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GENERALIZED PASCAL K-ELIMINATED FUNCTIONAL MATRIX WITH $2N$ VARIABLES*

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Abstract. In this paper, we introduce the Pascal k-eliminated functional matrix and the Pascal symmetric functional matrix with $2n$ variables. Some algebraic properties of these matrices are presented and proved. In addition, we demonstrate a direct application of these properties for LU decompositions of some well-known matrices (such as symmetric Pascal matrices).

Key words. Pascal matrix, Pascal k-eliminated functional matrix, Pascal symmetric functional matrix, LU decompositions, Cholesky factorization.

AMS subject classifications. 15A06, 34A30.

1. Introduction. In [1], the Pascal matrix $P_n[x]$ is defined by

$$(P_n[x])_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j \\ 0, & i < j \end{cases} \qquad (i,j = 0,1,\ldots,n). \tag{1.1}$$

Call and Velleman [1] discussed the inverse and exponential property of $P_n[x]$, i.e., $P_n[x+y] = P_n[x]P_n[y]$ for all x, y, and a few basic properties of this matrix. In [2], the symmetric Pascal matrix Q_n is defined by

$$(Q_n)_{ij} = \binom{i+j}{j} \qquad (i,j = 0,1,\ldots,n), \tag{1.2}$$

and it has been shown that Q_n can be expressed as the product of a lower triangular Pascal matrix, $P_n[1]$, and an upper triangular Pascal matrix, $P_n^T[1]$.

In [3], the extended generalized lower triangular Pascal matrix for two variables $\Phi_n[x,y]$ is defined by

$$(\Phi_n[x,y])_{ij} = \begin{cases} \binom{i}{j} x^{i-j} y^{i+j}, & i \geq j \\ 0, & i < j \end{cases} \qquad (i,j = 0,1,\ldots,n), \tag{1.3}$$

and the extended generalized rectangular Pascal matrix $\Psi_n[x,y]$ is defined by

$$(\Psi_n[x,y])_{ij} = x^{i-j} y^{i+j} \binom{i+j}{j} \qquad (i,j = 0,1,\ldots,n). \tag{1.4}$$

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In [3], the authors also demonstrated that $\Psi_n[x,y]$ has the LU decomposition $\Psi_n[x,y] = \Phi_n[x,y]P_n^T\left[\frac{y}{x}\right]$. The Pascal functional matrix was introduced in [4], by

$$(1.5)\qquad (P_{n,\lambda}[x])_{ij} = \begin{cases} \binom{i}{j}x^{(i-j)|\lambda}, & i \geq j \\ 0, & i < j \end{cases} \qquad (i,j = 0,1,\ldots,n),$$

where $x^{n|\lambda}$ is the generalized upper factorial, which is defined as follows:

$$x^{n|\lambda} = \begin{cases} x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda), & n \geq 1 \\ 0, & n = 0 \end{cases}.$$

In [5], the notion of Pascal functional matrix was extended to a more general Pascal functional matrix, $G_n[x]$, defined by

$$(1.6)\qquad (G_n[x])_{i,j} = \begin{cases} \binom{i}{j}g_{i-j}(x), & i \geq j \\ 0, & i < j \end{cases} \qquad (i,j = 0,1,\ldots,n),$$

where $\{g_n(x)\}$ is an arbitrary sequence of binomial-type polynomials, i.e., for all n, $g_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}g_k(x)g_{n-k}(y)$ for any x and y.

In [6], the Pascal $k-$eliminated functional matrix with two variables, is defined by

$$(1.7)\quad (P_{n,k}[x,y])_{i,j} = \begin{cases} \binom{i+k}{j+k}x^{i-j}y^j, & i \geq j \\ 0, & i < j \end{cases} \qquad (i,j = 0,1,\ldots,n,\ k \in \mathbb{N}\cup\{0\}).$$

Another variant of Pascal functional matrix was introduced in [8], defined by

$$(1.8)\qquad (\mathcal{P}_n[f(t,x)])_{i,j} = \begin{cases} \binom{i}{j}f^{(i-j)}(t,x), & i \geq j \\ 0, & i < j \end{cases} \qquad (i,j = 0,1,\ldots,n),$$

where $f^{(k)}(t,x)$ is kth order derivative of f with respect to t. In [8], it was shown that all well-known variants of Pascal matrices in [4,5] are special cases of this generalization of Pascal functional matrix. In [3–8], the authors proved some algebraic properties of such matrices and derived combinatorial identities from these properties.

