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INVARIANCE PROPERTIES OF AN OPERATOR PRODUCT INVOLVING GENERALIZED INVERSES*

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Abstract. Given bounded linear operators T_1, T_2 and T_3, this paper investigates certain invariance properties of the operator product T_1XT_3 with respect to the choice of bounded linear operator X, where X is a generalized inverse of T_2. Different types of generalized inverses are taken into account.

Key words. Bounded linear operators, Generalized inverse, Moore-Penrose inverse, Operator product, Invariance property.

AMS subject classifications. 15A09, 15A24, 47A05.

1. Introduction. Throughout this paper, $\mathbb{H}$, $\mathbb{K}$, and $\mathbb{L}$ denote arbitrary Hilbert spaces. We use $L(\mathbb{H},\ \mathbb{K})$ to denote the set of all bounded linear operators from $\mathbb{H}$ to $\mathbb{K}$. Also, $L(\mathbb{H})=L(\mathbb{H},\ \mathbb{H})$. I denotes the identity operator on Hilbert spaces and O is the zero operator on Hilbert spaces. For $T \in L(\mathbb{H},\ \mathbb{K})$, the symbols T^*, $R(T)$, and $N(T)$ will stand for the adjoint operator, the range, and the kernel of T, respectively.

Let $T \in L(\mathbb{H},\ \mathbb{K})$. If there exists an operator $X \in L(\mathbb{K},\ \mathbb{H})$ satisfying the following four operator equations:

$$(1)\ TXT = T, \quad (2)\ XTX = X, \quad (3)\ (TX)^* = TX, \quad (4)\ (XT)^* = XT,$$

then X is called a Moore-Penrose inverse of T and denoted by $T^\dagger$. As we know, T has a Moore-Penrose inverse if and only if $R(T)$ is closed and the Moore-Penrose inverse of T is unique (see, for example, [5, 7, 14, 17, 18, 19, 20, 21, 22, 23]). For a subset $\eta \subseteq \{1, 2, 3, 4\}$, the set of operators satisfying the equations contained in η is denoted by $T\eta$. An operator from $T\eta$ is called an η-inverse of T. For example, an operator X of the set $T\{1\}$ is called a $\{1\}$-inverse of T and denoted by $T^{(1)}$ or T^-. One usually denotes any $\{1, 3\}$-inverse of T as $T^{(1,3)}$ and any $\{1, 4\}$-inverse of T is denoted by $T^{(1,4)}$. The unique $\{1, 2, 3, 4\}$-inverse of T is the Moore-Penrose inverse of T. We

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refer the reader to [5, 6, 9, 10, 11, 15, 16, 23, 24] for basic results on the generalized inverses of operators.

Invariance properties of operator product involving generalized inverses are fundamental in the theory of operators. They have attracted considerable attention and many interesting results have been obtained (see, for example, [1, 2, 3, 4, 12, 13]). In this paper, given bounded linear operators $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$ and $T_3 \in L(\mathbb{H}, \mathbb{K})$, we investigate properties of the operator product $T_1T_2^-T_3$ for various types of generalized inverses T_2^- of T_2, where our interest is focused on invariance properties with respect to the choice of T_2^- concerning the value and range of $T_1T_2^-T_3$.

We first mention the following three results, which will be used in this paper.

Lemma 1.1. [9]. *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range. Then*

$$T\{1\} = \{T^\dagger + Y - T^\dagger TYTT^\dagger : \quad Y \in L(\mathbb{K}, \mathbb{H})\}.$$

Lemma 1.2. [8, 9]. *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range and $X \in L(\mathbb{K}, \mathbb{H})$. Then the following statements are equivalent:*

(1) $TXT = T$ and $(TX)^* = TX$;

(2) *there exists some $Y \in L(\mathbb{K}, \mathbb{H})$ such that $X = T^\dagger + (I - T^\dagger T)Y$.*

Lemma 1.3. [8, 9]. *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range and $X \in L(\mathbb{K}, \mathbb{H})$. Then the following statements are equivalent:*

(1) $TXT = T$ and $(XT)^* = XT$;

(2) *there exists some $Y \in L(\mathbb{K}, \mathbb{H})$ such that $X = T^\dagger + Y(I - TT^\dagger)$.*

2. Invariance properties of operator product $T_1T_2^{(1)}T_3$. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$ and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2 and T_3 have closed ranges. In this section, we will study several invariance properties of the operator product $T_1T_2^{(1)}T_3$ with respect to the choice of $T_2^{(1)} \in T_2\{1\}$. The main result is the following theorem.

