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THE P-LAPLACIAN SPECTRAL RADIUS OF WEIGHTED TREES WITH A DEGREE SEQUENCE AND A WEIGHT SET*

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Abstract. In this paper, some properties of the discrete p-Laplacian spectral radius of weighted trees have been investigated. These results are used to characterize all extremal weighted trees with the largest p-Laplacian spectral radius among all weighted trees with a given degree sequence and a positive weight set. Moreover, a majorization theorem with two tree degree sequences is presented.

Key words. Weighted tree, Discrete p-Laplacian, Degree sequence, Spectrum.

AMS subject classifications. 05C50, 05C05, 05C22.

1. Introduction. In the last decade, the p-Laplacian, which is a natural non-linear generalization of the standard Laplacian, plays an increasing role in geometry and partial differential equations. Recently, the discrete p-Laplacian, which is the analogue of the p-Laplacian on Riemannian manifolds, has been investigated by many researchers. For example, Amghibech in [1] presented several sharp upper bounds for the largest p-Laplacian eigenvalues of graphs. Takeuchi in [7] investigated the spectrum of the p-Laplacian and p-harmonic morphism of graphs. Luo et al. in [6] used the eigenvalues and eigenvectors of the p-Laplacian to obtain a natural global embedding for multi-class clustering problems in machine learning and data mining areas. Based on the increasing interest in both theory and application, the spectrum of the discrete p-Laplacian should be further investigated. The main purpose of this paper is to investigate some properties of the spectral radius and eigenvectors of the p-Laplacian of weighted trees.

In this paper, we only consider simple weighted graphs with a positive weight set. Let $G = (V(G), E(G), W(G))$ be a weighted graph with vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$, edge set $E(G)$ and weight set $W(G) = \{w_k > 0,\ k = 1,\ 2,\ \ldots,\ |E(G)|\}$. Let $w_G(uv)$ denote the weight of an edge uv. If $uv \notin E(G)$, define $w_G(uv) = 0$. Then $uv \in E(G)$ if and only if $w_G(uv) > 0$. The weight of a vertex u, denoted by

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$w_G(u)$, is the sum of weights of all edges incident to u in G.

Let $p > 1$. Then the *discrete p-Laplacian* $\triangle_p(G)$ of a function f on $V(G)$ is given by

$$\triangle_p(G)f(u) = \sum_{v,uv \in E(G)} (f(u) - f(v))^{[p-1]} w_G(uv),$$

where $x^{[q]} = sign(x)|x|^q$. When $p = 2$, $\triangle_2(G)$ is the well-known *(combinatorial) graph Laplacian* (see [4]), i.e., $\Delta_2(G) = L(G) = D(G) - A(G)$, where $A(G) = (w_G(v_i v_j))_{n \times n}$ denotes the weighted adjacency matrix of G and $D(G) = \text{diag}(w_G(v_0), w_G(v_1), \ldots, w_G(v_{n-1}))$ denotes the weighted diagonal matrix of G (see [8]).

A real number λ is called an *eigenvalue* of $\triangle_p(G)$ if there exists a function $f \neq 0$ on $V(G)$ such that for $u \in V(G)$,

$$\Delta_p(G)f(u) = \lambda f(u)^{[p-1]}.$$

The function f is called the *eigenfunction* corresponding to λ. The largest eigenvalue of $\Delta_p(G)$, denoted by $\lambda_p(G)$, is called the *p-Laplacian spectral radius*. Let $d(v)$ denote the degree of a vertex v, i.e., the number of edges incident to v. A nonincreasing sequence of nonnegative integers $\pi = (d_0, d_1, \cdots, d_{n-1})$ is called *graphic degree sequence* if there exists a simple connected graph having π as its vertex degree sequence. Zhang [9] in 2008 determined all extremal trees with the largest spectral radius of the Laplacian matrix among all trees with a given degree sequence. Further, Bıyıkoğlu, Hellmuth, and Leydold [2] in 2009 characterized all extremal trees with the largest p-Laplacian spectral radius among all trees with a given degree sequence. Let $\mathcal{T}_{\pi,W}$ be the set of trees with a given graphic degree sequence π and a positive weight set W. Recently, Tan [8] determined the extremal trees with the largest spectral radius of the weight Laplacian matrix in $\mathcal{T}_{\pi,W}$. Moreover, the adjacency, Laplacian and signless Laplacian eigenvalues of graphs with a given degree sequence have been studied (for example, see [3] and [10]). Motivated by the above results, we investigate the largest p-Laplacian spectral radius of trees in $\mathcal{T}_{\pi,W}$. The main result of this paper can be stated as follows:

