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NORM ESTIMATES FOR FUNCTIONS OF TWO NON-COMMUTING MATRICES*

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Abstract. A class of matrix valued analytic functions of two non-commuting matrices is considered. A sharp norm estimate is established. Applications to matrix and differential equations are also discussed.

Key words. Functions of non-commuting matrices, Norm estimate, Matrix equation, Differential equation.

AMS subject classifications. 15A54, 15A45, 15A60.

1. Introduction and statement of the main result. In the book [5], I.M. Gel'fand and G.E. Shilov have established an estimate for the norm of a regular matrix-valued function in connection with their investigations of partial differential equations. However, that estimate is not sharp; it is not attained for any matrix. The problem of obtaining a sharp estimate for the norm of a matrix-valued function has been repeatedly discussed in the literature, cf. [2]. In the paper [6] (see also [7]), the author has derived an estimate for regular matrix-valued functions, which is attained in the case of normal matrices. In [8], the results of the paper [6] were generalized to functions of two commuting matrices. In the present paper, we establish a sharp estimate for the norm of a matrix-valued function of two non-commuting matrices.

It should be noted that functions of many operators were investigated by many mathematicians, (cf. [1, 15, 16] and references therein) however the norm estimates were not considered, but as it is well-known, matrix valued functions give us representations of solutions of various differential, difference equations and matrix equations. This fact allows us to investigate stability, well-posedness and perturbations of these equations by norm estimates for matrix valued functions, cf. [2].

Let $\mathbb{C}^n$ be the Euclidean space with scalar product $(\cdot,\cdot)$, the Euclidean norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$ and the unit operator I. Unless otherwise stated A, K and $\tilde{A}$ will be $n \times n$ matrices. $\|A\| = \sup_{h\in\mathbb{C}^n} \|Ah\|/\|h\|$ is the spectral (operator) norm of A. By $\sigma(A)$ and $R_z(A) = (A - zI)^{-1}$ $(z \notin \sigma(A))$ we denote the spectrum and resolvent of A, respectively.

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Let Ω_A and $\Omega_{\tilde{A}}$ be open simple connected supersets of $\sigma(A)$ and $\sigma(\tilde{A})$, respectively, and f be a scalar function analytic on $\Omega_A \times \Omega_{\tilde{A}}$. We define the matrix valued function

$$F(f, A, K, \tilde{A}) := -\frac{1}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} f(z, w) R_z(A) K R_w(\tilde{A}) dw\, dz, \tag{1.1}$$

where $C_A \subset \Omega_A, C_{\tilde{A}} \subset \Omega_{\tilde{A}}$ are closed contour surrounding $\sigma(A)$ and $\sigma(\tilde{A})$, respectively. Such functions play an essential role in the theory of matrix equations. More specifically, consider the matrix equation

$$\sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} A^j X \tilde{A}^k = K, \tag{1.2}$$

where X should be found and c_{jk} are complex numbers. Put

$$p(z, w) = \sum_{j=0}^{m_1} \sum_{k=0}^{m_2} c_{jk} z^j \tilde{w}^k.$$

Then by Theorem 3.1 from [2, Chapter 1] a unique solution of equation (1.2) is given by the formula

$$X = F\left(\frac{1}{p(z, w)}, A, K, \tilde{A}\right) \tag{1.3}$$

provided $\lambda_k \neq \tilde{\lambda}_j$ $(j, k = 1, \ldots, n)$. Throughout the rest of this paper λ_k and $\tilde{\lambda}_j$ are the eigenvalues counted with their multiplicities of A and $\tilde{A}$, respectively. Equations of the type (1.2) naturally arose in various applications, cf. [2, 14, 12]. The Lyapunov equation $A^*X + XA = K$, cf. [2], and the Lyapunov type equation

$$X + A^*XA = K \tag{1.4}$$

which play an important role in the theory of difference equations, cf. [9] are the examples of equation (1.2). These equations recently attracted the attention of many mathematicians. Mainly, numerical methods for the solutions of matrix equations were developed, cf. [11, 13, 17]. In the paper [3], reflexive and anti-reflexive solutions of a linear matrix equation were explored. No estimates were established for solutions of these equations. Furthermore, suppose that

$$T(t) := -\frac{1}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} e^{t(z+w)} R_z(A) K R_w(\tilde{A}) dw\, dz. \tag{1.5}$$