Theorem 2.1. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$ and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2, and T_3 are not zero operators on Hilbert spaces. Then the following statements are equivalent:*

(1) *The operator product $T_1T_2^{(1)}T_3$ does not depend on the choice of $T_2^{(1)} \in T_2\{1\}$;*

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(2) $R(T_1^*) \subseteq R(T_2^*)$ *and* $R(T_3) \subseteq R(T_2)$.

Proof. According to the technique of block operator matrices in [7], we know that the operator T_2 has the following matrix form with respect to the orthogonal sum of subspaces:

$$T_2 = \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.1}$$

where T_2^{11} is invertible in $L(R(T_2^*),\ R(T_2))$, and

$$T_2^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}. \tag{2.2}$$

Also we have that the operator T_1 has the following form:

$$T_1 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \tag{2.3}$$

and

$$T_1^* = \begin{pmatrix} (T_1^{11})^* & O \\ (T_1^{12})^* & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}$$

and

$$T_1T_1^* = \begin{pmatrix} D & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix},$$

where $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ is positive and invertible in $L(R(T_1))$. In particular

$$T_1^\dagger = T_1^*(T_1T_1^*)^\dagger = \begin{pmatrix} (T_1^{11})^*D^{-1} & O \\ (T_1^{12})^*D^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}. \tag{2.4}$$

Furthermore, we obtain that the operator T_3 has the following matrix form with respect to the orthogonal sum of subspaces:

$$T_3 = \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \tag{2.5}$$

and

$$T_3^* = \begin{pmatrix} (T_3^{11})^* & (T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}$$

and

$$T_3^*T_3 = \begin{pmatrix} S & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix},$$

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where $S = (T_3^{11})^* T_3^{11} + (T_3^{21})^* T_3^{21}$ is positive and invertible in $L(R(T_3^*))$. Then

$$(2.6)\quad T_3^\dagger = (T_3^* T_3)^\dagger T_3^* = \begin{pmatrix} S^{-1}(T_3^{11})^* & S^{-1}(T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}.$$

Next, we will prove the facts that (1) and (2) in Theorem 2.1 are equivalent.

(2)⇒(1): The inclusion $R(T_1^*) \subseteq R(T_2^*)$ is equivalent to $T_1 T_2^\dagger T_2 = T_1$. Then from (2.1)-(2.6), we have

$$(2.7)\qquad T_1 T_2^\dagger T_2 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix}.$$

Hence, according to (2.3) and (2.7), the equality $T_1 T_2^\dagger T_2 = T_1$ is equivalent to $T_1^{12} = O$, that is

$$(2.8)\qquad R(T_1^*) \subseteq R(T_2^*) \Leftrightarrow T_1^{12} = O.$$

In the same manner, we can prove that the inclusion $R(T_3) \subseteq R(T_2)$ is equivalent to $T_2 T_2^\dagger T_3 = T_3$ and

$$(2.9)\qquad R(T_3) \subseteq R(T_2) \Leftrightarrow T_3^{21} = O.$$

On the other hand, from Lemma 1.1 it follows that arbitrary $T_2^{(1)} \in T_2\{1\}$ has the form

$$(2.10)\qquad T_2^{(1)} = \begin{pmatrix} (T_2^{11})^{-1} & U \\ V & W \end{pmatrix},$$

where U, V and W are bounded linear operators on appropriate spaces. Hence, from (2.2), (2.3), (2.5), (2.8), (2.9), (2.10), we have that

$$(2.11)\qquad T_1 T_2^{(1)} T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1} T_3^{11} & O \\ O & O \end{pmatrix} = T_1 T_2^\dagger T_3.$$

Combining (2.10) with (2.11), we have the result (2)⇒(1).

