

THE NUMERICAL RANGE OF LINEAR OPERATORS ON THE 2-DIMENTIONAL KREIN SPACE*

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Abstract. The aim of this note is to provide the complete characterization of the numerical range of linear operators on the 2-dimensional Krein space \mathbb{C}^2 .

Key words. Krein space, Numerical range, Rank one operator.

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1. Introduction. The concept of numerical range of linear operators on a Hilbert space was introduced by Toeplitz [16] and has been generalized in several directions. The theory of numerical ranges of linear operators on a Krein space has also been considered by some authors (see [2, 3, 10, 11, 13, 14, 15] and the references therein). There are many motivations for the study of the numerical range of linear operators on Hilbert spaces or Krein spaces. We enumerate some of them: the localization of the spectrum of an operator, related inequalities, control theory and applications to physics (cf. [8]). Recently, an application of a generalized numerical range to NMR spectroscopy has been discussed (cf. [7, 12]). The aim of this paper is the complete determination of the numerical range of linear operators on the 2-dimensional Krein space \mathbb{C}^2 . By addition of scalar operators, the study of the numerical range of operators on \mathbb{C}^2 is reduced to that of rank one operators. The numerical range of such rank one operators has been already investigated in [14] for non-neutral vectors.

Let C, A be (non-zero) rank one operators on the 2-dimensional Krein space \mathbb{C}^2 , endowed with the indefinite inner product space $[\cdot, \cdot]$ defined by $[\xi, \nu] = (J\xi, \nu) = \nu^* J\xi$ for $J = I_1 \oplus (-I_1)$. We refer [1, 4, 6] for general reference on Krein spaces or Krein spaces operators. For the rank one operators C, A, there exist non-zero vectors

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 $\eta, \zeta, \kappa, \tau$ such that

(1.1)
$$C\xi = [\xi, \eta]\zeta, \ A\xi = [\xi, \kappa]\tau, \ \xi \in \mathbb{C}^2.$$

Denote by SU(1,1) the group of 2×2 complex matrices U with determinant 1 such that $U^*JU = J$. We consider the indefinite C-numerical range of A denoted and defined by

(1.2)
$$W_C^J(A) = \{ [U\zeta, \kappa] \overline{[U\eta, \tau]} : U \in SU(1, 1) \},$$

which has been characterized in [14] for $\eta, \zeta, \kappa, \tau$, non-neutral vectors, that is, $[\eta, \eta]$, $[\zeta, \zeta]$, $[\kappa, \kappa]$, $[\tau, \tau]$ do not vanish. An analogous object for a 2-dimensional Hilbert space is the *C*-numerical range of *A* defined as

$$W_C(A) = \{ \kappa^* U \zeta \ \overline{\tau^* U \eta} : U \in SU(2) \},\$$

for rank one operators C, A. The range $W_C(A)$ is a (possible degenerate) closed elliptical disc (cf. [9]). This paper treats the analogous object for Krein spaces.

The main aim of this note is to complete the characterization of $W_C^J(A)$ considering the case of the vectors $\eta, \zeta, \kappa, \tau$ being neutral. For the range $W_C(A)$ of any dimensional matrices A, C, the numerical method to draw the boundary is given in [7] based on a result in [5]. A numerical algorithm is not known for Krein spaces numerical ranges except for some special cases. We give a complete characterization for operators on 2-dimensional spaces.

The following classification takes place, being the different cases treated in the next five sections.

First Case: All the vectors η , ζ , κ , τ are neutral.

- Second Case: One of the vectors η , ζ , κ , τ is non-neutral and the other three are neutral.
- Third Case: The vectors κ , τ are neutral and η , ζ are non-neutral.
- Fourth Case: The vectors κ , ζ are neutral and η , τ are non-neutral.
- *Fifth Case*: The vectors κ , η are neutral and ζ , τ are non-neutral.

Sixth Case: One of the vectors η , ζ , κ , τ is neutral and the other three are non-neutral.

Using the concrete description of $W_C^J(A)$ given in Sections 2–6 and in [14], we prove the following results in Section 7.

THEOREM 1.1. Let C, A be arbitrary linear operators on a 2-dimensional Krein space, and $J = I_1 \oplus -I_1$. If the boundary of $W_C^J(A)$ is non-empty, then it is a singleton or it lies on a possibly degenerate conic.

THEOREM 1.2. Let C, A be arbitrary linear operators on the Krein space \mathbb{C}^2 , and $J = I_1 \oplus -I_1$. Then the fundamental group $\pi_1(W_C^J(A))$ of $W_C^J(A)$ is a trivial group,



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or an abelian group isomorphic to the additive group of the integers, \mathbb{Z} . The number of the connected components of the complement $\mathbb{C}\backslash W^J_C(A)$ is 1 or 2.

