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PATH PRODUCT AND INVERSE M-MATRICES*

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Abstract. It is known that inverse M-matrices are strict path product (SPP) matrices, and that the converse is not true for matrices of order greater than 3. In this paper, given a normalized SPP-matrix A, some new values s' for which $A+s'I$ is an inverse M-matrix are obtained. Our values s' extend the values s given by Johnson and Smith [C.R. Johnson and R.L. Smith. Positive, path product, and inverse M-matrices. *Linear Algebra Appl.*, 421:328–337, 2007.]. The question whether or not a 4×4 SPP-matrix is a P-matrix is settled.

Key words. M-matrix, Inverse M-matrix, Path product matrix, P-matrix.

AMS subject classifications. 15A48, 15A57.

1. Introduction. An $n \times n$ matrix $A = (a_{ij})$ is an *M-matrix* if $a_{ij} \leq 0$ $(i \neq j)$ and $A^{-1} \geq 0$. A nonnegative matrix which is the inverse of an M-matrix is an *inverse M-matrix* (*IM-matrix*). Inverse M-matrices arise in mathematical modeling, random energy models in statistical physics [1], numerical integration and the Ising model of ferromagnetism [12]. There has been a great deal of work on special types of IM-matrices (see, for example, [3, 4, 9–11]).

Here we will be interested in the property

$$\frac{a_{ij}a_{jk}}{a_{jj}} \leq a_{ik}\,, \quad 1 \leq i, j, k \leq n \tag{1.1}$$

of an IM-matrix $A = (a_{ij})_{n\times n}$, $n \geq 3$, which was first noted in [12] and more fully developed in [7].

Following [7], we call (1.1) the path product conditions or PP conditions, for short. An $n \times n$ nonnegative matrix $A = (a_{ij})$, with $a_{ii} > 0$, satisfying these conditions is

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a *PP-matrix.* Moreover, if at least one strict inequality in (1.1) holds for $i = k$ and $i \neq j$, then A is a *strict path product* (SPP) *matrix.* In [7] (see also [12]), it is proved that an IM-matrix is an SPP-matrix. Furthermore, an SPP-matrix is an IM-matrix when $n \leq 3$, and this is not necessarily the case for larger n. Consequently, it was noted in [6] that an SPP-matrix may be made an IM-matrix by adding an appropriate nonnegative diagonal matrix.

We say that an $n \times n$ nonnegative matrix $A = (a_{ij})$ is *normalized* if $a_{ii} = 1$ and $a_{ij} < 1$, for $i \neq j$. It was noted in [7] that if A is an $n \times n$ SPP-matrix, then there exist positive diagonal matrices D and E such that $B = DAE$, where B is a normalized SPP-matrix.

Given an $n \times n$ matrix A and index sets $\alpha,\ \beta \subseteq N$, $N = \{1, \ldots, n\}$, we denote by $A[\alpha, \beta]$ the submatrix lying in rows α and columns β. Similarly, $A(\alpha, \beta)$ denotes the submatrix deleting rows α and columns β. If $\alpha = \beta$, then we denote the principal submatrix $A[\alpha, \alpha]$ (resp., $A(\alpha, \alpha)$) by $A[\alpha]$ (resp., $A(\alpha)$).

An *almost principal submatrix* (resp., *minor*) is a submatrix $A[\alpha, \beta]$ (resp., $\det A[\alpha, \beta]$) for which α and β have the same number of elements and differ just in one of their elements. Almost principal minors are exactly the numerators of off-diagonal entries of inverses of principal submatrices. Following [8], we abbreviate "almost principal minor" to APM.

In this paper, for an $n \times n$ normalized SPP-matrix $A = (a_{ij})$, we will give new values s' such that $A + s'I$ is an IM-matrix. Our values s' extend the values given by Johnson and Smith [6]. Examples are also given, and we will show that a 4×4 normalized SPP-matrix is necessarily a P-matrix; this answers a question raised in [7].

2. Main results. The results about SPP-matrices established by Johnson and Smith [7] that we shall use are the following.

LEMMA 2.1. *Let $A = (a_{ij})$ be a normalized SPP-matrix of order n. Then $A[\alpha]$ is a normalized SPP-matrix.*

LEMMA 2.2. *Let $A = (a_{ij})$ be a normalized SPP-matrix of order n. Then all 3×3 principal submatrices of A are IM-matrices.*

The following appear in [6].