(1)⇒(2): Since the Moore-Penrose inverse of a bounded linear operator is unique and belongs to the set of {1}-inverse, it is clear that $T_1 T_2^{(1)} T_3$ doesn't depend on the choice of $T_2^{(1)} \in T_2\{1\}$ if and only if the equality $T_1 T_2^{(1)} T_3 = T_1 T_2^\dagger T_3$ holds for every $T_2^{(1)} \in T_2\{1\}$. By Lemma 1.1, it follows that both

$$(2.12)\qquad M = \begin{pmatrix} (T_2^{11})^{-1} & (T_1^{11})^* \\ O & (T_1^{12})^* \end{pmatrix} \text{ and } N = \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_3^{11})^* & (T_3^{21})^* \end{pmatrix}$$

are {1}-inverses of T_2. Since

$$(2.13)\qquad T_1 M T_3 = T_1 N T_3 = T_1 T_2^\dagger T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1} T_3^{11} & O \\ O & O \end{pmatrix},$$

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from (2.2), (2.3), (2.5), (2.12), (2.13), we get

$$T_1^{11}(T_1^{11})^*T_3^{21} + T_1^{12}(T_1^{12})^*T_3^{21} = O \tag{2.14}$$

and

$$T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*(T_3^{21}) = O. \tag{2.15}$$

Combining (2.14), (2.15) with the equalities (2.3) and (2.5), we have

$$DT_3^{21} = O \;\text{ and }\; T_1^{12}S = O. \tag{2.16}$$

Since D and S are invertible, from (2.16) we obtain

$$T_3^{21} = O \;\text{ and }\; T_1^{12} = O,$$

which are respectively equivalent to $R(T_3) \subseteq R(T_2)$ and $R(T_1^*) \subseteq R(T_2^*)$. □

Corollary 2.2. *Let $T_1 \in L(\mathbb{L},\, \mathbb{H})$, $T_2 \in L(\mathbb{L},\, \mathbb{K})$, and $T_3 \in L(\mathbb{H},\, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2, and T_3 are not zero operators on Hilbert spaces. Then the identity $T_1T_2^{(1)}T_3 = O$ holds for every $T_2^{(1)} \in T_2\{1\}$ if and only if $R(T_1^*) \subseteq R(T_2^*)$, $R(T_3) \subseteq R(T_2)$, and $R(T_3) \subseteq N(T_1T_2^\dagger)$.*

Corollary 2.3. *Let $T_1 \in L(\mathbb{L},\, \mathbb{H})$, $T_2 \in L(\mathbb{L},\, \mathbb{K})$, and $T_3 \in L(\mathbb{H},\, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. If $R(T_1^*) \subseteq R(T_2^*)$ and $R(T_3) \subseteq R(T_2)$, then $R(T_1T_2^{(1)}T_3)$ is the same for every $T_2^{(1)} \in T_2\{1\}$.*

3. Invariance properties of products $T_1T_2^{(1,3)}T_3$ and $T_1T_2^{(1,4)}T_3$. Let $T_1 \in L(\mathbb{L},\, \mathbb{H})$, $T_2 \in L(\mathbb{L},\, \mathbb{K})$, and $T_3 \in L(\mathbb{H},\, \mathbb{K})$. In this section, we will investigate the invariance properties of the operator products $T_1T_2^{(1,3)}T_3$ and $T_1T_2^{(1,4)}T_3$ with respect to any $T_2^{(1,3)} \in T_2\{1,3\}$ and $T_2^{(1,4)} \in T_2\{1,4\}$.