2. The first case. In the sequel, we identify the complex plane with \mathbb{R}^2 and we denote by E_{ij} the 2×2 matrix with the (i, j)th entry equal to 1 and all the others 0. For $V, W \in U(1, 1)$, we have

$$(2.1) \ VAV^{-1}\xi = [V^{-1}\xi,\kappa]V\tau = [\xi,V\kappa]V\tau, \ WCW^{-1}\xi = [W^{-1}\xi,\eta]\zeta = [\xi,W\eta]W\zeta,$$

and so we may assume that $\zeta = (1, 1)^T$, $\tau = (1, -1)^T$. We also may consider that $\kappa = (\overline{k_1}, -\overline{k_2})$, $\eta = (\overline{q_1}, \overline{q_2})$, with $|k_1| = |k_2| = |q_1| = |q_2| = 1$. Under these assumptions, we find

$$W_C^J(A) = \{(k_1\alpha + k_1\overline{\beta} + k_2\overline{\alpha} + k_2\beta)(q_1\overline{\alpha} + q_2\beta + q_1\overline{\beta} + q_2\alpha) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}$$

Writing $k_1 = \exp(is) \exp(i\theta)$, $k_2 = \exp(is) \exp(-i\theta)$, $q_1 = \exp(it) \exp(i\phi)$, $q_2 = \exp(it) \exp(-i\phi)$, $s, t, \theta, \phi \in \mathbb{R}$, we obtain

(2.2)
$$W_C^J(A) = \{4\exp(is) \Re(\exp(i\theta)(\alpha + \overline{\beta}))\exp(it)\Re(\exp(-i\phi)(\alpha + \beta)) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.$$

Thus, $W_C^J(A)$ is contained in a straight line passing through the origin. We may assume that $\kappa = (\exp(-i\theta), -\exp(i\theta))^T$, $\eta = (\exp(-i\phi), \exp(i\phi))^T$, $\zeta = (1, 1)^T$ and $\tau = (1, -1)^T$. Under these assumptions, we prove the following.

PROPOSITION 2.1. Let $C = \exp(i\phi)(E_{11} + E_{21}) - \exp(-i\phi)(E_{12} + E_{22})$, $A = \exp(i\theta)(E_{11} - E_{21}) + \exp(-i\theta)(E_{12} - E_{22})$, $0 \le \theta, \phi \le 2\pi$. If either θ or ϕ is not congruent to 0 modulo π , then $W_C^J(A)$ is the real line. Otherwise, $W_C^J(A) = [0, +\infty)$ for $\theta = \phi = 0$ or $\theta = \phi = \pi$, and $W_C^J(A) = (-\infty, 0]$ for $\theta = 0$, $\phi = \pi$ or $\theta = \pi$, $\phi = 0$.

Proof. Having in mind (2.2), we easily find for $0 \le \theta$, $\phi \le 2\pi$,

$$W_C^J(A) = \{4\Re(\exp(i\theta)(\alpha + \overline{\beta}))\Re(\exp(-i\phi)(\alpha + \beta))\} = \{4(\cosh t \cos(u + \theta) + \sinh t \cos(v - \theta))(\cosh t \cos(u - \phi) + \sinh t \cos(v - \phi)): t \ge 0, 0 \le u, v \le 2\pi\}.$$

For fixed $0 \le u, v, \theta, \phi \le 2\pi$, we study the behavior of the real valued function

$$\psi(t) = 4(\cosh t \cos(u+\theta) + \sinh t \cos(v-\theta))(\cosh t \cos(u-\phi) + \sinh t \cos(v-\phi))$$

as $t \to +\infty$. For this purpose, we consider the sign of the function

$$f(u,v) = (\cos(u+\theta) + \cos(v-\theta))(\cos(u-\phi) + \cos(v-\phi))$$

= $4\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u+v}{2} - \phi\right)\cos\left(\frac{u-v}{2} + \theta\right)\cos\left(\frac{u-v}{2}\right), \quad u,v \in \mathbb{R}.$



Hence, it is sufficient to determine the sign of the function $\tilde{f}(x,y) = 4 \cos x \cos(x - \phi) \cos y \cos(y + \theta)$ for the variables ranging over the reals. We observe that function $\cos x \cos(x - \phi)$ takes positive and negative values except for $\phi \equiv 0 \mod \pi$, while $\cos y \cos(y + \theta)$ takes positive and negative values except for $\theta \equiv 0 \mod \pi$. Therefore, if either ϕ or θ is not congruent to 0 modulo π , then f(u, v) takes positive and negative values, and so $W_C^J(A)$ coincides with the real line.

To finish the proof we consider the following cases: (1) $\theta = \phi = 0$; (2) $\theta = 0$ and $\phi = \pi$; (3) $\theta = \pi$ and $\phi = 0$; (4) $\theta = \phi = \pi$. We concentrate on the case (1). Considering t = 0, $u = \pi/2$, we produce the origin. Taking u = v = 0, we get $[0, +\infty) \subset W_C^J(A)$ and the reverse inclusion is clear.