2. The Pascal k-eliminated functional matrix with $2n$ variables.

Definition 2.1. Using the sequence $t_0 = 1, t_1, t_2, \ldots$, we define the sequence $t^{[0]} = 1, t^{[1]}, t^{[2]}, \ldots$ by the relation

$$t^{[n]} = t_n t^{[n-1]}.$$

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Notation. For convenience, we accept the following notation:

$$t^{[i]+[j]} := t^{[i]}t^{[j]} \quad \text{and} \quad t^{[i]-[j]} := \frac{t^{[i]}}{t^{[j]}}.$$

DEFINITION 2.2. Let $x_1, \dots, x_n, y_1, \dots, y_n$ be any $2n$ variables. The Pascal k-eliminated functional matrix with $2n$ variables $\Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n]$ for $k \in \mathbb{N} \cup \{0\}$ is defined by

$$(\Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n])_{ij} = \binom{i+k}{j+k} x^{[i]-[j]} y^{[i]+[j]} \quad (i, j = 0, 1, \dots, n), \tag{2.1}$$

where it is assumed that

$$\binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

It can be shown that all well-known variants of Pascal matrices in [1,3,4,6,7] are special cases of this new generalization of Pascal functional matrix with $2n$ variables. At first we study a special case of this matrix where $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = y$ and $k \in \mathbb{N} \cup \{0\}$:

$$\Phi_{n,k}[x; y] = \Phi_{n,k}[x, \dots, x; y, \dots, y],$$

called the Pascal k-eliminated functional matrix with two variables [4]. It is easily seen that for $k = 0$, $\Phi_{n,0}[x, y]$ is the Pascal matrix $\Phi_n[x, y]$ in [3], and for $k = 0$ and $y = 1$, $\Phi_{n,0}[x, 1]$ is the Pascal matrix $P_n[x]$ in [1]. Also taking $x_1 = x$, $x_2 = x + \lambda$, $\dots$, $x_n = x + (n-1)\lambda$, $y_1 = y$, $y_2 = y + \lambda$, $\dots$, $y_n = y + (n-1)\lambda$ and $k = 0$ in (2.1), it yields (1.5).

THEOREM 2.3. *For any four real numbers x_1, x_2, x_3, x_4 and $k \in \mathbb{N} \cup \{0\}$, we have*

(i) $\Phi_{n,k}[x_1; y_1].\Phi_{n,k}[x_2; y_2] = \Phi_{n,k}\left[\frac{x_1}{y_2} + x_2y_1; y_1y_2\right]$.

Also for any two real numbers x and y, we have

(ii) $\Phi_{n,k}[-x; y] = \Phi_{n,k}[x; -y]$,
(iii) $\Phi^{-1}_{n,k}[x; y] = \Phi_{n,k}\left[-x; \frac{1}{y}\right] = \Phi_{n,k}\left[x, -\frac{1}{y}\right] \quad (y \neq 0)$,
(iv) $\prod_{i=1}^{m} \Phi_{n,k}[x_i; y_i] = \Phi_{n,k}\left[\left(x_1 + \sum_{i=2}^{k} x_i y_1 (y_2 \cdots y_{i-1})^2 y_i\right) \prod_{j=2}^{m} \frac{1}{y_j}, \prod_{i=1}^{m} y_i\right]$.