THEOREM 1.1. *For a given degree sequence π of some tree and a positive weight set W, $T^*_{\pi,W}$ (see in Section 3) is the unique tree with the largest p-Laplacian spectral radius in $\mathcal{T}_{\pi,W}$, which is independent of p.*

The rest of this paper is organized as follows. In Section 2, some notations and results are presented. In Section 3, we give a proof of Theorem 1.1 and a majorization theorem for two tree degree sequences.

2. Preliminaries. The following are several propositions and lemmas about the Rayleigh quotient and eigenvalues of the p-Laplacian for weighted graphs. The proofs

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are similar to unweighted graphs (see [2]). So we only present the result and omit the proofs.

Let f be a function on $V(G)$ and

$$R_G^p(f) = \frac{\sum\limits_{uv \in E(G)} \mid f(u) - f(v) \mid^p w_G(uv)}{\| f \|_p^p},$$

where $\|f\|_p = \sqrt[p]{\sum\limits_v \mid f(v) \mid^p}$. The following Proposition 2.1 generalizes the well-known Rayleigh-Ritz theorem.

Proposition 2.1. ([6])

$$\lambda_p(G) = \max_{||f||_p=1} R_G^p(f) = \max_{||f||_p=1} \sum_{uv \in E(G)} \mid f(u) - f(v) \mid^p w_G(uv).$$

Moreover, if $R_G^p(f) = \lambda_p(G)$, *then* f *is an eigenfunction corresponding to the p-Laplacian spectral radius* $\lambda_p(G)$.

Define the *signless p-Laplacian* $Q_p(G)$ of a function f on $V(G)$ by

$$Q_p(G)f(u) = \sum_{v, uv \in E(G)} (f(u) + f(v))^{[p-1]} w_G(uv)$$

and its Rayleigh quotient by

$$\Lambda_G^p(f) = \frac{\sum\limits_{uv \in E(G)} \mid f(u) + f(v) \mid^p w_G(uv)}{\| f \|_p^p}.$$

A real number μ is called an *eigenvalue* of $Q_p(G)$ if there exists a function $f \neq 0$ on $V(G)$ such that for $u \in V(G)$,

$$Q_p(G)f(u) = \mu f(u)^{[p-1]}.$$

The largest eigenvalue of $Q_p(G)$, denoted by $\mu_p(G)$, is called the *signless p-Laplacian spectral radius*. Then we have the following.

Proposition 2.2. ([2])

$$\mu_p(G) = \max_{||f||_p=1} \Lambda_G^p(f) = \max_{||f||_p=1} \sum_{uv \in E(G)} \mid f(u) + f(v) \mid^p w_G(uv).$$

Moreover, if $\Lambda_G^p(f) = \mu_p(G)$, *then* f *is an eigenfunction corresponding to* $\mu_p(G)$.

Corollary 2.3. *Let* G *be a connected weighted graph. Then the signless p-Laplacian spectral radius* $\mu_p(G)$ *of* $Q_p(G)$ *is positive. Moreover, if* f *is an eigenfunction of* $\mu_p(G)$, *then either* $f(v) > 0$ *for all* $v \in V(G)$ *or* $f(v) < 0$ *for all* $v \in V(G)$.

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Let f be an eigenfunction of $\mu_p(G)$. We call f a *Perron vector* of G if $f(v) > 0$ for all $v \in V(G)$.