The treatment of the remaining cases is similar and the proposition follows. \Box

3. The fourth and fifth cases. In the fourth case, we may consider $\zeta = (1, 1)^T$, $\kappa = (1, -1), \tau = (1, 0)^T$, and $\eta = (1, 0)^T$ or $\eta = (0, 1)^T$. In the fifth case, we may take $\eta = (1, 1)^T$, $\kappa = (1, 1)^T$, $\zeta = (1, 0)^T$, and $\tau = (1, 0)^T$ or $\tau = (0, 1)^T$.

PROPOSITION 3.1. If $C = E_{11} + E_{21}$ and $A = E_{11} + E_{12}$, then $W_C^J(A) = \mathbb{C} \setminus (-\infty, 1] \cup \{0\}.$

Proof. Some computations yield

$$(3.1) \quad W_C^J(A) = \{ (\alpha + \overline{\alpha} + \beta + \overline{\beta})\overline{\alpha} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ = \{ 2(\cosh t \cos \theta + \sinh t \cos \phi) \cosh t \exp(i\theta) : t \ge 0, 0 \le \theta, \phi \le 2\pi \}.$$

By choosing in (3.1) $\theta = \pi/2$ and $\phi = 0$ or $\phi = \pi$, we conclude that the imaginary axis is contained in $W_C^J(A)$. It can be easily seen that a non-zero real number λ belongs to $W_C^J(A)$ if and only if it is of the form

$$\lambda(t,\phi) = 2(\cosh t + \sinh t \cos \phi) \cosh t, \ t \ge 0, \ 0 \le \phi \le 2\pi.$$

For a fixed t, λ attains its minimum value when $\cos \phi = -1$ and this minimum equals $1 + \exp(-2t)$. Letting t vary on its domain, we find $\lambda > 1$. Thus,

$$W_C^J(A) \cap \mathbb{R} = \{0\} \cup (1, +\infty).$$

Considering $0 < \theta < \pi$ and $\phi = \pi$ in (3.1), we find

(3.3)
$$\{2(\cosh t \cos \theta - \sinh t) \cosh t \exp(i\theta) : t \ge 0\} = \{\lambda \exp(i\theta) : \lambda \le 2 \cos \theta\}$$
$$\subset W_C^J(A),$$

while for $\phi = \pi/2$ we get the inclusion

$$(3.4) \quad \{2\cosh t\cos\theta \cosh t\exp(i\theta): t\geq 0\} = \{\lambda\exp(i\theta): \lambda\geq 2\cos\theta\} \subset W_C^J(A).$$



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Therefore the proposition follows from (3.2), (3.3), (3.4).

PROPOSITION 3.2. If $C = -E_{12} - E_{22}$ and $A = E_{11} + E_{12}$, then $W_C^J(A) = \mathbb{C} \setminus (-\infty, -1]$.

Proof. We have

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$$\begin{split} W_C^J(A) &= \{ (\alpha + \overline{\alpha} + \beta + \overline{\beta})\beta : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ 2(\cosh t \cos \phi + \sinh t \cos \theta) \sinh t \exp(i\theta) : t \ge 0, 0 \le \theta, \phi \le 2\pi \}. \end{split}$$

Choosing $\phi = \pi/2$ and letting $\theta \neq 0$ vary in $[-\pi/2, \pi/2]$, we conclude that

(3.5) $\{2\sinh t\cos\theta\sinh t\exp(i\theta):t\geq 0\}=\{\lambda\exp(i\theta):0\leq\lambda\}\subset W_C^J(A),$

and considering $\phi = \pi$, we find

(3.6) $\{2(-\cosh t + \sinh t \cos \theta) \sinh t \exp(i\theta) : t \ge 0\} = \{\lambda \exp(i\theta) : \lambda \le 0\} \subset W_C^J(A).$

It follows from (3.5), (3.6) that every complex number with nonvanishing imaginary part belongs to $W_C^J(A)$.

Taking $\theta = \phi = 0$ and $t \to +\infty$, we conclude that $[0, +\infty) \subset W_C^J(A)$, because $2\sinh^2 t \in W_C^J(A)$. For a fixed ϕ different from 0 or π , we consider the real valued function of real variable $f(t) = 2(\cosh t \cos \phi + \sinh t) \sinh t$. Its derivative $f'(t) = (\exp(2t) + \exp(-2t)) \cos \phi + \exp(2t) - \exp(-2t)$ vanishes at $\exp(2t) = \sqrt{(1 - \cos \phi)/(1 + \cos \phi)}$, and the function attains here the minimum value $-1 + \sqrt{1 - \cos^2 \phi}$. Thus, the proposition follows. \Box

PROPOSITION 3.3. If $A = C = E_{11} - E_{12}$, then $W_C^J(A) = \mathbb{C} \setminus (-\infty, 0]$.