Proof. Let $(\Phi_{n,k}[x_1; y_1].\Phi_{n,k}[x_2; y_2])_{ij} = a_{ij}$. Then

$$a_{ij} = \sum_{t=0}^{n} \binom{i+k}{t+k} x_1^{i-t} y_1^{i+t} \binom{t+k}{j+k} x_2^{t-j} y_2^{t+j}$$

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$$= \sum_{t=0}^{n} \binom{i+k}{j+k}\binom{i-j}{t-j} x_1^{i-t} y_1^{i+t} x_2^{t-j} y_2^{t+j}$$
$$= \binom{i+k}{j+k}(y_1y_2)^{i+j} \sum_{t=0}^{n} \binom{i-j}{t-j} \left(\frac{x_1}{y_2}\right)^{i-t} (x_2y_1)^{t-j}$$
$$= \binom{i+k}{j+k}(y_1y_2)^{i+j} \left(\frac{x_1}{y_2} + x_2y_1\right)^{i-j}$$
$$= \left(\Phi_{n,k}\left[\frac{x_1}{y_2} + x_2y_1; y_1y_2\right]\right)_{ij}.$$

Now, using (i) and $\Phi_{n,k}[0;1] = I_{n+1}$, considering $y_1y_2 = 1$ and $\frac{x_1}{y_2} + x_2y_1 = 0$, we have $x_2 = -x_1$ and $y_2 = \frac{1}{y_1}$. For the proof of (iv), we apply induction on m. This completes the proof. ☐

Corollary 2.4. *It holds that*

$$\Phi^m_{n,k}[x;y] = \Phi_{n,k}\left[xy^{-(m-1)}(1 + y^2 + y^4 + \cdots + y^{2(m-1)}); y^m\right]$$
$$= \Phi_{n,k}\left[\frac{xy^{-(m-1)}(1 - y^{2(m)})}{1 - y^2}; y^m\right].$$

Example 2.5.

$$\Phi_{3,2}[x;y].\Phi_{3,2}\left[-x;\frac{1}{y}\right]$$
$$= \Phi_{3,2}[x;y].\Phi_{3,2}\left[x;-\frac{1}{y}\right]$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3xy & y^2 & 0 & 0 \\ 6x^2y^2 & 4xy^3 & y^4 & 0 \\ 10x^3y^3 & 10x^2y^4 & 5xy^5 & y^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3x}{y} & \frac{1}{y^2} & 0 & 0 \\ \frac{6x^2}{y^2} & -\frac{4x}{y^3} & \frac{1}{y^4} & 0 \\ -\frac{10x^3}{y^3} & \frac{10x^2}{y^4} & -\frac{5x}{y^5} & \frac{1}{y^6} \end{bmatrix} = I_4.$$

The matrices $\Phi_{n,0}[x;y]$ and $\Phi_{n,0}[x;1]$ are the same as $\Phi_n[x;y]$ and $P_n[x]$, respectively, which are defined in [1,3]. We also consider the $(n+1)\times(n+1)$ matrices

$$(W_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n])_{ij} = \begin{cases} \binom{k+i-j}{k} x^{[i]-[j]} y^{[i]+[j]}, & i \geq j \\ 0, & i < j \end{cases}$$

and

$$(U_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n])_{ij} = \begin{cases} (-1)^m \binom{k+1}{m} \frac{x^{[i]-[j]}}{y^{[i]+[j]}}, & i = j+m, m = 0,1,\ldots,k+1 \\ 0, & \text{otherwise} \end{cases},$$

where $i,j = 0,1,\ldots,n$.

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THEOREM 2.6. *For* $k \in \mathbb{N} \cup \{0\}$, *we have*

(i) $\Phi_{n,k}[-x_1, \ldots, -x_n; y_1, \ldots, y_n] = \Phi_{n,k}[x_1, \ldots, x_n; -y_1, \ldots, -y_n]$,

(ii) $\Phi_{n,k}^{-1}[x_1, \ldots, x_n; y_1, \ldots, y_n] = \Phi_{n,k}\left[-x_1, \ldots, -x_n; \frac{1}{y_1}, \ldots, \frac{1}{y_n}\right]$
$= \Phi_{n,k}\left[x_1, \ldots, x_n; -\frac{1}{y_1}, \ldots, -\frac{1}{y_n}\right]$,

(iii) $W_{n,k}^{-1}[x_1, \ldots, x_n; y_1, \ldots, y_n] = U_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n]$.

