

A NOTE ON MINIMUM RANK AND MAXIMUM NULLITY OF SIGN PATTERNS*

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Abstract. The minimum rank of a sign pattern matrix is defined to be the smallest possible rank over all real matrices having the given sign pattern. The maximum nullity of a sign pattern is the largest possible nullity over the same set of matrices, and is equal to the number of columns minus the minimum rank of the sign pattern. Definitions of various graph parameters that have been used to bound maximum nullity of a zero-nonzero pattern, including path cover number and edit distance, are extended to sign patterns, and the SNS number is introduced to usefully generalize the triangle number to sign patterns. It is shown that for tree sign patterns (that need not be combinatorially symmetric), minimum rank is equal to SNS number, and maximum nullity, path cover number and edit distance are equal, providing a method to compute minimum rank for tree sign patterns. The minimum rank of small sign patterns is determined.

Key words. Minimum rank, Maximum nullity, Sign pattern, Tree sign pattern, Asymmetric minimum rank, Path cover number, Edit distance, SNS sign pattern, SNS number, Ditree, Matrix.

AMS subject classifications. 15B35, 05C50, 15A03, 05C05, 94A05, 68Q17.

1. Introduction. The *minimum rank problem for a sign pattern* asks for a determination of the minimum rank among all real matrices whose pattern of signs is described by a given sign pattern matrix. This is a variant on the *symmetric minimum rank problem for a simple graph*, which asks for a determination of the minimum rank among all real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G (the diagonal of the matrix is free), and the *asymmetric minimum rank problem*, which asks for a determination of the minimum rank among all real matrices whose zero-nonzero pattern of entries is described by a given digraph or zero-nonzero pattern. The symmetric minimum rank problem arose from the study of possible eigenvalues of real symmetric matrices described by a graph and has received considerable attention over the last ten years (see [7] and references therein). Recently minimum rank problems for digraphs or zero-nonzero patterns have been receiving attention (see, for example, [1, 5]).

Minimum rank problems have found application to the study of communication complexity in computer science. Many of these connections involve the minimum rank of a sign pattern rather than a graph or digraph. For example, Forster [9] establishes a lower bound on the minimum rank of a sign pattern having no zero entries and uses this to establish a

*Received by the editors on December 15, 2009. Accepted for publication on February 27, 2011. Handling Editor: Bryan Shader.

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linear lower bound on unbounded error probabilistic communication complexity. The application to communication complexity gives added importance to the minimum rank problem for sign patterns.

One obvious strategy is to attempt to extend results for the minimum rank of graphs and digraphs to sign patterns. However, not all results remain valid for sign patterns, and in order to extend some other results new parameters are needed, since the straightforward extensions fail. For example, the triangle number is used to study digraphs, but it is not useful for sign patterns. We introduce the SNS number (Definition 3.1) to usefully generalize triangle number to sign patterns, and this allows us to extend the solutions of the minimum rank problem for combinatorially symmetric tree sign patterns [6] and for directed trees [1] to tree sign patterns (see Section 4). In Section 5 we consider extreme minimum rank of sign patterns and determine the minimum rank of small sign patterns having no entry equal to zero.

2. Definitions and terminology. A *sign pattern* is a matrix having entries in $\{+, -, 0\}$; a *full sign pattern* has entries in $\{+, -\}$. A *zero-nonzero pattern* is a matrix having entries in $\{*, 0\}$, where $*$ indicates a nonzero entry. For a real matrix A , $\text{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A . If X is an $n \times m$ sign pattern, the *sign pattern class* (or *qualitative class*) of X , denoted $\mathcal{Q}(X)$, is the set of all $A \in \mathbb{R}^{n \times m}$ such that $\text{sgn}(A) = X$. The *minimum rank* of a sign pattern X is

$$\text{mr}(X) = \min\{\text{rank}(A) : A \in \mathcal{Q}(X)\},$$

and the *maximum nullity* of X is

$$M(X) = \max\{\text{null}(A) : A \in \mathcal{Q}(X)\}.$$

If X is $n \times m$, then $\text{mr}(X) + M(X) = m$.