Proof. We easily find

$$\begin{split} W_C^J(A) &= \{ (\alpha - \beta)(\overline{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ (\cosh t + \sinh t \exp(i\theta))(\cosh t + \sinh t \exp(i\phi)) : 0 \le t, \ 0 \le \theta, \phi \le 2\pi \}. \end{split}$$

Setting

$$G = \{(\cosh t + \sinh t \exp(i\theta)) : 0 \le t, \ 0 \le \theta \le 2\pi\},\$$

we clearly have

$$\{z^2 : z \in G\} \subset W^J_C(A) = \{z_1 \, z_2 : z_1, z_2 \in G\}.$$

We show that G is the family of circles $(x-r)^2 + y^2 = r^2 - 1$, $r \ge 1$. In fact, if $(x,y) \in G$, then $2rx = x^2 + y^2 + 1$ and so x > 0. Conversely, if x > 0 and $y \in \mathbb{R}$, then the real number $r = \frac{1}{2}(x + \frac{1}{x}) + \frac{y^2}{2x}$ satisfies $(x-r)^2 + y^2 = r^2 - 1$. Thus, $G = \{z \in \mathbb{C} : \Re(z) > 0\}$. \Box



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PROPOSITION 3.4. If $C = E_{11} - E_{12}$ and $A = E_{21} - E_{22}$, then $W_C^J(A) = \mathbb{C} \setminus (-\infty, 0] \cup \{-1\}.$

Proof. We obtain

 $W_C^J(A) = \{(\sinh t + \cosh t \exp(i\theta))(\sinh t + \cosh t \exp(i\phi)) : t \in \mathbb{R}, 0 \le \theta, \phi \le 2\pi\}.$

By similar arguments to those used in Proposition 3.3, it can be easily seen that $\sinh t + \cosh t \exp(i\theta)$, $t \in \mathbb{R}$, $0 \le \theta \le 2\pi$, ranges over the complementary of the imaginary axis, taken in the complex plane, with i and -i deleted. \Box

4. The second case. In this case, we may assume that $\tau = (1,0)^T$, $\zeta = (1,1)^T$, $\kappa = (\overline{k_1}, -\overline{k_2})^T$, $\eta = (1,\overline{q})^T$, $k_1, k_2, q \in \mathbb{C}$, $|k_1| = |k_2| = |q| = 1$, and we have

$$W_C^J(A) = \{ (k_1\alpha + k_2\overline{\alpha} + k_1\overline{\beta} + k_2\beta)(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.$$

Writing $k_1 = \exp(i s) \exp(i\theta)$, $k_2 = \exp(i s) \exp(-i\theta)$, $s, \theta \in \mathbb{R}$, it follows that

$$W_C^J(A) = \{2\exp(is)\Re(\exp(-i\theta)(\overline{\alpha}+\beta))(\overline{\alpha}+q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}$$

(4.1)
$$= \{2k_1 \Re(\alpha+\beta) (\alpha+q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\},$$

and so we may assume that $k_1 = 1$.

PROPOSITION 4.1. Let $C = (E_{11} + E_{21}) - \overline{q}(E_{21} + E_{22}), |q| = 1$, and let $A = E_{11} + E_{12}$. The following hold:

1) If
$$q = 1$$
, then $W_C^J(A) = \{0\} \cup \{z \in \mathbb{C} : \Re(z) > 0\}.$

2) If $q \neq 1$, then $W_C^J(A) = \mathbb{C} \setminus \{\lambda(1-q) : \lambda \in \mathbb{R}\} \cup \{1-q\}.$

Proof. 1) It may be easily seen that the origin belongs to $W_C^J(A)$. We have

$$\Re(2\Re(\alpha+\beta)(\alpha+\beta)) = 2\Re(\alpha+\beta)^2 \ge 0$$

with occurrence of equality if and only if $\Re(\alpha + \beta) = 0$. Thus,

$$W_C^J(A) \subset \{0\} \cup \{z \in \mathbb{C} : \Re(z) > 0\}.$$

We prove that the reverse inclusion holds. For every $z \in \mathbb{C}$ with $\theta = \operatorname{Arg}(z) \in (-\pi/2, \pi/2)$, we may find $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$ and $z = 2\Re(\alpha + \beta)(\alpha + \beta)$. In fact, let $\alpha = \cosh t \exp(i\theta)$ and $\beta = \sinh t \exp(i\theta)$, $t \in \mathbb{R}$, so that $2\exp(2t)\cos\theta = |z|$. Since $\cos\theta > 0$, we have $2\exp(2t)\exp(i\theta)\cos\theta = z$.

2) Let $q = q_1 + i q_2 \neq 1$. From (4.1) we easily get

$$W_C^J(A) = \{2\Re(z) \ w : z, w \in \mathbb{C}, 2\Re(z\overline{w}) - 2\Re(q \ z\overline{w}) = |1 - q|^2\},\$$



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that is, the elements of $W_C^J(A)$ are the complex numbers of the form $2x_0 (x + iy)$ such that

$$(4.2) \ (q_2 x + (1 - q_1) y) y_0 = (1/2)|1 - q|^2 - (1 - q_1) x_0 x + q_2 x_0 y, x_0, y_0, x, y \in \mathbb{R}.$$

If $q_2 x + (1 - q_1) y \neq 0$, then y_0 may always be found such that (4.2) is satisfied, and thus

$$S = \{(x, y) \in \mathbb{R}^2 : q_2 x + (1 - q_1) y \neq 0\} \subset W_C^J(A).$$