Proof. Here we prove (iii). Let

$$(W_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n]\, U_{n,k}[x_1, \ldots, x_n; y_1, \ldots, y_n])_{ij} = a_{ij}.$$

Obviously, $a_{ii} = 1$ $(i = 0, 1, \ldots, n)$ and $a_{ij} = 0$ $(i < j)$. When $i > j$ we have

$$\begin{aligned} a_{ij} &= \sum_{t=0}^{i} \binom{k+i-t}{k} x^{[i]-[t]} y^{[i]+[t]} (-1)^{t-j} \binom{k+1}{t-j} \frac{x^{[t]-[j]}}{y^{[t]+[j]}} \\ &= x^{[i]-[j]} y^{[i]-[j]} \sum_{t=j}^{i} (-1)^{t-j} \binom{k+i-t}{k} \binom{k+1}{t-j} \\ &= x^{[i]-[j]} y^{[i]-[j]} \sum_{s=0}^{i-j} (-1)^{s} \binom{k+i-j-s}{k} \binom{k+1}{s} = 0. \quad \square \end{aligned}$$

EXAMPLE 2.7.

$$\begin{aligned} &W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] U_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 2x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2y_3 & 2x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{2x_1}{y_1} & \frac{1}{y_1^2} & 0 & 0 \\ \frac{x_1x_2}{y_1y_2} & -\frac{2x_2}{y_1^2y_2} & \frac{1}{y_1^2y_2^2} & 0 \\ 0 & \frac{x_2x_3}{y_1^2y_2y_3} & -\frac{2x_3}{y_1^2y_2^2y_3} & \frac{1}{y_1^2y_2^2y_3^2} \end{bmatrix} = I_4. \end{aligned}$$

Again, we need the matrices $P_n[x_1, \ldots, x_n]$, $S_n[x_1, \ldots, x_n]$, $\overline{P}_m[x_1, \ldots, x_m]$ and $G_n[x_1, \ldots, x_n]$:

$$P_n[x_1, \ldots, x_n] := \Phi_{n,0}[x_1, \ldots, x_n; 1, \ldots, 1],$$
$$S_n[x_1, \ldots, x_n] := W_{n,0}[x_1, \ldots, x_n; 1, \ldots, 1],$$
$$\overline{P}_m[x_1, \ldots, x_m] = \begin{bmatrix} 1 & O^T \\ O & P_m[x_1, \ldots, x_m] \end{bmatrix} \in \mathbb{R}^{(m+2)\times(m+2)} \qquad (m \geq 0),$$

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$$G_m[x_1,\ldots,x_m] = \begin{bmatrix} I_{n-m} & O^T \\ O & S_m[x_1,\ldots,x_m] \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \qquad (m = 0,1,\ldots,n-1),$$
$$G_n[x_1,\ldots,x_n] := S_n[x_1,\ldots,x_n].$$

LEMMA 2.8. *For $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we have*

$$W_{m,k}[x_1,\ldots,x_m; y_1,\ldots,y_m]\,\overline{P}_{m-1}\left[\frac{x_2}{y_2},\ldots,\frac{x_m}{y_m}\right] = P_{m,k}[x_1,\ldots,x_m; y_1,\ldots,y_m].$$

Proof. Let

$$\left(W_{m,k}[x_1,\ldots,x_m; y_1,\ldots,y_m]\,\overline{P}_{m-1}\left[\frac{x_2}{y_2},\ldots,\frac{x_m}{y_m}\right]\right)_{ij} = a_{ij}.$$

Obviously, $a_{ii} = (y^{[i]})^2$ $(i = 0,1,\ldots,m)$ and $a_{ij} = 0$ $(i < j)$. When $i > j$ we have

$$\begin{aligned} a_{ij} &= \sum_{t=0}^{i} \binom{k+i-t}{k} x^{[i]-[t]} y^{[i]+[t]} \binom{t-1}{j-1} \frac{x^{[t]-[j]}}{y^{[t]-[j]}} \\ &= x^{[i]-[j]} y^{[i]+[j]} \sum_{t=j}^{i} \binom{k+i-t}{k}\binom{t-1}{j-1} \\ &= x^{[i]-[j]} y^{[i]+[j]} \sum_{s=j-1}^{i-1} \binom{k+i-1-s}{k}\binom{s}{j-1} \\ &= x^{[i]-[j]} y^{[i]+[j]} \binom{i+k}{j+k}. \qquad \square \end{aligned}$$