A *signature pattern* is a diagonal sign pattern that does not have any zero entries on the main diagonal. A *permutation pattern* is a sign pattern that does not have any negative entries and that has exactly one positive entry in each row and column. Signs are multiplied in the obvious manner, and in this paper, sign patterns are multiplied only when no ambiguity arises; for example, we can multiply any sign pattern by a signature pattern. Pre- or post-multiplication of a sign pattern X by a signature pattern or permutation pattern does not change the minimum rank of X , nor does taking the transpose. A sign pattern \tilde{X} that is obtained from X by performing one or more of these operations is called *equivalent* to X ; if only one type of operation is used, \tilde{X} is referred to as *signature equivalent*, *permutation equivalent*, or *transpose equivalent* to X .

Let $X = [x_{ij}]$ be an $n \times n$ sign pattern. We say X is *sign nonsingular* (SNS) if X requires nonsingularity, i.e., if every matrix $A \in \mathcal{Q}(X)$ is nonsingular; otherwise X *allows singularity*. The *generic matrix* M_X of X is the matrix having i, j entry $x_{ij}z_{ij}$, where

$\{z_{ij}, i, j = 1, \dots, n\}$, is a set of independent indeterminates. It is well-known that X is an SNS sign pattern if and only if at least one of the $n!$ terms in the standard expansion of the determinant of M_X is nonzero and all nonzero terms have the same sign [4, p. 8]. Thus, one can determine whether X is an SNS sign pattern by evaluating $\det M_X$.

A *graph* is simple (no loops or multiple edges), whereas a *digraph* allows loops (but not multiple copies of the same arc) and is denoted by $\Gamma = (V_\Gamma, E_\Gamma)$ where V_Γ and E_Γ are the sets of vertices and arcs of Γ ; in both cases, the set of vertices is finite and nonempty. If $X = [x_{ij}]$ is an $n \times n$ sign pattern, the *signed digraph of X* , $\Gamma^\pm(X)$, is obtained from the digraph $(\{1, \dots, n\}, \{(i, j) : x_{ij} \neq 0\})$ by attaching the sign $x_{ij} \in \{+, -\}$ to the arc (i, j) . A digraph Γ is *symmetric* if $(v, w) \in \Gamma$ implies $(w, v) \in \Gamma$. A signed digraph Γ is *combinatorially symmetric* if the digraph obtained by ignoring the signs is symmetric, and a sign pattern X is *combinatorially symmetric* if $\Gamma^\pm(X)$ is combinatorially symmetric.

We use the term *(signed) digraph* to mean a digraph or a signed digraph. The *underlying simple graph* of a (signed) digraph Γ is the graph obtained from Γ by ignoring the signs in the case of a signed digraph, deleting all loops, and replacing each arc (v, w) or pair of arcs (v, w) and (w, v) by the edge $\{v, w\}$. A *path* is a (signed) digraph $P_k = (\{v_1, \dots, v_k\}, E)$ such that $i \neq j$ implies $v_i \neq v_j$ and $E = \{(v_i, v_{i+1}) : i = 1, \dots, k-1\}$. A *cycle* is (signed) digraph $C_k = (\{v_1, \dots, v_k\}, E)$ such that $i \neq j$ implies $v_i \neq v_j$ and $E = \{(v_i, v_{i+1}) : i = 1, \dots, k-1\} \cup \{(v_k, v_1)\}$; the order in which the vertices appear in the cycle is denoted (v_1, \dots, v_k) . A *generalized cycle* is the disjoint union of one or more cycles. The *length* of a path or cycle is the number of arcs. Let X be a sign pattern, let $B = [b_{ij}]$ be a matrix having sign pattern X (possibly the generic matrix of X). The *cycle product* in B of a cycle (v_1, \dots, v_k) in $\Gamma^\pm(X)$ is $b_{v_1, v_2} \dots b_{v_{k-1}, v_k} b_{v_k, v_1}$, and a *generalized cycle product* in B is the product of the cycle products corresponding to the cycles in the generalized cycle. A *pseudocycle* is a digraph from which a cycle of length at least three can be obtained by reversing the direction of one or more arcs. A *(signed) ditree* is a (signed) digraph that does not contain any pseudocycles. A square sign pattern T is a *tree sign pattern* if $\Gamma^\pm(T)$ is a signed ditree.