LEMMA 2.4. *Let $G = (V_1, V_2, E, W)$ be a bipartite weighted graph with bipartition V_1 and V_2. Then $\lambda_p(G) = \mu_p(G)$.*

Clearly, trees are bipartite graphs. So, Lemma 2.4 also holds for trees.

3. Main result. Let $G - uv$ denote the graph obtained from G by deleting an edge uv and $G + uv$ denote the graph obtained from G by adding an edge uv. The following lemmas will be used in the proof of the main result, Theorem 1.1.

LEMMA 3.1. *Let $T \in \mathcal{T}_{\pi,W}$ with $u, v \in V(T)$ and f be a Perron vector of T. Assume $uu_i \in E(T)$ and $vu_i \notin E(T)$ such that u_i is not in the path from u to v for $i = 1, 2, \ldots, k$. Let $T' = T - \bigcup_{i=1}^{k} uu_i + \bigcup_{i=1}^{k} vu_i$, $w_{T'}(vu_i) = w_T(uu_i)$ for $i = 1, 2, \ldots,$ k, and $w_{T'}(e) = w_T(e)$ for $e \in E(T) \setminus \{uu_1, uu_2, \ldots, uu_k\}$. In other words, T' is the weighted tree obtained from T by deleting the edges $uu_1, \ldots, uu_k$ and adding the edges $vu_1, \ldots, vu_k$ with their weights $w_T(uu_1), \ldots, w_T(uu_k)$, respectively. If $f(u) \le f(v)$, then $\mu_p(T) < \mu_p(T')$.*

Proof. Without loss of generality, assume $\| f \|_p = 1$. Then

$$\begin{aligned}\mu_p(T') - \mu_p(T) &\ge \Lambda^p_{T'}(f) - \Lambda^p_T(f)\\ &= \sum_{i=1}^{k}[(f(v) + f(u_i))^p - (f(u) + f(u_i))^p]w_T(uu_i)\\ &\ge 0.\end{aligned}$$

If $\mu_p(T') = \mu_p(T)$, then f must be an eigenfunction of $\mu_p(T')$. Clearly, by computing the values of the function f on $V(T)$ and $V(T')$ at the vertex u, we have

$$\begin{aligned}Q_p(T)f(u) &= \sum_{x, xu \in E(T)} (f(x) + f(u))^{[p-1]} w_T(ux)\\ &= \sum_{x, xu \in E(T')} (f(x) + f(u))^{[p-1]} w_T(ux) + \sum_{i=1}^{k} (f(u) + f(u_i))^{[p-1]} w_T(uu_i)\end{aligned}$$

and

$$Q_p(T')f(u) = \sum_{x, xu \in E(T')} (f(x) + f(u))^{[p-1]} w_T(ux).$$

Moreover, $Q_p(T)f(u) = \mu_p(T)f(u)^{[p-1]} = \mu_p(T')f(u)^{[p-1]} = Q_p(T')f(u)$. Hence $\sum_{i=1}^{k}(f(u) + f(u_i))^{[p-1]}w_T(uu_i) = 0$, which implies $f(u) + f(u_i) = 0$ for $i = 1, 2, \ldots, k$. This is impossible. So the assertion holds. □

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From Lemma 3.1 we can easily get the following corollary.

Corollary 3.2. *Let T be a weighted tree with the largest p-Laplacian spectral radius in $\mathcal{T}_{\pi,W}$ and $u, v \in V(T)$. Suppose that f is a Perron vector of T. Then we have the following:*

(1) if $f(u) \le f(v)$, then $d(u) \le d(v)$;

(2) if $f(u) = f(v)$, then $d(u) = d(v)$.