THEOREM 2.3. *Let $A = (a_{ij})$ be a normalized SPP matrix of order n, $n \geq 2$, whose proper principal minors are positive and whose APMs are signed as those of an IM-matrix. Then,*

1. *For each nonempty proper subset α of $N = \{1, 2, \ldots, n\}$ and for all indices*

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$i \in \alpha$ and $j \notin \alpha$, we have

$$\det A[\alpha] > \max\{|\det A[\alpha - i + j, \alpha]|,\ |\det A[\alpha, \alpha - i + j]|\};$$

2. $\det A > 0$*;*
3. *A is an IM-matrix.*

THEOREM 2.4. *Let $A = (a_{ij})$ be a 4×4 normalized SPP-matrix. Then $A + I$ is an IM-matrix. Furthermore, $A + sI$ need not be an IM-matrix when $s < 1$.*

Now we are ready to state the following result about 4×4 normalized SPP matrices.

THEOREM 2.5. *Let $A = (a_{ij})$ be a 4×4 normalized SPP-matrix. Then $A + s'I$ is an IM-matrix for all $s' \geq m$, where*

$$m = \max_{i \neq j} \frac{a_{ik} a_{kj}}{a_{ij}} \leq 1, \quad k = 1, \ldots, n, \quad k \neq i, j, \ \text{and} \ \ a_{ij} \neq 0.$$

Proof. Following the idea of Theorem 2.4, to show $A + mI$ is an IM-matrix, we will show that the $(4, 1)$ APM (i.e., the determinant of $A[\{1, 2, 3\}, \{2, 3, 4\}]$) is nonnegative. Note that

$$\begin{aligned}
\det(A + mI)(4, 1) &= \det \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ 1 + m & a_{23} & a_{24} \\ a_{32} & 1 + m & a_{34} \end{bmatrix} \\
&= (1 + m)^2 a_{14} - (1 + m) a_{12} a_{24} - (1 + m) a_{13} a_{34} + a_{12} a_{23} a_{34} \\
&\quad + a_{13} a_{32} a_{24} - a_{14} a_{23} a_{32} \\
&= (1 + m)(a_{14} - a_{12} a_{24} + m a_{14} - a_{13} a_{34}) + a_{12} a_{23} a_{34} \\
&\quad + a_{13} a_{32} a_{24} - a_{14} a_{23} a_{32} \\
&\geq (1 + m)(a_{14} - a_{12} a_{24} + m a_{14} - a_{13} a_{34}) \\
&\quad + a_{12} a_{23} a_{32} a_{24} + a_{13} a_{32} a_{23} a_{24} - a_{14} a_{23} a_{32},
\end{aligned}$$

where $m a_{14} - a_{13} a_{34} = a_{14} \left(m - \frac{a_{13} a_{34}}{a_{14}} \right) \geq 0$. If the sum of the last three terms is nonnegative, then the determinant is nonnegative by the path product inequalities. Otherwise, we have

$$\begin{aligned}
\det(A + mI)(4, 1) &\geq (1 + m)(a_{14} - a_{12} a_{24} + m a_{14} - a_{13} a_{34}) \\
&\quad + a_{12} a_{23} a_{32} a_{24} + a_{13} a_{32} a_{23} a_{24} - a_{14} a_{23} a_{32} \\
&= (1 + m)(a_{14} - a_{12} a_{24} + m a_{14} - a_{13} a_{34}) \\
&\quad + (a_{12} a_{24} + a_{13} a_{24} - a_{14}) a_{23} a_{32} \\
&\geq (1 + m)(a_{14} - a_{12} a_{24} + m a_{14} - a_{13} a_{34}) + (a_{12} a_{24} + a_{13} a_{24} - a_{14}) \\
&= m a_{14} - a_{13} a_{34} + m(a_{14} - a_{12} a_{24} + m a_{14} - a_{13} a_{34}) + a_{13} a_{24} \\
&\geq 0.
\end{aligned}$$

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As a consequence, $A + mI$ is an IM-matrix. Since $s' \geq m$, $A + s'I$ is necessarily an IM-matrix. □

EXAMPLE 2.6. Consider the following normalized SPP-matrix

$$A = \begin{bmatrix} 1 & 0.4 & 0.6 & 0.7 \\ 0.6 & 1 & 0.5 & 0.7 \\ 0.6 & 0.5 & 1 & 0.7 \\ 0.4 & 0.5 & 0.3 & 1 \end{bmatrix}.$$

Then A is not an IM-matrix, since $\det A(2,1) = -0.019 \leq 0$. By actual calculation, $m = \frac{a_{14}a_{42}}{a_{12}} = 0.875$, so $A + 0.875I$ is an IM-matrix. In fact, $A + mI$ is an IM-matrix if and only if $m \geq 0.11$.