EXAMPLE 2.9.

$$W_{3,1}[x_1,x_2,x_3; y_1,y_2,y_3]\,\overline{P}_2\left[\frac{x_2}{y_2},\frac{x_3}{y_3}\right]$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 2x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2y_3 & 2x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x_2}{y_2} & 1 & 0 \\ 0 & \frac{x_2x_3}{y_2y_3} & \frac{2x_3}{y_3} & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 3x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 6x_2x_3y_1^2y_2y_3 & 4x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix}$$
$$= P_{3,1}[x_1,x_2,x_3; y_1,y_2,y_3].$$

By Lemma 2.8 and the definition of $G_m[x_1,\ldots,x_m]$, we get the following result:

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THEOREM 2.10. *The extended generalized Pascal matrix* $\Phi_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ *can be factorized by* $W_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ *and* $G_m[x_1,\ldots,x_m]$ $(m=0,1,\ldots,n-1)$*:*

$$\begin{aligned}&\Phi_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]\\&=W_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]\,G_{n-1}\left[\frac{x_2}{y_2},\ldots,\frac{x_n}{y_n}\right]G_{n-2}\left[\frac{x_3}{y_3},\ldots,\frac{x_n}{y_n}\right]\cdots G_1\left[\frac{x_n}{y_n}\right].\end{aligned}$$

EXAMPLE 2.11.

$$\begin{aligned}&\Phi_{3,1}[x_1,x_2,x_3;y_1,y_2,y_3]\\&=\begin{bmatrix}1&0&0&0\\2x_1y_1&y_1^2&0&0\\3x_1x_2y_1y_2&3x_2y_1^2y_2&y_1^2y_2^2&0\\4x_1x_2x_3y_1y_2y_3&6x_2x_3y_1^2y_2y_3&4x_3y_1^2y_2^2y_3&y_1^2y_2^2y_3^2\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\frac{x_2}{y_2}&1&0\\0&\frac{x_2x_3}{y_2y_3}&\frac{2x_3}{y_3}&1\end{bmatrix}\\&\times\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{x_3}{y_3}&1\end{bmatrix}=W_{3,1}[x_1,x_2,x_3;y_1,y_2,y_3]\,G_2\left[\frac{x_2}{y_2},\frac{x_3}{y_3}\right]G_1\left[\frac{x_3}{y_3}\right].\end{aligned}$$

If we take $x_1=\cdots=x_n=x$, $y_1=\cdots=y_n=y$ and $k=0$ (or $x_1=\cdots=x_n=x$, $y_1=\cdots=y_n=1$) in Theorem 2.10, then we get the following results [3,4]:

COROLLARY 2.12. *It holds that*

$$\Phi_n[x;y]=W_{n,k}[x;y]\,G_{n-1}\left[\frac{x}{y}\right]G_{n-2}\left[\frac{x}{y}\right]\cdots G_1\left[\frac{x}{y}\right],$$

where

$$\Phi_n[x;y]=P_{n,0}[x,\ldots,x;y,\ldots,y]\quad\text{and}\quad W_n[x;y]=W_{n,0}[x,\ldots,x;y,\ldots,y].$$

COROLLARY 2.13. *It holds that*

$$P_{n,k}[x]=(G_n[x])^k\;G_{n-1}[x]\,G_{n-2}[x]\cdots G_1[x],$$

where

$$P_{n,k}[x]=P_{n,k}[x,\ldots,x]\quad\text{and}\quad(G_n[x])^k=W_{n,k}[x,\ldots,x;x,\ldots,x].$$

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For the inverse of the extended generalized Pascal matrix $\Phi_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]$, applying Theorems 2.6 and 2.10 yields the following.