A sign pattern X is an L -matrix if every matrix in $\mathcal{Q}(X)$ has linearly independent rows [4, p. 6]. Clearly, an $n \times m$ sign pattern with $n < m$ has $\text{mr}(X) = n$ if and only if X is an L -matrix. No full $n \times n$ sign pattern is SNS for $n \geq 3$ [3, p. 39]. See [3, 4] for more information on L -matrices.

For an $n \times m$ sign pattern X and $R \subseteq \{1, \dots, n\}$, $C \subseteq \{1, \dots, m\}$, define $X[R|C]$ to be the submatrix of X lying in the rows that have indices in R and columns that have indices in C . In a square sign pattern, the principal submatrix $X[R|R]$ is denoted $X[R]$, and the principal submatrix $X[\overline{R}]$ is usually denoted by $X(R)$, or in the case R is a single index k , by $X(k)$ (where $\overline{R} = \{1, \dots, n\} \setminus R$). For a (signed) digraph $\Gamma = (V_\Gamma, E_\Gamma)$ and $R \subseteq V_\Gamma$, the *induced subdigraph* $\Gamma[R]$ is the digraph with vertex set R and arc set $\{(v, w) \in E_\Gamma \mid v, w \in R\}$. The induced subdigraph $\Gamma^\pm(X)[R]$ is naturally associated with the signed digraph of the

the principal submatrix for R , i.e., $\Gamma^\pm(X[R])$.

There is a one-to-one correspondence between square zero-nonzero patterns and digraphs, and likewise between square sign patterns and signed digraphs. The *associated* sign pattern of a signed digraph Γ is the sign pattern X such that $\Gamma^\pm(X) = \Gamma$. We apply digraph terminology to square sign patterns and vice versa. Specifically, a *component* of a square sign pattern X is a principal submatrix $X[R]$ of X such that $\Gamma^\pm(X)[R]$ is a component of $\Gamma^\pm(X)$, i.e., $\Gamma^\pm(X)[R]$ is the signed digraph induced by the vertices of a connected component of the underlying simple graph of $\Gamma^\pm(X)$. The *generic matrix* of a signed digraph Γ is the generic matrix of its associated sign pattern, and a signed digraph *requires nonsingularity* if its associated sign pattern requires nonsingularity.

3. SNS number. One of the parameters that played a major role in the study of minimum rank of digraphs in [1] was the triangle number, and it was shown that if T is a ditree then $\text{mr}(T) = \text{tri}(T)$. It is easy to give an example of a tree sign pattern T for which $\text{mr}(T) > \text{tri}(T)$, e.g., $T = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$. However, the SNS number (see Definition 3.1) is a generalization of triangle number that retains the property of being equal to minimum rank for tree sign patterns (see Theorem 4.12).

DEFINITION 3.1. The *SNS number* of a sign pattern X , denoted $\text{SNS}(X)$, is the maximum size of an SNS sign pattern submatrix $X[R|C]$ of X .

An $n \times n$ zero-nonzero pattern requires nonsingularity (or equivalently, has minimum rank n) if and only if it is permutationally similar to a triangle zero-nonzero pattern [1, Proposition 4.6]. Thus, the SNS number is a generalization of triangle number to sign patterns.

OBSERVATION 3.2. For any sign pattern X , $\text{SNS}(X) \leq \text{mr}(X)$.

As is the case with triangle number, the inequality in Observation 3.2 can be strict, as the next example shows.