For convenience, let $n \geq 3$, and, for $i \neq j$, define

$$u_{ij}(A) = \begin{cases} \frac{1}{a_{ij}} \sum\limits_{k=1,k\neq i,j}^{n} a_{ik}a_{kj}, & a_{ij} \neq 0, \\ 0, & a_{ij} = 0, \end{cases}$$

$U(A) = \max_{i\neq j} u_{ij}(A)$, i.e., the largest value among $u_{ij}(A)$, where $i \neq j$,
$u(A)$ the second largest value among $u_{ij}(A)$, where $i \neq j$,
$\varepsilon = U(A) - u(A)$,
$\varepsilon' = U(A[\alpha]) - u(A[\alpha])$.

In [6, Theorem 3], a lower bound is given for the numbers s such that $A + sI$ is an IM-matrix. If $U(A) > 1$, then this bound is zero and it cannot be improved. But for $U(A) \leq 1$ Theorem 2.7 improves the lower bound $U(A) - 1$ given in [6, Theorem 3].

THEOREM 2.7. *Let $A = (a_{ij})$ be a normalized SPP matrix of order n, $n \geq 3$, and let $l = max\{U(A), 1\}$. Then $A + s'I$ is an IM-matrix for all $s' \geq |l - \varepsilon - 1|$.*

Proof. We use a proof technique analogous to that in [6, Theorem 3], and induction on n. If $n = 3$, A is an IM-matrix and thus $A + s'I$ is an IM-matrix for all

$$s' \geq |l - \varepsilon - 1|.$$

When $n > 3$, proceeding inductively, let

$$C = A + s'I = (c_{ij})_{n\times n}.$$

It follows that the $(n-1) \times (n-1)$ principal minors of C are positive since for any principal submatrix $A[\alpha]$ of A, $A[\alpha] + s''I$ is an IM-matrix so that $A[\alpha] + s'I$ is an IM-matrix, as $s' \geq s''$, where

$$s'' = \begin{cases} 0, & U(A[\alpha]) \leq 1, \\ |U(A[\alpha]) - \varepsilon' - 1|, & U(A[\alpha]) > 1. \end{cases}$$

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Using Theorem 2.3 and permutation similarity, it is enough to prove that the complement of the (1,2)-entry is nonnegative, that is,

$$c_{21} \det C(\{1,2\}) - \begin{bmatrix} c_{23} \cdots c_{2n} \end{bmatrix} \operatorname{adj} C(\{1,2\}) \begin{bmatrix} c_{31} \\ \vdots \\ c_{n1} \end{bmatrix} \geq 0,$$

or

$$c_{21} \det C(\{1,2\}) \geq \begin{bmatrix} c_{23} \cdots c_{2n} \end{bmatrix} \operatorname{adj} C(\{1,2\}) \begin{bmatrix} c_{31} \\ \vdots \\ c_{n1} \end{bmatrix}.$$

Dividing by $\det C(\{1,2\})$, we obtain

$$c_{21} \geq \begin{bmatrix} c_{23} \cdots c_{2n} \end{bmatrix} C(\{1,2\})^{-1} \begin{bmatrix} c_{31} \\ \vdots \\ c_{n1} \end{bmatrix}. \tag{2.1}$$

Let b_{ij}, $i,j = 3,\dots,n$, be the entries of $C(\{1,2\})^{-1}$. By induction, we verify that $C^{-1} = B = (b_{ij})$ is an M-matrix. Obviously, the right hand side of (2.1) is

$$\sum_{i,j=3}^{n} c_{2i} b_{ij} c_{j1} = \sum_{i \neq j} c_{2i} b_{ij} c_{j1} + \sum_{i=3}^{n} c_{2i} b_{ii} c_{i1}.$$