Moreover, $\Re(-i(x+iy)\overline{1-q}) = q_2 x + (1-q_1) y = 0$ if and only if $z = x+iy = \lambda (1-q)$, $\lambda \in \mathbb{R}$. Clearly,

$$S = \mathbb{C} \setminus \{ \lambda (1 - q) : \lambda \in \mathbb{R} \}.$$

Some calculations show that there exist complex numbers α, β satisfying $\lambda(1-q) = 2\Re(\alpha + \beta) (\alpha + q\beta)$ and $|\alpha|^2 - |\beta|^2 = 1$ if and only if $\lambda = 1$. Finally, it can easily be seen that 1-q belongs to $W_C^J(A)$. \Box

5. The sixth case. In this case, we may assume that κ is neutral. By replacing the inner product $[\xi_1, \xi_2]$ by $-[\xi_1, \xi_2]$, we may consider (1): $\zeta = \tau = (1, 0)^T$ or (2): $\zeta = (0, 1)^T$, $\tau = (1, 0)^T$. In either case, we may suppose that $\kappa = (1, 1)^T$, by replacing A by diag $(1, \exp(i\theta))$ A diag $(1, \exp(-i\theta))$ for some $\theta \in \mathbb{R}$. By performing a transformation of this type, the components of the vector $\eta = (\eta_1, \eta_2)$ may be assumed to be real and such that $|\eta_1| \neq |\eta_2|$. Thus, one of the following situations occurs:

 $\begin{array}{ll} 1st \ Subcase. \ A\xi = [\xi, (1,1)^T] \ (1,0)^T, \ C\xi = [\xi, (1,q)^T] \ (1,0)^T, & -1 < q < 1. \\ 2nd \ Subcase. \ A\xi = [\xi, (1,1)^T] \ (1,0)^T, \ C\xi = [\xi, (q,1)^T] \ (1,0)^T, & -1 < q < 1. \\ 3rd \ Subcase. \ A\xi = [\xi, (1,1)^T] \ (1,0)^T, \ C\xi = [\xi, (1,q)^T] \ (0,1)^T, & -1 < q < 1. \\ 4rd \ Subcase. \ A\xi = [\xi, (1,1)^T] \ (1,0)^T, \ C\xi = [\xi, (q,1)^T] \ (0,1)^T, & -1 < q < 1. \\ \end{array}$

Firstly, we treat the 1st Subcase.

PROPOSITION 5.1. If
$$C = E_{11} - qE_{12}$$
 and $A = E_{11} - E_{12}$, then
 $W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : x > 1/2 - |q|/2 \cosh t, \ y = \sqrt{1 - q^2}/2 \sinh t, \ t \in \mathbb{R}\}.$

Proof. By some computations we get

$$\begin{split} W_C^J(A) &= \{ (\alpha - \beta)(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ (\cosh t + \sinh t \exp(i\theta))(\cosh t + q \sinh t \exp(i\phi)) : t \ge 0, 0 \le \theta, \phi \le 2\pi \}. \end{split}$$

The case q = 0 is easily treated, so we may assume that 0 < q < 1. The above set is the union of the family of circles centered at $z = z(t, \theta) = x_0 + i y_0 = \cosh^2 t + i y_0$

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 $\sinh t \cosh t \exp(i\theta)$ whose radii $r = r(t,\theta)$ satisfy $R = r^2 = q^2 \sinh^2 t (1+2\sinh^2 t + 2\sinh t \cosh t \cos \theta)$, and $R = q^2 ((x_0 - 1)^2 + y_0^2)$. It may be shown that the centers of these circles describe the open half-plane x > 1/2. Thus,

$$W_C^J(A) = \{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = q^2((x_0-1)^2 + y_0^2), x_0 > 1/2, y_0 \in \mathbb{R} \}.$$

The boundary of $W_C^J(A)$ is the envelope of the family of circles when their centers run over x = 1/2 (cf. Proposition 2.3 in [14]) and the result follows. \Box

Next, the 2nd Subcase is studied.

PROPOSITION 5.2. If $C = qE_{11} - E_{12}$, -1 < q < 1, $A = E_{11} - E_{12}$, then $W_C^J(A) = \mathbb{C}$.

Proof. Some computations yield

$$\begin{split} W_C^J(A) &= \{ (\alpha - \beta)(q\overline{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ (\cosh t + \sinh t \exp(i\theta))(q \cosh t + \sinh t \exp(i\phi)) : t \ge 0, 0 \le \theta, \phi \le 2\pi \}, \end{split}$$

where we may assume q > 0. Therefore, $W_C^J(A)$ is the union of the family of circles centered at $z(t, \theta) = x_0 + i y_0 = q (\cosh t + \sinh t \exp(i\theta)) \cosh t$, whose radii $r = r(t, \theta)$ satisfy $R = r^2 = \sinh^2 t (1 + 2 \sinh^2 t + 2 \sinh t \cosh t \cos \theta)$, and the following relation holds

$$R = \frac{1}{q^2}((x_0 - q)^2 + y_0^2).$$

The centers of the circles describe the open half-plane x > q/2. Thus,

$$W_C^J(A) = \left\{ (x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = \frac{1}{q^2} ((x_0 - q)^2 + y_0^2) x_0 > q/2, \ y_0 \in \mathbb{R} \right\}$$