Take into account that $zR_z(A) = AR_z(A) - I$. Then simple calculations show that

$$T'(t) = -\frac{1}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} (z + w) e^{t(z+w)} R_z(A) K R_w(\tilde{A}) dw\, dz =$$

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$$-\frac{1}{4\pi^2}\int_{C_{\tilde{A}}}\int_{C_A} e^{t(z+w)}[AR_z(A)KR_w(\tilde{A})+R_z(A)KR_w(\tilde{A})\tilde{A}]dw\,dz.$$

So

$$T'(t) = AT(t) + T(t)\tilde{A}. \tag{1.6}$$

Such equations arise in numerous applications, in particular in the theory of vector differential equations, cf. [10, p. 509], [2, Section VI.4, equation (4.15) and Section VI.2], [4, Section XV.10]. Additional examples are given in Section 3.

The following quantity plays a key role in this article:

$$g(A) = \left[N_2^2(A) - \sum_{k=1}^{n}|\lambda_k|^2\right]^{1/2},$$

where $N_2(A) = (Trace\ AA^*)^{1/2}$ is the Frobenius (Hilbert-Schmidt norm) of A. Here, A^* is adjoint to A. The following relations are checked in [7, Section 2.1]:

$$g^2(A) \le N_2^2(A) - |Trace\ A^2| \ \text{ and } \ g^2(A) \le \frac{N_2^2(A-A^*)}{2} = 2N_2^2(A_I), \tag{1.7}$$

where $A_I = (A-A^*)/2i$. If A is a normal matrix: $AA^* = A^*A$, then $g(A) = 0$.

By $co(A)$ we denote the closed convex hull of $\sigma(A)$. Let $f(z,w)$ be regular on a neighborhood of $co(A)\times co(\tilde{A})$. Put

$$f^{(j,k)}(z,w) = \frac{\partial^{j+k} f(z,w)}{\partial z^j \partial w^k},$$

and let the numbers $\eta_{jk} = \eta_{jk}(f, A, \tilde{A})$ be given by

$$\eta_{00} = \sup_{z\in\sigma(A), w\in\sigma(\tilde{A})} |f(z,w)|;\ \eta_{jk} = \frac{1}{(j!k!)^{3/2}} \sup_{z\in co(A), w\in co(\tilde{A})} |f^{(j,k)}(z,w)|;$$

$$\eta_{0j} := \frac{1}{(j!)^{3/2}} \sup_{z\in\sigma(A), w\in co(\tilde{A})} \left|\frac{\partial^j f(z,w)}{\partial w^j}\right|,$$

and

$$\eta_{j0} := \frac{1}{(j!)^{3/2}} \sup_{z\in co(A), w\in\sigma(\tilde{A})} \left|\frac{\partial^j f(z,w)}{\partial z^j}\right| \quad (j,k \ge 1).$$

Now we are in a position to formulate the main result of the paper.

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THEOREM 1.1. *Let both A and $\tilde{A}$ be non-normal matrices and $f(z,w)$ be regular on a neighborhood of $co(A) \times co(\tilde{A})$. Then*

$$\|F(f, A, K, \tilde{A})\| \le N_2(K) \sum_{j,k=0}^{n-1} \eta_{jk} g^j(A) g^k(\tilde{A}).$$

If A is normal, $\tilde{A}$ is non-normal and $f(z,w)$ is regular on a neighborhood of $\sigma(A) \times co(\tilde{A})$, then

$$\|F(f, A, K, \tilde{A})\| \le N_2(K) \sum_{j=0}^{n-1} \eta_{0j} g^j(\tilde{A}).$$

If $\tilde{A}$ is normal, A is non-normal and $f(z,w)$ is regular on a neighborhood of $\sigma(\tilde{A}) \times co(A)$, then

$$\|F(f, A, K, \tilde{A})\| \le N_2(K) \sum_{j=0}^{n-1} \eta_{j0} g^j(A).$$

If both A and $\tilde{A}$ are normal and $f(z,w)$ is regular on a neighborhood of $\sigma(A) \times \sigma(\tilde{A})$, then

$$\|F(f, A, K, \tilde{A})\| \le N_2(K) \max_{j,k} |f(\lambda_j, \tilde{\lambda}_k)|.$$