Since $b_{ij} \leq 0$, by path product

$$\sum_{i \neq j} c_{2i} b_{ij} c_{j1} \leq \sum_{i \neq j} c_{2i} b_{ij} c_{ji} c_{i1};$$

applying Fischer's inequality [5] to the IM-matrix $C(\{1,2\})$, we have

$$\det C(\{1,2\}) \leq c_{ii} \det C(\{1,2,i\}) = (1+s') \det C(\{1,2,i\}).$$

So

$$\frac{1}{1+s'} \leq \frac{\det C(\{1,2,i\})}{\det C(\{1,2\})} = b_{ii}.$$

From the above inequalities, we obtain

$$\sum_{i=3}^{n} \sum_{j=3}^{n} c_{2i} b_{ij} c_{j1} = \sum_{i=3}^{n} \sum_{j=3, j \neq i}^{n} c_{2i} b_{ij} c_{j1} + \sum_{i=3}^{n} (c_{2i} b_{ii} c_{i1} + c_{2i} b_{ii} c_{ii} c_{i1} - c_{2i} b_{ii} c_{ii} c_{i1}).$$

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Since $c_{j1} = a_{j1} \geq a_{ji}a_{i1} = c_{ji}c_{i1} \geq 0$ and $b_{ij} \leq 0,\ i \neq j$, we obtain

$$\begin{aligned}\sum_{i=3}^{n}\sum_{j=3}^{n} c_{2i}b_{ij}c_{j1} &\leq \sum_{i=3}^{n}\sum_{j=3}^{n} c_{2i}b_{ij}c_{ji}c_{i1} + \sum_{i=3}^{n}(1-c_{ii})c_{2i}b_{ii}c_{i1}\\ &= \sum_{i=3}^{n} c_{2i}c_{i1}\sum_{j=3}^{n} b_{ij}c_{ji} + \sum_{i=3}^{n}(-s')c_{2i}b_{ii}c_{i1}.\end{aligned}$$

Observing that $\sum_{j=3}^{n} b_{ij}c_{ji} = 1$, the (i,i) entry of BB^{-1}, we get

$$\begin{aligned}\sum_{i=3}^{n}\sum_{j=3}^{n} c_{2i}b_{ij}c_{j1} &\leq \sum_{i=3}^{n} c_{2i}c_{i1}(1+(-s')b_{ii})\\ &\leq \sum_{i=3}^{n} c_{2i}c_{i1}\left(1+(-s')\tfrac{1}{1+s'}\right)\\ &= \tfrac{1}{1+s'}\textstyle\sum_{i=3}^{n} c_{2i}c_{i1}\\ &= \tfrac{1}{1+s'}\sum_{i=3}^{n} a_{2i}a_{i1}\\ &\leq \tfrac{1}{1+s'}(U(A)-\varepsilon)a_{21}\\ &= a_{21} = c_{21}. \quad \square\end{aligned}$$

EXAMPLE 2.8. [6] Consider the 4×4 normalized SPP-matrix

$$A = \begin{bmatrix} 1 & 0.1 & 0.4 & 0.3\\ 0.4 & 1 & 0.4 & 0.65\\ 0.1 & 0.2 & 1 & 0.6\\ 0.15 & 0.3 & 0.6 & 1 \end{bmatrix}.$$

As seen in [12], A is not an IM-matrix (the $(2,3)$-entry of A^{-1} is positive). By actual calculation, $U(A) = \frac{1}{a_{31}}(a_{32}a_{21} + a_{34}a_{41}) = 1.7 > 1$. Hence, $A + sI$ is IM for all $s \geq 0.7$ according to Theorem 3 of [6].

However, $\varepsilon = \max\{0, (U(A)-u(A))\}$=0.325. So according to Theorem 2.7 $A+s'I$ is an IM-matrix for all $s' \geq 0.375$. (In fact, $A + s'I$ is an IM-matrix if and only $s' \geq 0.18$.)

REMARK 2.9. If $U(A) = u(A)$, then Theorem 2.7 is the same as Theorem 3 of [6].

Similar to [6, Theorem 4], we have:

THEOREM 2.10. *Let $A = (a_{ij})$ be a normalized SPP matrix of order n, $n \geq 3$. Then $A + s'I$ is an IM-matrix for all $s' \geq |n-3-\varepsilon|$.*

Proof. The result follows from Lemma 2.2 (ii) of [6] and Theorem 2.7. $\square$

A consequence of Theorem 2.10 is as follows.