Lemma 3.3. ([2]) *Let $0 \le \varepsilon \le \delta \le z$ and $p > 1$. Then $(z+\epsilon)^p + (z-\epsilon)^p \le (z+\delta)^p + (z-\delta)^p$. Equality holds if and only if $\epsilon = \delta$.*

Lemma 3.4. *Let $T \in \mathcal{T}_{\pi,W}$ and $uv, xy \in E(T)$ such that v and y are not in the path from u to x. Let f be a Perron vector of T and $T' = T - uv - xy + uy + xv$ with $w_{T'}(uy) = \max\{w_T(uv), w_T(xy)\}$, $w_{T'}(xv) = \min\{w_T(uv), w_T(xy)\}$, and $w_{T'}(e) = w_T(e)$ for $e \in E(T) \setminus \{uv, xy\}$. If $f(u) \ge f(x)$ and $f(y) \ge f(v)$, then $T' \in \mathcal{T}_{\pi,W}$ and $\mu_p(T) \le \mu_p(T')$. Moreover, $\mu_p(T) < \mu_p(T')$ if one of the two inequalities is strict.*

Proof. Without loss of generality, assume $\parallel f \parallel_p = 1$.

Claim : $(f(u)+f(y))^p + (f(x)+f(v))^p \ge (f(u)+f(v))^p + (f(x)+f(y))^p$.

Assume $f(u)+f(y) = z+\delta$, $f(x)+f(v) = z-\delta$, $\max\{f(u)+f(v), f(x)+f(y)\} = z+\epsilon$, $\min\{f(u)+f(v), f(x)+f(y)\} = z-\epsilon$. Without loss of generality, assume $f(u)+f(v) \ge f(x)+f(y)$. Then $\delta - \epsilon = f(y) - f(v) \ge 0$. By Lemma 3.3, the Claim holds. Without loss of generality, assume $w_T(uv) \ge w_T(xy)$. Then, by the Claim and $w_{T'}(uy) = w_T(uv)$ and $w_{T'}(xv) = w_T(xy)$, we have

$$
\begin{aligned}
\mu_p(T') - \mu_p(T) &\ge \Lambda^p_{T'}(f) - \Lambda^p_T(f) \\
&= (f(u)+f(y))^p w_{T'}(uy) + (f(x)+f(v))^p w_{T'}(xv) \\
&\quad -(f(u)+f(v))^p w_T(uv) - (f(x)+f(y))^p w_T(xy) \\
&= [(f(u)+f(y))^p - (f(u)+f(v))^p] w_T(uv) \\
&\quad +[(f(x)+f(v))^p - (f(x)+f(y))^p] w_T(xy) \\
&\ge [(f(u)+f(y))^p + (f(x)+f(v))^p - (f(u)+f(v))^p \\
&\quad -(f(x)+f(y))^p] w_T(uv) \\
&\ge 0.
\end{aligned}
$$

If $\mu_p(T') = \mu_p(T)$, then $\epsilon = \delta$ by Lemma 3.3, and f must be an eigenfunction of

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$\mu_p(T')$. So $f(y) = f(v)$. Moreover, since $w_{T'}(uy) = w_T(uv) \geq w_T(xy)$ and

$$\begin{aligned}
Q_p(T)f(y) &= \sum_{z,zy \in E(T)\setminus\{xy\}} (f(z)+f(y))^{[p-1]} w_T(zy) + (f(x)+f(y))^{[p-1]} w_T(xy) \\
&= \mu_p(T) f(y)^{[p-1]} = \mu_p(T') f(y)^{[p-1]} = Q_p(T')f(y) \\
&= \sum_{z,zy \in E(T)\setminus\{xy\}} (f(z)+f(y))^{[p-1]} w_T(zy) + (f(u)+f(y))^{[p-1]} w_{T'}(uy),
\end{aligned}$$

we have $f(x) \geq f(u)$. Hence $f(x) = f(u)$, and the assertion holds. □

Lemma 3.5. *Let $T \in \mathcal{T}_{\pi,W}$ with $uv, xy \in E(T)$ and f be a Perron vector of T. If $f(u) + f(v) \geq f(x) + f(y)$ and $w_T(uv) < w_T(xy)$, then there exists a tree $T' \in \mathcal{T}_{\pi,W}$ such that $\mu_p(T') > \mu_p(T)$.*