2. Proof of Theorem 1.1. We need the following result proved in [8].

LEMMA 2.1. *Let Ω and $\tilde{\Omega}$ be the closed convex hulls of the complex points $x_0, x_1, \dots, x_n$ and $y_0, y_1, \dots, y_m$, respectively, and let a scalar-valued function $f(z,w)$ be regular on a neighborhood of $\Omega \times \tilde{\Omega}$. Additionally, let L and $\tilde{L}$ be the boundaries of Ω and $\tilde{\Omega}$, respectively. Then with the notation*

$$Y(x_0, \dots, x_n;\ y_0, \dots, y_m) = -\frac{1}{4\pi^2} \int_L \int_{\tilde{L}} \frac{f(z,w)dz\, dw}{(z-x_0)\cdots(z-x_n)(w-y_0)\cdots(w-y_m)},$$

we have

$$|Y(x_0, \dots, x_n;\ y_0, \dots, y_m)| \le \frac{1}{n!m!} \sup_{z \in \Omega, w \in \tilde{\Omega}} |f^{(n,m)}(z,w)|.$$

Let $\{e_k\}$ and $\{\tilde{e}_k\}$ be the orthogonal normal bases of the triangular representation (Schur's bases) to A and $\tilde{A}$, respectively. So,

$$Ae_k = \sum_{j=1}^{k} a_{jk} e_j.$$

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We can write

$$A = D_A + V_A, \quad \tilde{A} = D_{\tilde{A}} + V_{\tilde{A}}, \tag{2.1}$$

where $D_A, D_{\tilde{A}}$ are the diagonal parts, V_A and $V_{\tilde{A}}$ are the nilpotent parts of A and $\tilde{A}$, respectively. Namely,

$$D_A e_k = \lambda_k e_k; \quad V_A e_k = \sum_{j=1}^{k-1} a_{jk} e_j.$$

Similarly, $D_{\tilde{A}}$ and $V_{\tilde{A}}$ are defined. Furthermore, let $|V_A|$ be the operator whose entries in $\{e_k\}$ are the absolute values of the entries of a matrix V_A. That is, $(|V_A| e_j, e_k) = |(V_A e_j, e_k)|$ and

$$|V_A| = \sum_{k=1}^{n} \sum_{j=1}^{k-1} |a_{jk}|(\cdot, e_k) e_j.$$

Similarly, $|V_{\tilde{A}}|$ is defined with respect to $\{\tilde{e}_k\}$. In addition, $|K|$ is defined by

$$|K| \tilde{e}_j = \sum_{k=1}^{n} |(K\tilde{e}_j, e_k)| e_k.$$

LEMMA 2.2. *Under the hypothesis of Theorem 1.1, the inequality*

$$\|F(f, A, K, \tilde{A})\| \le \| |K| \| \sum_{j,k=1}^{n-1} \sqrt{k!j!}\,\eta_{jk} \| |V_{\tilde{A}}|^j \| \| \, |V_{\tilde{A}}|^k \|$$

is true, where V_A and $V_{\tilde{A}}$ are the nilpotent parts of A and $\tilde{A}$, respectively.

Proof. It is not hard to see that the representation (2.1) implies the equality

$$(A - I\lambda)^{-1} = (D_A + V_A - \lambda I)^{-1} = (I + R_\lambda(D_A) V_A)^{-1} R_\lambda(D_A)$$

for all regular λ. According to Lemma 1.7.1 from [7] $R_\lambda(D_A)V_A$ is a nilpotent operator, because V_A and $R_\lambda(D_A)$ the same invariant subspaces. Hence, $(R_\lambda(D_A)V_A)^n = 0$. Therefore, from (1.1) it follows

$$F(f, A, K, \tilde{A}) = \sum_{j,k=0}^{n-1} M_{jk}, \tag{2.2}$$

where

$$M_{jk} = \frac{(-1)^{k+j}}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} f(z,w) (R_z(D_A)V_A)^j R_z(D_A) K (R_w(D_{\tilde{A}}) V_{\tilde{A}})^k R_w(D_{\tilde{A}}) dz\, dw.$$