EXAMPLE 3.3. Let

$$H = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \quad \text{and} \quad H' = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \end{bmatrix}. \quad (3.1)$$

Since no full $n \times n$ pattern is SNS for $n \geq 3$ [3, p. 39], $\text{mr}(H) \leq 3$ and $\text{SNS}(H) \leq 2$. Since $\det M_H[\{1, 2\}] = -z_{11}z_{22} - z_{12}z_{21}$, $\text{SNS}(H) = 2$. It is known [3, p. 42] that H' is an L -matrix, so $\text{mr}(H') = 3$. Since H' is a submatrix of H , this implies $\text{mr}(H) = 3$.

Note that H in Example 3.3 is a *Hadamard sign pattern*, i.e., the sign pattern of a Hadamard matrix, and Hadamard sign patterns are the sign patterns to which Forster ap-

plies his lower bound on minimum rank of a full sign pattern to obtain a linear bound on unbounded error probabilistic communication complexity. Forster's lower bound on minimum rank of a full $n \times n$ sign pattern X is

$$\frac{n}{\|A_X\|}, \quad (3.2)$$

where A_X is the matrix obtained from a sign pattern X by replacing $+$ by 1 and $-$ by -1 , and $\|A\|$ is the spectral norm of A [9]. For any $n \times n$ Hadamard matrix H_n , Forster's bound (3.2) gives $\sqrt{n} \leq \text{mr}(\text{sgn}(H_n))$, since $\|H_n\| = \sqrt{n}$. Thus, Example 3.3 also shows that the minimum rank of a Hadamard sign pattern can be strictly greater than that given by Forster's lower bound, and suggests further investigation of the minimum ranks of sign patterns of Hadamard matrices.

The next result is used in place of [1, Observation 4.8] in Section 4.2.

PROPOSITION 3.4. *Let X' be obtained from the $n \times m$ sign pattern X by deleting one row. Then $\text{SNS}(X') \geq \text{SNS}(X) - 1$.*

Proof. Let X' be obtained from X by deleting row r . Let $R \subseteq \{1, \dots, n\}, C \subseteq \{1, \dots, m\}$ be such that $|R| = |C| = \text{SNS}(X)$ and $X[R|C]$ is SNS. If $r \notin R$ then $\text{SNS}(X') = \text{SNS}(X)$. So assume that $r \in R$. Let $M_X = [x_{ij}z_{ij}]$ be the generic matrix for $X = [x_{ij}]$. Since $\det M_X[R|C] \neq 0$, by computing this determinant by the Laplace expansion on row r , it is clear that there is an index $c \in C$ such that $x_{rc} \neq 0$ and $\det(M_X[R \setminus \{r\}|C \setminus \{c\}]) \neq 0$. Since $X[R|C]$ is SNS, all the terms in $\det(M_X[R \setminus \{r\}|C \setminus \{c\}])$ have the same sign, i.e., $X[R \setminus \{r\}|C \setminus \{c\}]$ is SNS. Thus, $\text{SNS}(X') \geq \text{SNS}(X) - 1$. \square

4. Tree sign patterns. In this section, we adapt edit distance and path cover number to square sign patterns, establish relationships analogous to those proved in [1], and show these parameters and SNS number can be used to compute the minimum rank of a tree sign pattern.

4.1. Other parameters.

DEFINITION 4.1. Let X be a square sign pattern. The (row) *edit distance to nonsingularity*, $\text{ED}(X)$, of X is the minimum number of rows that must be changed to obtain an SNS pattern.

THEOREM 4.2. *For any $n \times n$ sign pattern X , $\text{SNS}(X) + \text{ED}(X) = n$.*

Proof. Observe that $\text{ED}(X) \leq n - \text{SNS}(X)$, because if $X[R|C]$ is SNS and $|R| = |C| = \text{SNS}(X)$, then we can edit the $n - \text{SNS}(X)$ rows with indices in \overline{R} to get a sign pattern permutation equivalent to the SNS sign pattern

$$\begin{bmatrix} X[R|C] & X[R|\overline{C}] \\ 0 & I \end{bmatrix},$$

where $I = \text{diag}(+, +, \dots, +)$.