Clearly, we have

$$(x - x_0)^2 + (y - y_0)^2 = \frac{1}{q^2}((x_0 - q)^2 + y_0^2).$$

If $(x - x_0)^2 = \frac{1}{q^2}(x_0 - q)^2$ and $(y - y_0)^2 = \frac{1}{q^2}y_0^2$, then (x, y) belongs to $W_C^J(A)$. It is always possible to find a real y_0 such that the first equation holds, while the second equation is satisfied if $x_0 = \frac{q}{1\pm q}(1\pm x)$. Since $x_0 > q/2$, this equation has a real solution x_0 for x > (q-1)/2 and also for x < (1+q)/2. \Box

The 3rd Subcase is investigated in the following.

PROPOSITION 5.3. If $C = E_{12} - qE_{22}$, -1 < q < 1, and $A = E_{11} - E_{12}$, then $W_C^J(A) = \left\{ (x, y) \in \mathbb{R}^2 : (x + q/2)^2 + \frac{y^2}{(1 - q^2)} > \frac{1}{4} \right\}.$



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Proof. We easily find

$$\begin{split} W_C^J(A) &= \{ (\overline{\beta} - \overline{\alpha})(q\overline{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ (\sinh t + \cosh t \exp(i\theta))(q \sinh t + \cosh t \exp(i\phi)) : t \ge 0, 0 \le \theta, \phi \le 2\pi \}, \end{split}$$

and we may consider $q \ge 0$. Since $W_C^J(A)$ is invariant under rotations around the origin in the case q = 0, we may concentrate on the case q > 0. Thus, $W_C^J(A)$ is the union of the family of circles centered at $z = x_0 + i y_0 = q \sinh t (\sinh t + \cosh t \exp(i\theta))$ and whose radii $r = r(t, \theta)$ satisfy $R = r^2 = \cosh^2 t (1 + 2 \sinh^2 t + 2 \sinh t \cosh t \cos \theta)$. The following relation holds

$$R = \frac{1}{q^2}((x+q)^2 + y^2)$$

The centers $z = z(t, \theta)$ of the circles describe the half-plane x > -q/2. Thus,

$$W_C^J(A) = \{ (x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = 1/q^2 ((x_0 + q)^2 + y_0^2), x_0 > -q/2, \ y_0 \in \mathbb{R} \}.$$

For a fixed complex number x + i y, we consider the Apolonius circle

$$\{(X,Y) \in \mathbb{R}^2 : (X+q)^2 + Y^2 = q^2((X-x)^2 + (Y-y)^2)\}$$

with center $(-\frac{q(1+qx)}{1-q^2}, -\frac{q^2y}{1-q^2})$, and radius $\frac{q}{1-q^2}\sqrt{(x+q)^2+y^2}$. Hence,

$$M(x,y) = \max\{x_0 : |(x_0 + iy_0) + \frac{q(1+qx)}{1-q^2} + i\frac{q^2y}{1-q^2}| = \frac{q}{1-q^2}\sqrt{(x+q)^2 + y^2}\}$$
$$= -\frac{q(1+qx)}{1-q^2} + \frac{q\sqrt{(x+q)^2 + y^2}}{1-q^2}.$$

The point x + iy belongs to $W_C^J(A)$ if and only if M(x,y) > -q/2. Thus, we conclude

$$W^J_C(A) = \{(x,y) \in \mathbb{R}^2 : 4((x+q)^2+y^2) - (1+q^2+2q\,x)^2 > 0\}. \quad \Box$$

The 4th Subcase is analysed in the following.

PROPOSITION 5.4. If
$$C = qE_{21} - E_{22}$$
, $-1 < q < 1$, and $A = E_{11} - E_{12}$, then
 $W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : y = \sqrt{1 - q^2}/2\sinh t, \ x > -1/2 - |q|/2\cosh t, \ t \in \mathbb{R}\}.$

Proof. We find

$$\begin{split} W_C^J(A) &= \{ (\overline{\beta} - \overline{\alpha})(q\overline{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ (\sinh t + \cosh t \exp(i\theta))(\sinh t + q \cosh t \exp(i\phi)) : t \ge 0, 0 \le \theta, \phi \le 2\pi \}. \end{split}$$



The case q = 0 is trivial, so we assume 0 < q < 1. Under this assumption, $W_C^J(A)$ is the union of the family of circles centered at $z = z(t, \theta) = x + iy = \sinh t (\sinh t + \cosh t \exp(i\theta))$, whose radii $r = r(t, \theta)$ satisfy $R = r^2 = q^2 \cosh^2(1 + 2\sinh^2 t + 2\sinh t \cosh t \cos \theta)$, and also $R = q^2((x+1)^2 + y^2)$. The centers of the circles range over the half-plane x > -1/2 and the proposition easily follows from Proposition 5.1. \Box