Proof. Without loss of generality, assume $\| f \|_p = 1$. Let T' be the tree obtained from T with vertex set $V(T)$, edge set $E(T)$, $w_{T'}(uv) = w_T(xy)$, $w_{T'}(xy) = w_T(uv)$ and $w_{T'}(e) = w_T(e)$ for $e \in E(T) \setminus \{uv, xy\}$. Then we have

$$\begin{aligned}
\mu_p(T') - \mu_p(T) &\geq \Lambda^p_{T'}(f) - \Lambda^p_T(f) \\
&= [(f(u)+f(v))^p - (f(x)+f(y))^p](w_T(xy) - w_T(uv)) \\
&\geq 0.
\end{aligned}$$

If $\mu_p(T') = \mu_p(T)$, then f must be an eigenfunction of $\mu_p(T')$. Without loss of generality, assume $u \neq x$ and $u \neq y$. Since

$$\begin{aligned}
Q_p(T')f(u) &= \sum_{ut \in E(T)\setminus\{uv\}} (f(u)+f(t))^{[p-1]} w_T(ut) + (f(u)+f(v))^{[p-1]} w_T(xy) \\
&= Q_p(T)f(u) \\
&= \sum_{ut \in E(T)\setminus\{uv\}} (f(u)+f(t))^{[p-1]} w_T(ut) + (f(u)+f(v))^{[p-1]} w_T(uv),
\end{aligned}$$

we have $w_T(uv) = w_T(xy)$, which is a contradiction. So $\mu_p(T') > \mu_p(T)$. □

Let v_0 be the root of a tree T and $h(v_i)$ be the distance between v_i and v_0.

Definition 3.6. Let $T = (V(T), E(T), W(T))$ be a weighted tree with a positive weight set $W(T)$ and root v_0. Then a well-ordering $\prec$ of the vertices is called a *weighted breadth-first-search ordering* (*WBFS*-ordering for short) if the following holds for all vertices $u, v, x, y \in V(T)$:

(1) $v \prec u$ implies $h(v) \leq h(u)$;

(2) $v \prec u$ implies $d(v) \geq d(u)$;

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(3) Let $uv, uy \in E(T)$ with $h(v) = h(y) = h(u) + 1$. If $v \prec y$, then $w_T(uv) \geq w_T(uy)$;

(4) Let $uv, xy \in E(T)$ with $h(u) = h(v) - 1$ and $h(x) = h(y) - 1$. If $u \prec x$, then $v \prec y$ and $w_T(uv) \geq w_T(xy)$.

A weighted tree is called a *WBFS-tree* if its vertices have a WBFS-ordering. For a given degree sequence and a positive weight set, it is easy to see that the WBFS-tree is uniquely determined up to isomorphism by Definition 3.6 (for example, see [9]).

Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ be a degree sequence of tree such that $d_0 \geq d_1 \geq \cdots \geq d_{n-1}$ and $W = \{w_1, w_2, \ldots, w_{n-1}\}$ be a positive weight set with $w_1 \geq w_2 \geq \cdots \geq w_{n-1} > 0$. We now construct a weighted tree $T^*_{\pi,W}$ with the degree sequence π and the positive weight set W as follows. Select a vertex $v_{0,1}$ as the root and begin with $v_{0,1}$ of the zero-th layer. Let $s_1 = d_0$ and select s_1 vertices $v_{1,1}, v_{1,2}, \ldots, v_{1,s_1}$ of the first layer such that they are adjacent to $v_{0,1}$ and $w_{T^*_{\pi,W}}(v_{0,1}v_{1,k}) = w_k$ for $k = 1, 2, \ldots, s_1$. Assume that all vertices of the t-st layer have been constructed and are denoted by $v_{t,1}, v_{t,2}, \ldots, v_{t,s_t}$. We construct all the vertices of the $(t+1)$-st layer by the induction hypothesis. Let $s_{t+1} = d_{s_1+\cdots+s_{t-1}+1} + \cdots + d_{s_1+\cdots+s_t} - s_t$ and select s_{t+1} vertices $v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,s_{t+1}}$ of the$(t+1)$-st layer such that $v_{t,1}$ is adjacent to $v_{t+1,1}, \ldots, v_{t+1,d_{s_1+\cdots+s_{t-1}+1}-1}, \cdots$, v_{t,s_t} is adjacent to $v_{t+1,s_{t+1}-d_{s_1+\cdots+s_t}+2}, \ldots, v_{t+1,s_{t+1}}$ and if there exists $v_{t,l}$ with $v_{t,l}v_{t+1,i} \in E(T^*_{\pi,W})$,