To show $\text{SNS}(X) \geq n - \text{ED}(X)$, let $e = \text{ED}(X)$. Perform edits on rows r_1, \dots, r_e to obtain an SNS sign pattern \tilde{X} . Let X' be obtained from X (or equivalently from \tilde{X}) by deleting rows r_1, \dots, r_e . By starting with \tilde{X} and applying Lemma 3.4 repeatedly, $n - e \leq \text{SNS}(X') \leq \text{SNS}(X)$. \square

COROLLARY 4.3. *For any square sign pattern X , $M(X) \leq \text{ED}(X)$.*

We extend the definition of path cover number given in [1] to square sign patterns. Following the definition in that paper, paths are not required to be induced, whereas in many papers studying symmetric minimum rank, paths are required to be induced (see [7] and the references therein). As with zero-nonzero patterns, allowing paths that are not induced is necessary to obtain $P(X) \leq \text{ED}(X)$ (Theorem 4.5). However, the distinction is irrelevant for tree sign patterns, to which Theorem 4.5 will be applied in Section 4.2.

DEFINITION 4.4. Let X be a square sign pattern. The path cover number $P(X)$ of X is the minimum number of vertex-disjoint paths whose deletion from $\Gamma^\pm(X)$ leaves a signed digraph that requires nonsingularity (or the empty set), i.e., the deletion of the rows and columns corresponding to the vertices of the paths leaves an SNS sign pattern (or the empty set).

The proof that $P(\Gamma) \leq \text{ED}(\Gamma)$ given in [1] uses the zero forcing number, which we have not been able to adapt to sign patterns in a useful way. In the next theorem, we give a different proof of the analogous inequality for sign patterns.

THEOREM 4.5. *For any square sign pattern X , $P(X) \leq \text{ED}(X)$.*

Proof. Let \tilde{X} be an SNS sign pattern obtained from X by editing rows r_1, \dots, r_e , where $e = \text{ED}(X)$, let $\tilde{\Gamma} = \Gamma^\pm(\tilde{X})$, and let $\Gamma = \Gamma^\pm(X)$. Since \tilde{X} is an SNS pattern, all terms in the determinant of the generic matrix $M_{\tilde{X}}$ of \tilde{X} have the same sign. Select one nonzero term t in $\det M_{\tilde{X}}$. Note that t is a generalized cycle product. Let C_i , $i = 1, \dots, f \leq e$ be the cycles in $\tilde{\Gamma}$ associated with the simple cycle products in t that contain entries from rows r_1, \dots, r_e . If C_i contains $r_{k_1}, \dots, r_{k_{s_i}}$ (in that order on the cycle), then denote the cycle vertex immediately following r_{k_j} by $u_{k_{j+1}}$ where $s_i + 1$ is interpreted as 1. If there are no other vertices between the vertices r_{k_j} and $r_{k_{j+1}}$, then $u_{k_{j+1}} = r_{k_{j+1}}$. All the vertices of cycle C_i can be deleted from X by deleting the s_i paths in Γ consisting of the vertices and cycle arcs from u_{i_j} to r_{i_j} , $j = 1, \dots, s_i$. Note that the arcs involved in these paths are all in Γ , because the only cycle arcs of $\tilde{\Gamma}$ that may not exist in Γ are the arcs $(r_{k_j}, u_{k_{j+1}})$, which are not used in these paths. Let $V_C = \cup_{i=1}^f V_{C_i}$. We claim that $\tilde{X}(V_C) = X(V_C)$ is an SNS sign pattern or the empty set; once this is established, it is clear that $P(X) \leq \text{ED}(X)$.

Assume that $\{1, \dots, n\} \setminus V_C$ is nonempty. Since we have removed vertices of complete cycles in the generalized cycle product t , t can be factored as $t = t_1 t_2$, where all of the

indices in cycle products in t_1 are in V_C and all of the indices in cycle products in t_2 are in $\{1, \dots, n\} \setminus V_C$. Thus, t_2 is a nonzero term in $\det M_{\tilde{X}}(V_C)$. Suppose that there is a term t_3 in $\det M_{\tilde{X}}(V_C)$ that has the opposite sign from t_2 . Then $t_1 t_3$ would be a term of opposite sign from $t = t_1 t_2$ in $\det M_{\tilde{X}}$, contradicting the fact that \tilde{X} is an SNS sign pattern. Thus, all the terms in $\det M_{\tilde{X}}(V_C) = \det M_{\tilde{X}(V_C)}$ have the same sign and $\tilde{X}(V_C) = X(V_C)$ is an SNS sign pattern. \square

It is easy to find an example of a sign pattern that has path cover number strictly less than edit distance and maximum nullity.

