = n-2$, then $S_Q(F) \leq n-3$.*

Let F be a tricyclic graph with $\Delta = n-1$. Then $F \cong F_n^i$ $(i = 1, 2, \ldots, 5)$ depicted in Figure 4.

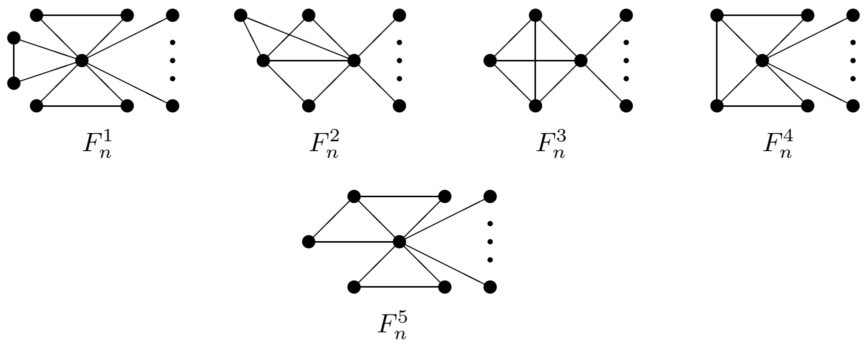

Figure 4. Tricyclic graphs with $\Delta = n-1$ and n vertices.

Lemma 5.3. *If $n \geq 9$, then $S_Q(F_n^1) = r_1 - 3 > n - 3 > S_Q(F_n^i)$ $(i = 2, \ldots, 5)$, where r_1 is the largest solution of the equation $x^3 - (n+3)x^2 + 3nx - 12 = 0$.*

Proof. By computer calculations or [7], the Q-characteristic polynomials of F_n^i are as follows.

The characteristic polynomial of $Q(F_n^1)$ is

$$\Phi(Q(F_n^1); x) = (x-1)^{n-5}(x-3)^2[x^3 - (n+3)x^2 + 3nx - 12].$$

Let $f_1(x) = x^3 - (n+3)x^2 + 3nx - 12$. Note that $n \geq 9$. Then

$$f_1(n+1) = n^2 - n - 14 > 0, \quad f_1(n) = -12 < 0, \quad f_1(3) = -12 < 0,$$

$$f_1(2) = 2(n-8) > 0 \text{ and } f_1(0) = -12 < 0.$$

The roots of the equation $f_1(x) = 0$ lie in $(0, 2)$, $(2, 3)$ and $(n, n+1)$. Then $n < q_1(F_n^1) < n+1$ and $q_2(F_n^1) = 3$. Hence $n - 3 < S_Q(F_n^1)$.

The characteristic polynomials of $Q(F_n^2)$ and $Q(F_n^3)$ are equal,

$$\Phi(Q(F_n^2); x) = \Phi(Q(F_n^3); x) = (x-1)^{n-5}(x-2)^2[x^3 - (n+5)x^2 + 5nx - 12].$$

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Let $f_2(x) = f_3(x) = x^3 - (n+5)x^2 + 5nx - 12$. Note that $n \geq 9$. Then

$$f_2\left(n+\frac{1}{3}\right) = \frac{1}{27}(9n^2 - 39n - 338) > 0, \quad f_2(n) = -12 < 0, \quad f_2(5) = -12 < 0,$$

$$f_2\left(\frac{10}{3}\right) = \frac{2}{27}(75n - 412) > 0 \text{ and } f_2(0) = -12 < 0.$$

The roots of the equation $f_2(x) = 0$ lie in $(0, \frac{10}{3})$, $(\frac{10}{3}, 5)$ and $(n, n+\frac{1}{3})$. Then $n < q_1(F_n^2) = q_1(F_n^3) < n + \frac{1}{3}$ and $\frac{10}{3} < q_2(F_n^1) = q_1(F_n^3) < 5$. Hence $S_Q(F_n^2) = S_Q(F_n^3) < n + \frac{1}{3} - \frac{10}{3} = n - 3$.

The characteristic polynomial of $Q(F_n^4)$ is

$$\Phi(Q(F_n^4); x) = (x-1)^{n-5}(x-3)[x^4 - (n+6)x^3 + (6n+7)x^2 - (7n+12)x + 20].$$

Let $f_4(x) = x^4 - (n+6)x^3 + (6n+7)x^2 - (7n+12)x + 20$. Note that $n \geq 9$. Then

$$f_4\left(n+\frac{1}{2}\right) = \frac{1}{16}(8n^3 - 36n^2 - 178n + 241) > 0, \quad f_4(n) = -12n + 20 < 0,$$

$$f_4(5) = -10(n-1) < 0, \quad f_4\left(\frac{7}{2}\right) = \frac{1}{16}(98n - 695) > 0,$$

$$f_4(1) = -2n + 10 < 0 \text{ and } f_4(0) = 20 > 0.$$

The roots of the equation $f_4(x) = 0$ lie in $(0,1)$, $(1, \frac{7}{2})$, $(\frac{7}{2}, 5)$ and $(n, n+\frac{1}{2})$. Then $n < q_1(F_n^4) < n + \frac{1}{2}$ and $\frac{7}{2} < q_2(F_n^4) < 5$. Hence $S_Q(F_n^4) < n + \frac{1}{2} - \frac{7}{2} = n - 3$.

The characteristic polynomial of $Q(F_n^5)$ is

$$\Phi(Q(F_n^5); x) = (x-1)^{n-5}(x-2)[x^4 - (n+7)x^3 + (7n+12)x^2 - (12n+12)x + 40].$$

Let $f_5(x) = x^4 - (n+7)x^3 + (7n+12)x^2 - (12n+12)x + 40$. Note that $n \geq 9$. Then

$$f_5\left(n+\frac{1}{2}\right) = \frac{1}{16}(8n^3 - 44n^2 - 146n + 579) > 0, \quad f_5(n) = -12n + 40 < 0,$$

$$f_5(4) = -8 < 0, \quad f_5\left(\frac{7}{2}\right) = \frac{1}{16}(14n - 81) > 0,$$

$$f_5(2) = -4(n-6) < 0 \text{ and } f_5(0) = 40 > 0.$$

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The roots of the equation $f_5(x) = 0$ lie in $(0, 2)$, $(2, \frac{7}{2})$, $(\frac{7}{2}, 4)$ and $(n, n + \frac{1}{2})$. Then $n < q_1(F_n^5) < n + \frac{1}{2}$ and $\frac{7}{2} < q_2(F_n^5) < 4$. Hence $S_Q(F_n^5) < n + \frac{1}{2} - \frac{7}{2} = n - 3$. ∎

By Lemmas 2.5 and 5.1–5.3, we have:

THEOREM 5.4. *Let F be a tricyclic graph with n vertices. If $n \geq 9$, then $S_Q(F) \leq S_Q(F_n^1) = r_1 - 3$, where r_1 is the largest solution of the equation $x^3 - (n+3)x^2 + 3nx - 12 = 0$, and equality holds if and only if $U \cong F_n^1$.*

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