$$w_{T^*_{\pi,W}}(v_{t,l}v_{t+1,i}) = w_{d_0+d_1+\cdots+d_{s_1+s_2+\cdots+s_{t-1}}-(s_1+s_2+\cdots+s_{t-1})+i}$$

for $1 \leq i \leq s_{t+1}$. In this way, we obtain only one tree $T^*_{\pi,W}$ with the degree sequence π and the positive weight set W (see Fig. 3.1 for an example). In the following we are ready to present a proof of Theorem 1.1.

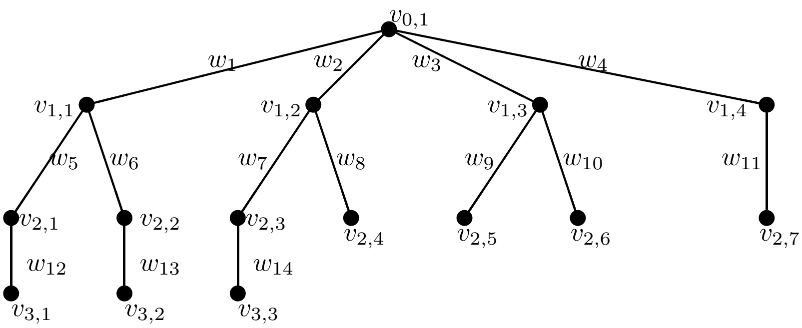

FIG. 3.1. $T^*_{\pi,W}$ *with* $\pi = (4, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1)$ *and* $W = \{w_1, \ldots, w_{14}\}$.

*P*roof of Theorem 1.1. Let T be a weighted tree with the largest p-Laplacian spec-

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tral radius in $\mathcal{T}_{\pi,W}$, where $\pi = (d_0, d_1, \ldots, d_{n-1})$ with $d_0 \geq d_1 \geq \cdots \geq d_{n-1}$. Let f be a Perron vector of T. Without loss of generality, assume $V(T) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that $f(v_i) \geq f(v_j)$ for $i < j$. By Corollary 3.2 we have $d(v_0) \geq d(v_1) \geq \cdots \geq d(v_{n-1})$. So $d(v_0) = d_0$. Let v_0 be the root of T. Suppose $\max\limits_{v \in V(T)} h(v) = h(T)$. Let $V_i = \{v \in V(T) | h(v) = i\}$ and $| V_i |= s_i$ for $i = 0, 1, \ldots, h(T)$. In the following we will relabel the vertices of T.

Let $V_0 = \{v_{0,1}\}$, where $v_{0,1} = v_0$. Obviously, $s_1 = d_0$. The vertices of V_1 are relabeled $v_{1,1}, v_{1,2}, \ldots, v_{1,s_1}$ such that $f(v_{1,1}) \geq f(v_{1,2}) \geq \cdots \geq f(v_{1,s_1})$. Assume that the vertices of V_t have been already relabeled $v_{t,1}, v_{t,2}, \ldots, v_{t,s_t}$. The vertices of V_{t+1} can be relabeled $v_{t+1,1}, v_{t+1,2}, \ldots, v_{t+1,s_{t+1}}$ such that they satisfy the following conditions: If $v_{t,k}v_{t+1,i}, v_{t,k}v_{t+1,j} \in E(T)$ and $i < j$, then $f(v_{t+1,i}) \geq f(v_{t+1,j})$; if $v_{t,k}v_{t+1,i}, v_{t,l}v_{t+1,j} \in E(T)$ and $k < l$, then $i < j$. In this way we can obtain a well ordering $\prec$ of vertices of T as follows:

$$v_{i,j} \prec v_{k,l}, \text{ if } i < k \text{ or } \ i = k \text{ and } j < l.$$

Clearly, $f(v_{1,1}) \geq \cdots \geq f(v_{1,s_1})$, and $f(v_{t+1,i}) \geq f(v_{t+1,j})$ when $i < j$ and $v_{t+1,i}, v_{t+1,j}$ have the same neighbor.