6. The third case. In this case, we are assuming that κ , τ are neutral and η , ζ are non- neutral. We may consider $\zeta = (1,0)^T$, $\tau = (1,-1)^T$, $\kappa = (1,-k)$, $k \in \mathbb{C}$ with |k| = 1 and $\eta = (\eta_1, \eta_2)$, $\eta_1, \eta_2 \in \mathbb{R}$, $|\eta_1| \neq |\eta_2|$. Under these assumptions, we obtain

$$W^J_C(A) = \{ (\overline{\alpha} + \overline{k\beta})(\eta_1(\alpha + \beta) + \eta_2(\overline{\alpha} + \overline{\beta})) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.$$

Firstly, we consider the three special Subcases: (1) $\eta_2 = 0$, $\eta_1 = 1$; (2) $\eta_1 = 0$, $\eta_2 = 1$; (3) k = 1. Finally, in Proposition 6.4 we treat the case $k \neq 1$ and $\eta_1 \neq \eta_2 \neq 0$.

If the Subcase (1) occurs, then

$$W_C^J(A) = \{ (\alpha + \beta) \overline{(\alpha + k \beta)} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}.$$

If k = 1, then $W_C^J(A)$ is the positive real line. Let $k \neq 1$ and let k_1 be a complex number such that $k_1^2 = k$. Thus, $|k_1| = 1$, $k_1 \neq 1$, $k_1 \neq -1$ and

$$\begin{split} W_C^J(A) &= \overline{k_1} \{ (\alpha + \beta)(k_1 \overline{\alpha} + \overline{k_1 \beta}) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \overline{k_1} \{ k_1 + (k_1 + \overline{k_1}) |\beta|^2 t + 2\Re(k_1 \overline{\alpha} \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}. \end{split}$$

Multiplying $W_C^J(A)$ by a complex number k_1 , and performing some calculations, the next proposition follows.

PROPOSITION 6.1. Let $C = E_{11}, A = k_1(E_{11} - E_{21}) + \overline{k_1}(E_{21} - E_{22}), k_1 = \exp(i\phi), -\pi < \phi < \pi, \phi \neq 0$. Then $W_C^J(A) = \{i \sin \phi + t : t \in \mathbb{R}\}.$

In the Subcase (2), we replace $(1, -k)^T$ by $(1, -\overline{k})^T$ and we prove.

PROPOSITION 6.2. Let $C = -E_{12}$, $A = (E_{11} - E_{21}) + k(E_{12} - E_{22})$, $k = \exp(i\phi)$, $-\pi < \phi \leq \pi$. Then $W_C^J(A) = \{z \in \mathbb{C} : |z| \geq \sin(|\phi|/2)\}$ for $\phi \neq 0$, and $W_C^J(A) = \mathbb{C} \setminus \{0\}$ for $\phi = 0$.

Proof. We get

$$W_C^J(A) = \{ (\alpha + k\beta)(\alpha + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \},\$$

being this set invariant under the rotation $z \mapsto \exp(i\theta) z$, $\theta \in \mathbb{R}$. Therefore, to conclude the proof it is sufficient to show that the function defined on $[0, +\infty) \times [0, 2\pi]$ by $f(t, \theta) = |\cosh t + \exp(i\theta) \sinh t|^2 |\cosh t + \exp(i\theta) \sinh t|^2$ ranges over

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 $[\sin^2(|\phi|/2), +\infty)$ for $\phi \neq 0$, and $(0, +\infty)$ in the case $\phi = 0$. This can be done by simple standard arguments. \Box

Next we consider the Subcase (3).

PROPOSITION 6.3. Let $C = E_{11} - qE_{12}$, $q \in \mathbb{R}$, $q \neq 1$, $q \neq -1$, and let $A = E_{11} + E_{12} - E_{21} - E_{22}$. The following hold:

1) If |q| > 1, then $W_C^J(A) = \mathbb{C} \setminus \{0\}$.

2) If
$$-1 < q < 1$$
, then $W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : x > 0, |y| \le |q| x/\sqrt{1-q^2}\}.$

Proof. Performing some computations, we obtain

$$W_C^J(A) = \{ (\alpha + \beta)(\overline{\alpha} + \overline{\beta} + q \alpha + q \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \}$$
$$= \{ |z| + |q| z : z \in \mathbb{C} \setminus \{0\} \}.$$

1) Let |q| > 1. The equation |z| + |q| z = 0 holds if and only if $z \neq 0$. Given a non-zero complex number $z_0 = x_0 + iy_0$ it is always possible to find a complex zsatisfying $z_0 = |z| + |q| z$, since the real system $x_0 = \sqrt{x^2 + y^2} + |q| x$, $y_0 = |q|y$ is possible.