THEOREM 2.14. *It holds that*

$$\begin{aligned}
&\Phi^{-1}_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]\\
&=\Phi_{n,k}\left[-x_1,\ldots,-x_n;\frac{1}{y_1},\ldots,\frac{1}{y_n}\right]\\
&=\Phi_{n,k}\left[x_1,\ldots,x_n;-\frac{1}{y_1},\ldots,-\frac{1}{y_n}\right]\\
&=G_1^{-1}\left[\frac{x_n}{y_n}\right]G_2^{-1}\left[\frac{x_{n-1}}{y_{n-1}},\frac{x_n}{y_n}\right]\cdots G_{n-1}^{-1}\left[\frac{x_2}{y_2},\cdots,\frac{x_n}{y_n}\right]W^{-1}_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]\\
&=F_1\left[\frac{x_n}{y_n}\right]F_2\left[\frac{x_{n-1}}{y_{n-1}},\frac{x_n}{y_n}\right]\cdots F_{n-1}\left[\frac{x_2}{y_2},\ldots,\frac{x_n}{y_n}\right]U_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n],
\end{aligned}$$

where

$$\begin{aligned}
F_m\left[\frac{x_{n-m+1}}{y_{n-m+1}},\ldots,\frac{x_n}{y_n}\right]&=G_m^{-1}\left[\frac{x_{n-m+1}}{y_{n-m+1}},\ldots,\frac{x_n}{y_n}\right]\\
&=\begin{bmatrix} I_{n-k} & O^T\\ O & D_m\left[\frac{x_{n-m+1}}{y_{n-m+1}},\ldots,\frac{x_n}{y_n}\right]\end{bmatrix}
\end{aligned}$$

for $m=1,\ldots,n-1$, *and* $D_n[x_1,\ldots,x_n]=U_{n,0}[x_1,\ldots,x_n;1,\ldots,1]$.

In particular,

$$P^{-1}_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]=J_n[y_1,\ldots,y_n]\,P_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n]\,J_n[y_1,\ldots,y_n],$$

where

$$J_n[y_1,\ldots,y_n]=\operatorname{diag}\left(1,-\frac{1}{y_1^2},\frac{1}{y_1^2y_2^2},\ldots,(-1)^n\frac{1}{y_1^2\cdots y_n^2}\right).$$

3. The symmetric Pascal matrix with $2n$ variables. In this section, we define the extended generalized symmetric Pascal matrix $\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n]$ by

$$(\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n])_{ij}=\binom{i+j}{i}x^{[i]-[j]}y^{[i]+[j]},$$

where $i,j=0,1,\ldots,n$.

THEOREM 3.1. *It holds that*

$$\begin{aligned}
P_n^T\left[\frac{y_1}{x_1},\ldots,\frac{y_n}{x_n}\right]&=F_1\left[\frac{x_n}{y_n}\right]F_2\left[\frac{x_{n-1}}{y_{n-1}},\frac{x_n}{y_n}\right]\cdots F_{n-1}\left[\frac{x_2}{y_2},\ldots,\frac{x_n}{y_n}\right]\\
&\quad\times U_{n,0}[x_1,\ldots,x_n;y_1,\ldots,y_n]\,\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n],
\end{aligned}$$

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$$P_{n,0}^T[x_1,\ldots,x_n;y_1,\ldots,y_n] = F_1\left[\frac{x_n}{y_n}\right] F_2\left[\frac{x_{n-1}}{y_{n-1}},\frac{x_n}{y_n}\right]\cdots F_{n-1}\left[\frac{x_2}{y_2},\ldots,\frac{x_n}{y_n}\right] \times U_{n,0}[1,\ldots,1;y_1,\ldots,y_n]\,\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n],$$

and the Cholesky factorization of $\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n]$ *is given by*

$$\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n] = P_{n,0}^T[1,\ldots,1;x_1y_1,\ldots,x_ny_n] \times P_{n,0}^T\left[y_1,\ldots,y_n;\frac{1}{x_1},\ldots,\frac{1}{x_n}\right], \tag{3.1}$$

$$\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n] = P_{n,0}^T[x_1,\ldots,x_n;y_1,\ldots,y_n]\,P_n^T\left[\frac{y_1}{x_1},\ldots,\frac{y_n}{x_n}\right]. \tag{3.2}$$