In the following we will prove that T is isomorphic to $T^*_{\pi,W}$ by proving that the ordering $\prec$ is a WBFS-ordering.

Claim: $f(v_{h,1}) \geq f(v_{h,2}) \geq \cdots \geq f(v_{h,s_h}) \geq f(v_{h+1,1})$ for $0 \leq h \leq h(T)$.

We will prove that the Claim holds by induction on h. Obviously, the Claim holds for $h = 0$. Assume that the Claim holds for $h = r - 1$. We now prove that the assertion holds for $h = r$. If there exist two vertices $v_{r,i} \prec v_{r,j}$ with $f(v_{r,i}) < f(v_{r,j})$, then there exist two vertices $v_{r-1,k}, v_{r-1,l} \in V_{r-1}$ with $k < l$ such that $v_{r-1,k}v_{r,i}, v_{r-1,l}v_{r,j} \in E(T)$. By the induction hypothesis, $f(v_{r-1,k}) \geq f(v_{r-1,l})$. Let

$$T_1 = T - v_{r-1,k}v_{r,i} - v_{r-1,l}v_{r,j} + v_{r-1,k}v_{r,j} + v_{r-1,l}v_{r,i}$$

with

$$w_{T_1}(v_{r-1,k}v_{r,j}) = \max\{w_T(v_{r-1,k}v_{r,i}), w_T(v_{r-1,l}v_{r,j})\},$$

$$w_{T_1}(v_{r-1,l}v_{r,i}) = \min\{w_T(v_{r-1,k}v_{r,i}), w_T(v_{r-1,l}v_{r,j})\},$$

and $w_{T_1}(e) = w_T(e)$ for $e \in E(T) \setminus \{v_{r-1,k}v_{r,i}, v_{r-1,l}v_{r,j}\}$. Then $T_1 \in \mathcal{T}_{\pi,W}$. By Lemma 3.4, $\mu_p(T) < \mu_p(T_1)$, which is a contradiction to our assumption that T has the largest p-Laplacian spectral radius in $\mathcal{T}_{\pi,W}$. So $f(v_{r,i}) \geq f(v_{r,j})$. Now assume $f(v_{r,s_r}) < f(v_{r+1,1})$. Note that $d(v_0) \geq 2$. It is easy to see that $v_{r,s_r}v_{r-1,s_{r-1}}$, $v_{r,1}v_{r+1,1} \in E(T)$. By the induction hypothesis, $f(v_{r-1,s_{r-1}}) \geq f(v_{r,1})$. Then, by

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similar proof, we can also get a new tree T_2 such that $T_2 \in \mathcal{T}_{\pi,W}$ and $\mu_p(T_2) > \mu_p(T)$, which is also a contradiction. So the Claim holds.

By the Claim and Corollary 3.2, the condition (2) in Definition 3.6 holds.

Assume that $uv, uy \in E(T)$ with $h(v) = h(y) = h(u) + 1$. If $v \prec y$, then $f(v) \geq f(y)$ and $w_T(uv) \geq w_T(uy)$ by Lemma 3.5. So the condition (3) in Definition 3.6 holds.

Let $uv, xy \in E(T)$ with $u \prec x$, $h(v) = h(u) + 1$ and $h(y) = h(x) + 1$. Then $v \prec y$. By the Claim, $f(u) \geq f(x)$ and $f(v) \geq f(y)$, which implies $f(u) + f(v) \geq f(x) + f(y)$. Further, by Lemma 3.5, we have $w_T(uv) \geq w_T(xy)$. Therefore, " $\prec$ " is a WBFS-ordering, i.e., T is a WBFS-tree. So $T^*_{\pi,W}$ is the unique tree with the largest p-Laplacian spectral radius in $\mathcal{T}_{\pi,W}$. Hence, the proof is completed. $\square$

Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{n-1})$ be two nonincreasing positive sequences. If $\sum_{i=0}^{t} d_i \leq \sum_{i=0}^{t} d'_i$ for $t = 0, 1, \ldots, n-2$ and $\sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} d'_i$, then π' is said to *majorize* π, and is denoted by $\pi \trianglelefteq \pi'$.