EXAMPLE 4.6. (cf. [1, Example 4.21]) Let $[+]_n$ denote the $n \times n$ sign pattern consisting entirely of positive entries. If $n \geq 3$, then

$$P([+]_n) = 1 < n - 1 = M([+]_n) = \text{ED}([+]_n).$$

4.2. Tree sign patterns. The proofs of the results in this section are omitted because they can be adapted from the proofs of the corresponding results in [1, Section 5], using results in previous sections of this paper to replace those in [1, Section 4]; the proofs are also available in the on-line appendix [10].

THEOREM 4.7. (cf. [1, Theorem 5.1]) *For every tree sign pattern T , $\text{ED}(T) \leq P(T)$.*

Using Theorems 4.5 and 4.7, we have the following corollary (cf. [1, Corollary 5.2]).

COROLLARY 4.8. *For every tree sign pattern T ,*

$$P(T) = \text{ED}(T).$$

LEMMA 4.9. (cf. [1, Lemma 5.5]) *Let T be a combinatorially symmetric $n \times n$ tree sign pattern, let $v \in \{1, \dots, n\}$, and let $S \subseteq \{1, \dots, n\}$ such that*

1. $T[S]$ is a component of $T - v$,
2. $T[S]$ allows singularity, and
3. *if $x \in S$, then $T - x$ has at most one component that is a submatrix of $T[S]$ and allows singularity.*

Then there is a path P in $\Gamma^\pm(T)$ from v to a vertex $u \in S$ such that every component of $T - V_P$ that is a submatrix of $T[S]$ is SNS.

THEOREM 4.10. (cf. [1, Theorem 5.6]) *If T is a combinatorially symmetric tree sign pattern, then*

$$M(T) = P(T) = \text{ED}(T) \quad \text{and} \quad \text{mr}(T) = \text{SNS}(T).$$

LEMMA 4.11. (cf. [1, Lemma 5.7]) Let Z be a sign pattern of the form $Z = \begin{bmatrix} X & O \\ U & W \end{bmatrix}$ where U is $k \times m$, U has exactly one nonzero entry, and that entry is in the $1, m$ -position of U . Let \tilde{X} be obtained from X by replacing the last column of X by 0s and \tilde{W} be obtained from W by replacing the first row of W by 0s. If $\text{mr}(X) = \text{SNS}(X)$, $\text{mr}(W) = \text{SNS}(W)$, $\text{mr}(\tilde{X}) = \text{SNS}(\tilde{X})$ and $\text{mr}(\tilde{W}) = \text{SNS}(\tilde{W})$, then $\text{mr}(Z) = \text{SNS}(Z)$.

THEOREM 4.12. (cf. [1, Theorem 5.8]) If T is a tree sign pattern, then

$$M(T) = P(T) = \text{ED}(T) \quad \text{and} \quad \text{mr}(T) = \text{SNS}(T).$$

Note that whereas the parameters $\text{mr}(T)$ and $M(T)$ involve optimizing over an infinite set of matrices, the computation of $\text{SNS}(T)$ or $P(T)$ involves only a finite (but possibly very large) number of subsets of vertices. Thus, Theorem 4.12 allows (at least in theory) the computation of $\text{mr}(T)$ and $M(T)$ for a tree sign pattern T .

5. Extreme minimum rank and minimum rank of small patterns. In this section, we examine extreme minimum ranks of sign patterns and determine minimum ranks of small sign patterns. Minimum rank $0, 1, 2, n, n - 1$ has been characterized for both graphs and digraphs of order n (and $n - 2$ has been characterized for graphs), but the situation is more complicated for sign patterns, except in the most trivial cases (minimum rank $0, 1$, and n for an $n \times n$ sign pattern).