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COROLLARY 2.11. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with positive diagonal entries and let D and E be positive diagonal matrices such that $DE = |n-3-\varepsilon|[diag(A)]^{-1}$. Then, if $DAE - |n-3-\varepsilon|I$ is an SPP-matrix, A is an IM-matrix.*

Following [6], the *Hadamard dual* of the IM-matrices, denoted by IM^{D}, is defined to be the set of all matrices B such that $A \circ B$ is an IM-matrix for all IM-matrices A.

We may obtain the following results which are similar to those in [6].

LEMMA 2.12. *Let $A = (a_{ij})$ be a normalized IM-matrix of order n. Then $A + |n-3-\varepsilon|I \in \mathrm{IM}^{\mathrm{D}}$.*

THEOREM 2.13. *Let $A = (a_{ij})$ be an IM-matrix of order n and let D and E be positive diagonal matrices such that $A_1 = DAE$ is normalized. Then*

$$A + |n-3-\varepsilon|D^{-1}E^{-1} \in \mathrm{IM}^{\mathrm{D}}.$$

A real $n \times n$ matrix A is called a P-matrix if the principal minors of A are all positive. Obviously, IM-matrices are P-matrices. SPP-matrices are not necessarily P-matrices for $n \geq 6$, but for $n \leq 3$ they are [7]. Here we will answer the question whether a 4×4 SPP-matrix is a P-matrix or not. We need the following lemma [2, Lemma 2.3].

LEMMA 2.14. *Let $A = (a_{ij})$ be an IM-matrix of order n, whose columns are denoted by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then for any $x = (x_1, x_2, \ldots, x_n)^T$, the functions*

$$f(x) = \det(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, x) \text{ and } g(x) = \det(x, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n)$$

have the following properties:

1) *If $x = (x_1, x_2, \ldots, x_n)^T \leq y = (y_1, y_2, \ldots, y_n)^T$ and $x_n = y_n$, then it holds that $f(x) \geq f(y)$;*
2) *If $x = (x_1, x_2, \ldots, x_n)^T \leq y = (y_1, y_2, \ldots, y_n)^T$ and $x_1 = y_1$, then it holds that $g(x) \geq g(y)$.*

THEOREM 2.15. *Let $A = (a_{ij})$ be a 4×4 SPP matrix. Then A is a P-matrix.*

Proof. Recall that a P-matrix is a real $n \times n$ matrix whose principal minors are all positive. From Lemma 2.1 and Lemma 2.2, we know that all 2×2 and 3×3 principal minors of A are positive. It suffices to prove that $\det A > 0$.

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Set $\alpha = \{2,3\} = N \setminus \{1,4\}$, and let A be partitioned as

$$A = \begin{bmatrix} a_{11} & A[1,\alpha] & a_{14} \\ A[\alpha,1] & A[\alpha] & A[\alpha,4] \\ a_{41} & A[4,\alpha] & a_{44} \end{bmatrix}.$$

We have

$$b_{14} = (-1)^{4+1} \det \begin{bmatrix} A[1,\alpha] & a_{14} \\ A[\alpha] & A[\alpha,4] \end{bmatrix} = -\det \begin{bmatrix} a_{14} & A[1,\alpha] \\ A[\alpha,4] & A[\alpha] \end{bmatrix},$$

$$b_{41} = (-1)^{4+1} \det \begin{bmatrix} A[\alpha,1] & A[\alpha] \\ a_{41} & A[4,\alpha] \end{bmatrix} = -\det \begin{bmatrix} A[\alpha] & A[\alpha,1] \\ A[4,\alpha] & a_{41} \end{bmatrix}.$$

If $b_{14}b_{41} \le 0$, then from (1.5) of [8] and $\det A[\alpha] > 0$, we have $\det A > 0$. If $b_{14}b_{41} \ge 0$, since $a_{i1}a_{14} \le a_{11}a_{i4}$, $a_{i4}a_{41} \le a_{44}a_{i1}$ $(\forall\, i \in \alpha)$, we obtain

$$a_{14}A[\alpha,1] \le a_{11}A[\alpha,4], \; a_{41}A[\alpha,4] \le a_{44}A[\alpha,1].$$