LEMMA 3.7. ([5]) *Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{n-1})$ be two nonincreasing graphic degree sequences. If $\pi \trianglelefteq \pi'$, then there exist graphic degree sequences $\pi_1, \pi_2, \ldots, \pi_k$ such that $\pi \trianglelefteq \pi_1 \trianglelefteq \pi_2 \trianglelefteq \cdots \trianglelefteq \pi_k \trianglelefteq \pi'$, and only two components of π_i and π_{i+1} are different by 1.*

THEOREM 3.8. *Let π and π' be two degree sequences of trees. Let $\mathcal{T}_{\pi,W}$ and $\mathcal{T}_{\pi',W}$ denote the set of trees with the same weight set W and degree sequences π and π', respectively. If $\pi \trianglelefteq \pi'$, then $\mu_p(T^*_{\pi,W}) \leq \mu_p(T^*_{\pi',W})$. The equality holds if and only if $\pi = \pi'$.*

Proof. By Lemma 3.7, without loss of generality, assume $\pi = (d_0, d_1, \ldots, d_{n-1})$ and $\pi' = (d'_0, d'_1, \ldots, d'_{n-1})$ such that $d_i = d'_i - 1$, $d_j = d'_j + 1$ with $0 \leq i < j \leq n-1$, and $d_k = d'_k$ for $k \neq i, j$. Then $T^*_{\pi,W}$ has a WBFS-ordering $\prec$ consistent with its Perron vector f such that $f(u) \geq f(v)$ implies $u \prec v$ by the proof of Theorem 1.1. Let $v_0, v_1, \ldots, v_{n-1} \in V(T^*_{\pi,W})$ with $v_0 \prec v_1 \prec \cdots \prec v_{n-1}$. Then $f(v_0) \geq f(v_1) \geq \cdots \geq f(v_{n-1})$ and $d(v_t) = d_t$ for $0 \leq t \leq n-1$. Since $d_j = d'_j + 1 \geq 2$, there exists a vertex v_s with $s > j$, $v_j v_s \in E(T^*_{\pi,W})$, $v_i v_s \notin E(T^*_{\pi,W})$ and v_s is not in the path from v_i to v_j. Let $T_1 = T^*_{\pi,W} - v_j v_s + v_i v_s$ with $w_{T_1}(v_i v_s) = w_{T^*_{\pi,W}}(v_j v_s)$ and $w_{T_1}(e) = w_{T^*_{\pi,W}}(e)$ for $e \in E(T_1) \setminus \{v_i v_s\}$. Then $T_1 \in \mathcal{T}_{\pi',W}$. Since $i < j$, we have $f(v_i) \geq f(v_j)$. By Lemma 3.1, $\mu_p(T^*_{\pi,W}) < \mu_p(T_1) \leq \mu_p(T^*_{\pi',W})$. The proof is completed. $\square$

COROLLARY 3.9. *Let $\mathcal{T}_{n,k}$ be the set of trees of order n with k pendent vertices and the same weight set W. Let $\pi_1 = \{k, 2, \ldots, 2, 1, \ldots, 1\}$, where the number of 1 is*

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k. Then $T^*_{\pi_1,W}$ is the unique tree with the largest p-Laplacian spectral radius in $\mathcal{T}_{n,k}$.

Proof. Let $T \in \mathcal{T}_{n,k}$ with degree sequence $\pi = (d_0, d_1, \ldots, d_{n-1})$. Obviously, $\pi \trianglelefteq \pi_1$. By Theorem 3.8, the assertion holds. □

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