A zero-nonzero pattern Y has minimum rank zero if and only if all entries are zero, and the same is obviously true for a sign pattern. After eliminating any zero rows and/or zero columns, a zero-nonzero pattern Y has $\text{mr}(Y) = 1$ if and only if every entry of Y is $*$. A similar result is true for sign patterns, accounting for signature equivalence.

PROPOSITION 5.1. For a sign pattern X that has no zero row or column, $\text{mr}(X) = 1$ if and only if $X = D_1[+]D_2$ where D_1, D_2 are signature patterns and $[+]$ denotes a sign pattern consisting entirely of positive entries.

Proof. Clearly, $\text{mr}(D_1[+]D_2) = 1$. Let X be a sign pattern such that every row and every column has a nonzero entry. If X has a zero entry then X has a 2-triangle so $\text{mr}(X) > 1$. So suppose X is a full sign pattern. Then there exists a signature pattern D_1 such that every entry of the first column of D_1X is $+$. For column $j > 1$, if $(D_1X)_{ij} = +$ and $(D_1X)_{kj} = -$, then $(D_1X)[\{i, k\}|\{1, j\}]$ is SNS, so $\text{mr}(X) = \text{mr}(D_1X) > 1$. Thus, if $\text{mr}(X) = 1$, then D_1X is striped and there exists a signature pattern D_2 such that $D_1XD_2 = [+]$. \square

For a zero-nonzero pattern Y , $\text{tri}(Y) \leq 2$ implies $\text{mr}(Y) = \text{tri}(Y)$ [1, Theorem 4.5], but Example 3.3 shows that this need not be the case for sign patterns, since $\text{SNS}(H) = 2 < 3 = \text{mr}(H)$.

For a $n \times m$ zero-nonzero pattern Y with $n \leq m$, $\text{mr}(Y) = n$ if and only if $\text{tri}(Y) = n$ [1, Proposition 4.6]. For an $n \times n$ sign pattern X , obviously $\text{mr}(X) = n$ if and only if $\text{SNS}(X) = n$. However, if X is an $n \times m$ pattern with $n < m$, then it is possible to have $\text{SNS}(X) < \text{mr}(X) = n$, as for the sign pattern H' in Example 3.3.

For symmetric minimum rank of a graph, the minimum rank is always less than the order of the graph (since the main diagonal of a matrix described by the graph is free). Fiedler's Theorem shows that symmetric minimum rank of a graph is equal to order minus one if and only if the graph is a path [8, 7]. The situation for minimum rank equal to order minus one is more complicated for digraphs (equivalently, square zero-nonzero patterns). For example, [5, Example 1.11] presents an example of a 7×7 nonzero pattern, here denoted by Y_7 , that has $\text{mr}(Y_7) = 6 = |Y_7| - 1$ and $\text{tri}(Y_7) = 5$. A characterization of digraphs Γ having $\text{mr}(\Gamma) = |\Gamma| - 1$ has been obtained [2]; this characterization is quite complex and the situation for sign patterns is likely to be much more complicated.

We now consider the minimum rank of small sign patterns. Since it is straightforward to determine whether a $n \times m$ sign pattern has minimum rank equal to 0 or 1, or equal to n in the case that $m = n$, the minimum rank of any $n \times m$ sign pattern with $n, m \leq 3$ can be easily computed. Square full sign patterns are of particular interest due to their application to communication complexity.

We now determine the minimum rank of all 4×4 full sign patterns. Since sign patterns having minimum rank equal to 1 have been characterized (Proposition 5.1) and no full sign 4×4 sign pattern is SNS, it suffices to determine which 4×4 full sign patterns have minimum rank equal to 3. A sign pattern has *duplicate* rows (respectively, columns) if it has two rows (columns) that are equal.