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Since D_A is a diagonal matrix with respect to the Schur basis $\{e_k\}$ and its diagonal entries are the eigenvalues of A, we obtain

$$R_z(D_A) = \sum_{j=1}^{n} \frac{Q_j}{\lambda_j(A) - z},$$

where $Q_k = (\cdot, e_k)e_k$. Similarly,

$$R_z(D_{\tilde{A}}) = \sum_{j=1}^{n} \frac{\tilde{Q}_j}{\lambda_j(\tilde{A}) - z},$$

where $\tilde{Q}_k = (\cdot, \tilde{e}_k)\tilde{e}_k$. Taking into account that $Q_s V_A Q_m = 0,\ \tilde{Q}_s V_{\tilde{A}} \tilde{Q}_m = 0\ (s \geq m)$, we get

$$M_{jk} = \sum_{1 \leq s_1 < s_2 < \cdots < s_{j+1} \leq n} Q_{s_1} V_A Q_{s_2} V_A \cdots V_A Q_{s_{j+1}} K \times$$

$$\times \sum_{1 \leq m_1 < m_2 < \cdots < m_{k+1} \leq n} \tilde{Q}_{m_1} V_{\tilde{A}} \tilde{Q}_{m_2} V_{\tilde{A}} \cdots V_{\tilde{A}} \tilde{Q}_{m_{k+1}} \hat{I}(s_1, \ldots, s_{j+1}, m_1, \ldots, m_{k+1}),$$

where $0 \leq j, k \leq n-1$ and

$$\hat{I}(s_1, \ldots, s_{j+1}, m_1, \ldots m_{k+1}) =$$

$$\frac{(-1)^{k+j}}{4\pi^2} \int_{C_A} \int_{C_{\tilde{A}}} \frac{f(z,w) dz\, dw}{(\lambda_{s_1}(A) - z) \cdots (\lambda_{s_{k+1}}(A) - z)(\lambda_{m_1}(\tilde{A}) - w) \cdots (\lambda_{m_{k+1}}(\tilde{A}) - w)}.$$

Hence, with $M_{jk} = M$, we have

$$|(M\tilde{e}_m, e_s)| = |\sum_{s < s_2 < \cdots < s_{j+1} \leq n} \sum_{1 \leq m_1 < m_2 < \cdots < m} \hat{I}(s, \ldots, s_{j+1}, m_1, \ldots, m) \times$$

$$\times (Q_s V_A Q_{s_2} V_A \cdots V_A Q_{s_{j+1}} K \tilde{Q}_{m_1} V_{\tilde{A}} \tilde{Q}_{m_2} V_{\tilde{A}} \cdots V_{\tilde{A}} \tilde{Q}_m \tilde{e}_m, e_s)| \leq J_{jk} \sum_{s < s_2 < \cdots < s_{j+1} \leq n}$$

$$\times \sum_{1 \leq m_1 < m_2 < \cdots < m} (Q_s |V_A| Q_{s_2} |V_A| \cdots Q_{s_{j+1}} |K| \tilde{Q}_{m_1} |V_{\tilde{A}}| \tilde{Q}_{m_2} |V_{\tilde{A}}| \cdots \tilde{Q}_m \tilde{e}_m, e_s),$$

where

$$J_{jk} := \max_{1 \leq s_1 < \cdots < s_{j+1} \leq n; 1 \leq m_1 < \cdots < m_{k+1} \leq n} |\hat{I}(s_1, \ldots, s_{j+1}, m_1, \ldots m_{k+1})|.$$

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Thus $|(M\tilde e_m, e_s)| \le (T\tilde e_m, e_s)$, where

$$T = J_{jk} \sum_{s_1<s_2<\cdots<s_{j+1}\le n} \; \sum_{1\le m_1<m_2<\cdots<m_{k+1}\le n} Q_{s_1}|V_A|Q_{s_2}|V_A|\cdots|V_A|Q_{s_{j+1}}|K|\times$$

$$\times \tilde Q_{m_1}|V_{\tilde A}|\tilde Q_{m_2}|V_{\tilde A}|\cdots|V_{\tilde A}|\tilde Q_{m_{k+1}}. \tag{2.3}$$

Take into account that

$$Mx = \sum_{k=1}^n (x,\tilde e_k)M\tilde e_k = \sum_{j=1}^n\sum_{k=1}^n (x,\tilde e_k)(M\tilde e_k, e_j)e_j \quad (x\in\mathbb{C}^n).$$