Theorem 3.1. *Let $T_1 \in L(\mathbb{L},\, \mathbb{H})$, $T_2 \in L(\mathbb{L},\, \mathbb{K})$, and $T_3 \in L(\mathbb{H},\, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2, and T_3 are not zero operators on Hilbert spaces. Then the following statements are equivalent:*

(1) *The equality $T_1T_2^{(1,3)}T_3 = T_1T_2^\dagger T_3$ holds for every $T_2^{(1,3)} \in T_2\{1,3\}$;*

(2) *$R(T_1^*) \subseteq R(T_2^*)$.*

Proof. By Lemma 1.2, we have that arbitrary $T_2^{(1,3)} \in T_2\{1,3\}$ has the form

$$T_2^{(1,3)} = \begin{pmatrix} (T_2^{11})^{-1} & O \\ W_{21} & W_{22} \end{pmatrix}, \tag{3.1}$$

where W_{21} and W_{22} are bounded linear operators on appropriate subspaces. Then from the equalities (2.2), (2.3), (2.5) and (3.1), we know that for any $T_2^{(1,3)} \in T_2\{1,3\}$

$$T_1T_2^{(1,3)}T_3 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix}\begin{pmatrix} (T_2^{11})^{-1} & O \\ W_{21} & W_{22} \end{pmatrix}\begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix}$$

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$$= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21} & O \\ O & O \end{pmatrix}$$

and

$$T_1T_2^{\dagger}T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix}. \tag{3.2}$$

We now prove that (1) and (2) in Theorem 3.1 are equivalent.

(2)⇒(1): Since

$$R(T_1^*) \subseteq R(T_2^*) \Leftrightarrow T_1^{12} = O,$$

we have that for arbitrary $T_2^{(1,3)} \in T_2\{1,3\}$,

$$\begin{aligned} T_1T_2^{(1,3)}T_3 &= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix} \\ &= T_1T_2^{\dagger}T_3. \end{aligned}$$

(1)⇒(2): From Lemma 1.2, it follows that

$$M_1 = \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_3^{11})^* & (T_3^{21})^* \end{pmatrix}$$

is a $\{1,3\}$-inverses of T_2. Then

$$\begin{aligned} T_1M_1T_3 &= \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_3^{11})^* & (T_3^{21})^* \end{pmatrix} \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} \\ &= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*T_3^{21} & O \\ O & O \end{pmatrix} \end{aligned} \tag{3.3}$$

and

$$T_1M_1T_3 = T_1T_2^{\dagger}T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix}. \tag{3.4}$$

Hence, from (3.3) and (3.4), we have

$$T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*T_3^{21} = O. \tag{3.5}$$

Combining (3.5) with the equality (2.5), we have $T_1^{12}S = O$. Since S is invertible, $T_1^{12} = O$, that is $R(T_1^*) \subseteq R(T_2^*)$. □

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Corollary 3.2. *Let $T_1 \in L(\mathbb{L},\ \mathbb{H})$, $T_2 \in L(\mathbb{L},\ \mathbb{K})$, and $T_3 \in L(\mathbb{H},\ \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2, and T_3 are not zero operators on Hilbert spaces. Then the identity $T_1T_2^{(1,3)}T_3 = O$ holds for every $T_2^{(1,3)} \in T_2\{1,3\}$ if and only if $R(T_1^*) \subseteq R(T_2^*)$ and $R(T_3) \subseteq N(T_1T_2^{\dagger})$.*

Next, we will investigate the invariance of the range of the operator product $T_1T_2^{(1,3)}T_3$ with respect to the choices of $T_2^{(1,3)} \in T_2\{1,3\}$.