THEOREM 5.2. *Let $X = [x_{ij}]$ be a 4×4 full sign pattern. Then $\text{mr}(X) = 3$ if and only if X contains a submatrix equivalent to H' in equation (3.1).*

Proof. If X contains a submatrix equivalent to H' , then $\text{mr}(X) = 3$ because $\text{mr}(X) \leq 3$ and $\text{mr}(X) \geq \text{mr}(H') = 3$. For the converse, we assume that X does not contain a submatrix equivalent to H' , and show that $\text{mr}(X) \leq 2$. Without loss of generality, all the entries of the first row and first column of X are positive.

If X has duplicate rows, then delete one of the equal rows. The resulting full 3×4 sign pattern X' is not equivalent to H' , and thus is not a L -matrix [3, p. 42]. So there is a matrix $A \in \mathcal{Q}(X')$ that has linearly dependent rows, i.e., $\text{rank } A \leq 2$. Thus, $\text{mr}(X) \leq 2$. The argument is the same for columns by using transpose equivalence.

So assume that X has neither duplicate rows nor duplicate columns. Since the submatrix of X obtained by deleting row 4 is not equivalent to H' , two columns of this 3×4 submatrix are equal. By multiplying by a permutation pattern and/or signature pattern, we may assume that $x_{22} = x_{32} = +$. Since the first two columns of X are not equal, $x_{42} = -$. Since

rows 2 and 3 of X are not equal, the entries $x_{23}, x_{24}, x_{33}, x_{34}$ cannot all be the same. Thus, one must be $+$, and without loss of generality, we can assume that $x_{23} = +$. Then $x_{24} = -$ (because rows 1 and 2 are not equal) and $x_{33} = -$, because if not, two of the first three columns of X would be equal. Thus, X is equivalent to

$$\begin{bmatrix} + & + & + & + \\ + & + & + & - \\ + & + & - & x_{34} \\ + & - & x_{43} & x_{44} \end{bmatrix}. \quad (5.1)$$

The assumption that no submatrix is equivalent to H' eliminates four of the eight possibilities for the patterns of the form (5.1), as shown in Table 5.1.

TABLE 5.1

x_{34}	x_{43}	x_{44}	Rows or columns equivalent to H'
+	+	+	Rows 2, 3, 4
-	+	-	Rows 1, 3, 4
+	-	-	Columns 1, 3, 4
-	-	+	Rows 1, 3, 4

Let

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{9}{5} & -\frac{1}{5} \\ 1 & 3 & -\frac{11}{5} & \frac{29}{5} \\ 1 & -\frac{1}{4} & 3 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & \frac{1}{5} & -\frac{3}{5} \\ 1 & 3 & -\frac{3}{5} & -\frac{11}{5} \\ 1 & -\frac{1}{4} & 2 & 3 \end{bmatrix}$$

$$\text{and } A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{7}{22} & -\frac{4}{11} \\ 1 & \frac{1}{10} & -\frac{5}{22} & -\frac{16}{11} \\ 1 & -\frac{1}{10} & -\frac{1}{2} & -2 \end{bmatrix}.$$

The only 4×4 full sign patterns that have neither duplicate rows nor duplicate columns and do not contain a submatrix equivalent to H' are $\text{sgn}(A_1)$, $\text{sgn}(A_2)$, $\text{sgn}(A_2)^T$, and $\text{sgn}(A_3)$. The matrices A_1, A_2, A_3 show that

$$\text{mr}(\text{sgn}(A_1)) = \text{mr}(\text{sgn}(A_2)) = \text{mr}(\text{sgn}(A_2)^T) = \text{mr}(\text{sgn}(A_3)) = 2. \quad \square$$

Acknowledgments: This paper was inspired by the work done at the American Institute of Mathematics (AIM) SQuaRE “Minimum rank of matrices described by a graph,” and the author thanks AIM and NSF for their support of the SQuaRE. The author also

thanks Bryan Shader and Venkatesh Srinivasan for discussions of the connections between minimum rank problems and communication complexity, and the anonymous referee for simplifying Theorem 5.2 and its proof, and other helpful suggestions.

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