So

$$\|Mx\|^2 = \sum_{j=1}^n \left|\sum_{k=1}^n (x,\tilde e_k)(M\tilde e_k, e_j)\right|^2 \le$$

$$\sum_{j=1}^n \left(\sum_{k=1}^n (x,\tilde e_k)(T\tilde e_k, e_j)\right)^2.$$

Since $\|x\| = \|y\|$ for

$$y = \sum_{k=1}^n |(x,\tilde e_k)|\tilde e_k,$$

we obtain $\|M\| \le \|T\|$. But

$$\sum_{1\le s_1<s_2<\cdots<s_{j+1}\le n} Q_{s_1}|V_A|Q_{s_2}|V_A|\cdots|V_A|Q_{s_{j+1}} = |V_A|^j$$

and

$$\sum_{1\le m_1<m_2<\cdots<m_{k+1}\le n} \tilde Q_{m_1}|V_{\tilde A}|\tilde Q_{m_2}|V_{\tilde A}|\cdots|V_{\tilde A}|Q_{m_{k+1}} = |V_{\tilde A}|^k.$$

So by (2.3)

$$\|M_{jk}\| \le \|T\| \le J_{jk}\| \,|V_A|^j|K||V_{\tilde A}|^k \,\| \quad (j,k\ge 0). \tag{2.4}$$

Due to Lemma 2.1

$$J_{jk} \le \sup_{z\in co(A), w\in co(\tilde A)} \frac{|f^{(j,k)}(z,w)|}{j!k!} = \sqrt{j!k!}\,\eta_{jk} \quad (j,k\ge 1).$$

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Thus,

$$\|M_{jk}\| \le \sqrt{j!k!}\eta_{jk}\| \, |V_A|^j |K| |V_{\tilde{A}}|^k \, \| \quad (j,k \ge 0). \tag{2.5}$$

This inequality and (2.2) imply the required result. □

Proof of Theorem 1.1. Theorem 2.5.1 from [7] implies

$$\|W^k\| \le \frac{1}{\sqrt{k!}} N_2^k(W) \tag{2.6}$$

for any $n \times n$ nilpotent matrix W. Take into account that $N_2(|V_A|) = N_2(V_A)$. Moreover, by Lemma 2.3.2 from [7], $N_2(V_A) = g(A)$. Thus,

$$\| \, |V_A|^k \, \| \le \frac{1}{\sqrt{k!}} g^k(A) \quad (k = 1, \dots, n-1).$$

The similar inequality holds for $V_{\tilde{A}}$. In addition,

$$N_2^2(|K|) = \sum_{j=1}^n \| |K| \tilde{e}_j \|^2 = \sum_{j=1}^n \sum_{k=1}^n |(K\tilde{e}_j, e_k)|^2 = \sum_{j=1}^n \sum_{k=1}^n \|K\tilde{e}_j\|^2 = N_2^2(K).$$

Now the previous lemma yields the required result. □

3. Examples. Consider the equation

$$AX - X\tilde{A} = K \tag{3.1}$$

assuming that

$$\delta := dist\ (co(A), co(\tilde{A})) > 0.$$

Take $f(z,w) = \frac{1}{z-w}$. Then

$$\eta_{jk} \le \frac{(k+j)!}{\delta^{j+k+1}(k!j!)^{3/2}} \quad (j,k = 0,1,\dots,n-1).$$

Hence, by Theorem 1.1 and (1.3) a solution of (3.1) satisfies the inequality

$$\|X\| \le N_2(K) \sum_{j,k=0}^{n-1} \frac{(k+j)!}{\delta^{j+k+1}(k!j!)^{3/2}} g^j(A) g^k(\tilde{A}).$$

Finally, consider the function

$$S(x) := -\frac{1}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} sin\ (x(z+w)) R_z(A) K R_w(\tilde{A}) dw\, dz \quad (x \in \mathbb{R}).$$

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We have

$$S''(x) = \frac{1}{4\pi^2} \int_{C_{\tilde{A}}} \int_{C_A} (z+w)^2 sin\ (x(z+w)) R_z(A) K R_w(\tilde{A}) dw\ dz.$$

But $zR_z(A) = AR_z(A) - I$ and therefore,

$$z^2 R_z(A) = zAR_z(A) - zI = A(AR_z(A) - I) - zI = A^2 R_z(A) - I - zI.$$

So, $S(x)$ is a solution of the equation

$$S'' = A^2 S + AS\tilde{A} + S\tilde{A}^2.$$

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