Proof. Let $\left(P_{n,0}^T[1,\ldots,1;x_1y_1,\ldots,x_ny_n]\,P_{n,0}^T\left[y_1,\ldots,y_n;\frac{1}{x_1},\ldots,\frac{1}{x_n}\right]\right)_{ij} = a_{ij}$. Then

$$a_{ij} = \begin{cases} \sum_{k=0}^{j}\binom{i}{k}\binom{j}{k}x^{[i]-[j]}y^{[i]+[j]}, & i \geq j \\ \sum_{k=0}^{i}\binom{i}{k}\binom{j}{k}x^{[i]-[j]}y^{[i]+[j]}, & i < j \end{cases},$$

which by using the Vandermonde identities

$$\sum_{k=0}^{i}\binom{i}{k}\binom{j}{k} = \sum_{k=0}^{i}\binom{i}{k}\binom{j}{j-k} = \binom{i+j}{j}$$

and

$$\sum_{k=0}^{j}\binom{i}{k}\binom{j}{k} = \sum_{k=0}^{j}\binom{i}{i-k}\binom{j}{k} = \binom{i+j}{j},$$

implies (3.1). Similarly we get (3.2). ◻

EXAMPLE 3.2.

$$W_{3,1}[x_1,x_2,x_3;y_1,y_2,y_3]\,\overline{P}_2\left[\frac{x_2}{y_2},\frac{x_3}{y_3}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 2x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2y_3 & 2x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x_2}{y_2} & 1 & 0 \\ 0 & \frac{x_2x_3}{y_2y_3} & \frac{2x_3}{y_3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 3x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 6x_2x_3y_1^2y_2y_3 & 4x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix},$$

$$= P_{3,1}[x_1,x_2,x_3;y_1,y_2,y_3].$$

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By using Theorems 2.6 and 3.1, we have:

COROLLARY 3.3. *It holds that*

$$\Psi_n^{-1}[x_1,\ldots,x_n;y_1,\ldots,y_n] = P_n^T\left[-\frac{y_1}{x_1},\ldots,-\frac{y_n}{x_n}\right]P_{n,0}\left[x_1,\ldots,x_n;-\frac{1}{y_1},\ldots,-\frac{1}{y_n}\right]$$
$$= P_{n,0}^T[y_1,\ldots,y_n;-x_1,\ldots,-x_n]\,P_{n,0}\left[1,\ldots,1;-\frac{1}{x_1y_1},\ldots,-\frac{1}{x_ny_n}\right].$$

Applying Theorems 2.14 and 3.1, we get the following.

COROLLARY 3.4. *It holds that*

$$\Psi_n^{-1}[x_1,\ldots,x_n;y_1,\ldots,y_n] = J_n[1,\ldots,1]\,P_n^T\left[\frac{y_1}{x_1},\ldots,\frac{y_n}{x_n}\right]J_n[1,\ldots,1]$$
$$\times J_n[y_1,\ldots,y_n]\,P_{n,0}[x_1,\ldots,x_n;y_1,\ldots,y_n]\,J_n[y_1,\ldots,y_n]$$
$$= J_n\left[\frac{1}{x_1},\ldots,\frac{1}{x_n}\right]P_{n,0}^T\left[y_1,\ldots,y_n;\frac{1}{x_1},\ldots,\frac{1}{x_n}\right]J_n\left[\frac{1}{x_1},\ldots,\frac{1}{x_n}\right]$$
$$\times J_n[y_1,\ldots,y_n]\,P_{n,0}^T[1,\ldots,1;x_1y_1,\ldots,x_ny_n]\,J_n[y_1,\ldots,y_n].$$

For the previous two kinds of extended generalized Pascal matrices, we can get:

COROLLARY 3.5. *It holds that*

$$\det\Phi_{n,k}[x_1,\ldots,x_n;y_1,\ldots,y_n] = y_1^{2n}y_2^{2(n-1)}\cdots y_n^2,$$
$$\det\Phi_{n,k}^{-1}[x_1,\ldots,x_n;y_1,\ldots,y_n] = y_1^{-2n}y_2^{-2(n-1)}\cdots y_n^{-2},$$
$$\det\Psi_n[x_1,\ldots,x_n;y_1,\ldots,y_n] = y_1^{2n}y_2^{2(n-1)}\cdots y_n^2,$$
$$\det\Psi_n^{-1}[x_1,\ldots,x_n;y_1,\ldots,y_n] = y_1^{-2n}y_2^{-2(n-1)}\cdots y_n^{-2}.$$

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