From Lemma 2.2, we observe that each principal submatrix A of order 3 is an inverse M-matrix. According to Lemma 2.14, we deduce that

$$\begin{aligned} a_{14} \det \begin{bmatrix} a_{11} & A[1,\alpha] \\ A[\alpha,1] & A[\alpha] \end{bmatrix} &= \det \begin{bmatrix} a_{11}a_{14} & A[1,\alpha] \\ a_{14}A[\alpha,1] & A[\alpha] \end{bmatrix} \\ &\ge \det \begin{bmatrix} a_{11}a_{14} & A[1,\alpha] \\ a_{11}A[\alpha,4] & A[\alpha] \end{bmatrix} \\ &= a_{11} \det \begin{bmatrix} a_{14} & A[1,\alpha] \\ A[\alpha,4] & A[\alpha] \end{bmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} a_{41} \det \begin{bmatrix} A[\alpha] & A[\alpha,4] \\ A[4,\alpha] & a_{44} \end{bmatrix} &= \det \begin{bmatrix} A[\alpha] & a_{41}A[\alpha,4] \\ A[4,\alpha] & a_{41}a_{44} \end{bmatrix} \\ &\ge \det \begin{bmatrix} A[\alpha] & a_{44}A[\alpha,1] \\ A[4,\alpha] & a_{41}a_{44} \end{bmatrix} \\ &= a_{44} \det \begin{bmatrix} A[\alpha] & A[\alpha,1] \\ A[4,\alpha] & a_{41} \end{bmatrix}. \end{aligned}$$

By the above inequalities, we have

$$\begin{aligned} &\det \begin{bmatrix} A[1,\alpha] & a_{14} \\ A[\alpha] & A[\alpha,4] \end{bmatrix} \det \begin{bmatrix} A[\alpha,1] & A[\alpha] \\ a_{41} & A[4,\alpha] \end{bmatrix} \\ &= (-1)^{n-2} \det \begin{bmatrix} a_{14} & A[1,\alpha] \\ A[\alpha,4] & A[\alpha] \end{bmatrix} (-1)^{n-2} \det \begin{bmatrix} A[\alpha] & A[\alpha,1] \\ A[4,\alpha] & a_{41} \end{bmatrix} \\ &= \tfrac{1}{a_{11}a_{44}} a_{11} \det \begin{bmatrix} a_{14} & A[1,\alpha] \\ A[\alpha,4] & A[\alpha] \end{bmatrix} a_{44} \det \begin{bmatrix} A[\alpha] & A[\alpha,1] \\ A[4,\alpha] & a_{41} \end{bmatrix} \\ &\le \tfrac{a_{14}a_{41}}{a_{11}a_{44}} \det \begin{bmatrix} a_{11} & A[1,\alpha] \\ A[\alpha,1] & A[\alpha] \end{bmatrix} \det \begin{bmatrix} A[\alpha] & A[\alpha,4] \\ A[4,\alpha] & a_{44} \end{bmatrix}. \end{aligned}$$

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Applying (1.5) of [8], it follows that

$$\begin{aligned}\det A \det A[\alpha] &= \det \begin{bmatrix} a_{11} & A[1,\alpha] \\ A[\alpha,1] & A[\alpha] \end{bmatrix} \det \begin{bmatrix} A[\alpha] & A[\alpha,4] \\ A[4,\alpha] & a_{44} \end{bmatrix} \\ &\quad - \det \begin{bmatrix} A[1,\alpha] & a_{14} \\ A[\alpha] & A[\alpha,4] \end{bmatrix} \det \begin{bmatrix} A[\alpha,1] & A[\alpha] \\ a_{41} & A[4,\alpha] \end{bmatrix} \\ &\geq \left(1 - \frac{a_{14}a_{41}}{a_{11}a_{44}}\right) \det \begin{bmatrix} a_{11} & A[1,\alpha] \\ A[\alpha,1] & A[\alpha] \end{bmatrix} \det \begin{bmatrix} A[\alpha] & A[\alpha,4] \\ A[4,\alpha] & a_{44} \end{bmatrix} \\ &> 0.\end{aligned}$$

Consequently, $\det A > 0$, all 2×2 and 3×3 principal minors of A are positive, so A is P-matrix. □

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