2) Let -1 < q < 1. We may assume $q \neq 0$. It can be easily checked that the equations of the above system are satisfied for real numbers x_0, y_0 if and only if $x_0 > 0$ and $\frac{y_0^2}{x_0^2} < \frac{q^2}{1-q^2}$. \Box

PROPOSITION 6.4. Let $C = E_{11} - qE_{12}, q \in \mathbb{R}, q \neq 1, q \neq -1, A = k_1(E_{11} - E_{21}) + \overline{k_1}(E_{12} - E_{22}), k_1 = \exp(i\psi), -\pi < \psi < \pi, \psi \neq 0$. The following hold:

1) If |q| > 1, then

(6.1)
$$W_C^J(A) = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{q^2 - 1} + \frac{(y - \sin\psi)^2}{q^2} \ge \sin^2(\psi) \right\}.$$

2) If 0 < |q| < 1, then

$$W_C^J(A) = \left\{ (x, y) \in \mathbb{R}^2 : \frac{(y - \sin \psi)^2}{q^2} - \frac{x^2}{1 - q^2} \le \sin^2(\psi) \right\}.$$

Proof. Some computations yield

$$\begin{split} W_C^J(A) &= \{ (k_1 \alpha + \overline{k_1} \beta) (\overline{\alpha} + \overline{\beta} + q\alpha + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1 \} \\ &= \{ [(k_1 \cosh t + \overline{k_1} \sinh t \exp(i\phi)] [\cosh t + \sinh t \exp(-i\phi)] + q [k_1 \cosh t + \overline{k_1} \sinh \exp(i\phi)] [\cosh t + \sinh t \exp(i\phi)] \exp(i\theta) : t \ge 0, 0 \le \phi, \theta \le 2\pi \}. \end{split}$$



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Then, $W_C^J(A)$ is the union of the family of circles centered at

$$z = z(t, \phi, \psi) = x + iy = [(k_1 \cosh t + \overline{k_1} \sinh t \exp(i\phi)][\cosh t + \sinh t \exp(-i\phi)]$$
$$= k_1 + 2\cos(\psi) \sinh^2 t + 2\sinh t \cosh t \cos(\phi - \psi),$$

where $\psi = \arg k_1$. The radii of the above family, $r = r(t, \theta)$, satisfy $R = r^2 = q^2 [\cosh^2 t + \sinh^2 t + 2 \sinh t \cosh t \cos(\phi - 2\psi)] [\cosh^2 t + \sinh^2 t + 2 \sinh t \cosh t \cos \phi]$. Thus, $R = q^2 (x^2 + \sin^2(\psi))$, $y = \sin \psi$, and we easily get

(6.2)
$$W_C^J(A) = \{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-\sin\psi)^2 = q^2 (x_0^2 + \sin^2\psi), x \in \mathbb{R}\}.$$

1) Let |q| > 1. If x + iy is a boundary point of $W_C^J(A)$, then it satisfies

$$F(x, y; x_0) = (x - x_0)^2 + (y - \sin \psi)^2 - q^2 (x_0^2 + \sin^2 \psi) = 0,$$

and $F_x(x, y; x_0) = -2(x + (q^2 - 1)x_0) = 0$. By eliminating x_0 in the above equations, we get

$$\frac{x^2}{q^2 - 1} + \frac{(y - \sin\psi)^2}{q^2} = \sin^2\psi,$$

and so the boundary of $W_C^J(A)$ is contained in this ellipse. Conversely, we *claim* that every point of this ellipse is contained in the boundary of $W_C^J(A)$. Indeed, every point (x, y) of the ellipse is parametrically represented as $x = \sqrt{(q^2 - 1)} \sin \psi \cos \theta$, $y = \sin \psi + q \sin \psi \sin \theta$, $\theta \in \mathbb{R}$, and satisfies

$$\left(x + \frac{\sin\psi\cos\theta}{\sqrt{q^2 - 1}}\right)^2 + \left(y - \sin\psi\right)^2 - q^2\left(\frac{\sin^2\psi\cos^2\theta}{q^2 - 1} + \sin^2\psi\right) = 0.$$

Thus, (x, y) is an element of (6.2). To finish the proof, we observe that if $x_0 \to \infty$, then the circle $F(x, y; x_0) = 0$ contains points x + i 0 with $x \to \infty$. Let 0 < |q| < 1. By similar arguments to the above, we find that the boundary of $W_C^J(A)$ is the hyperbola:

$$\frac{(y-\sin\psi)^2}{q^2} - \frac{x^2}{1-q^2} = \sin^2(\psi).$$

If $x_0 \to +\infty$, then the circle $F(x, y; x_0) = 0$ contains points x of the real line with $x \to +\infty$. \Box

7. Proof of main theorems and a concluding remark. As a consequence of the descriptions of the shape of $W_C^J(A)$ in Sections 2–6 and [14], we obtain Theorem 1.1. We remark that $W_C^J(A)$, or each connected component of its complement $\mathbb{C}\setminus W_C^J(A)$, is convex. This property does not hold for 3-dimensional Krein spaces. From Theorem 1.2 it follows that $W_C^J(A)$ has at most 1 hole.



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