Theorem 3.3. *Let $T_1 \in L(\mathbb{L},\ \mathbb{H})$, $T_2 \in L(\mathbb{L},\ \mathbb{K})$, and $T_3 \in L(\mathbb{H},\ \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2, and T_3 are not zero operators on Hilbert spaces and $T_1T_2^{\dagger}T_3 \neq O$. Then the following statements are equivalent:*

(1) *$R(T_1T_2^{(1,3)}T_3)$ is the same for every $T_2^{(1,3)} \in T_2\{1,3\}$;*

(2) *$R(T_1^*) \subseteq R(T_2^*)$.*

Proof. According to the above proof in Theorem 3.1, it follows that for any $T_2^{(1,3)} \in T_2\{1,3\}$

$$T_1T_2^{(1,3)}T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21} & O \\ O & O \end{pmatrix} \tag{3.6}$$

and

$$(T_1T_2^{(1,3)}T_3)^{\dagger} = \begin{pmatrix} (T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21})^{\dagger} & O \\ O & O \end{pmatrix}, \tag{3.7}$$

where W_{21} and W_{22} are bounded linear operators on appropriate subspaces. Furthermore, from (3.2) we have

$$(T_1T_2^{\dagger}T_3)^{\dagger} = \begin{pmatrix} (T_1^{11}(T_2^{11})^{-1}T_3^{11})^{\dagger} & O \\ O & O \end{pmatrix}. \tag{3.8}$$

(2)$\Rightarrow$ (1): Clearly the invariance of the product $T_1T_2^{(1,3)}T_3$ with respect to $T_2^{(1,3)} \in T_2\{1,3\}$ is sufficient for the invariance of $R(T_1T_2^{(1,3)}T_3)$ with respect to the choices of $T_2^{(1,3)} \in T_2\{1,3\}$. Then from Theorem 3.1, we have the result "(2)$\Rightarrow$ (1)" in Theorem 3.3.

(1)$\Rightarrow$ (2): From (1) in Theorem 3.3, we know that the equality $R(T_1T_2^{(1,3)}T_3) = R(T_1T_2^{\dagger}T_3)$ holds for any $T_2^{(1,3)} \in T_2\{1,3\}$. Under the definition of the range of operators, it follows that the equality $R(T_1T_2^{(1,3)}T_3) = R(T_1T_2^{\dagger}T_3)$ holds for any $T_2^{(1,3)}$ if and only if the following two inclusions:

$$R(T_1T_2^{(1,3)}T_3) \subseteq R(T_1T_2^{\dagger}T_3)$$

and

$$R(T_1T_2^{\dagger}T_3) \subseteq R(T_1T_2^{(1,3)}T_3)$$

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hold for any $T_2^{(1,3)} \in T_2\{1,3\}$, which are respectively equivalent to the following two identities:

$$T_1T_2^{\dagger}T_3(T_1T_2^{\dagger}T_3)^{\dagger}T_1T_2^{(1,3)}T_3 = T_1T_2^{(1,3)}T_3 \tag{3.9}$$

and

$$T_1T_2^{(1,3)}T_3(T_1T_2^{(1,3)}T_3)^{\dagger}T_1T_2^{\dagger}T_3 = T_1T_2^{\dagger}T_3 \tag{3.10}$$

valid for any $T_2^{(1,3)} \in T_2\{1,3\}$. Hence, from the equalities (3.2), (3.6), (3.7), (3.8) (3.9), (3.10), we have

$$(T_1^{11}(T_2^{11})^{-1}T_3^{11})(T_1^{11}(T_2^{11})^{-1}T_3^{11})^{\dagger}\mu = \mu \tag{3.11}$$

and

$$\mu\mu^{\dagger}T_1^{11}(T_2^{11})^{-1}T_3^{11} = T_1^{11}(T_2^{11})^{-1}T_3^{11}, \tag{3.12}$$

where W_{21} and W_{22} are bounded linear operators on appropriate subspaces and

$$\mu = T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21}. \tag{3.13}$$

Combining (3.11), (3.12), with (3.13), we get the following equality:

$$(T_1^{11}(T_2^{11})^{-1}T_3^{11})(T_1^{11}(T_2^{11})^{-1}T_3^{11})^{\dagger} = \mu\mu^{\dagger} \tag{3.14}$$

is valid for arbitrary bounded linear operators W_{21} and W_{22}.

On the other hand, since $T_1T_2^{\dagger}T_3 \neq O$, it follows that

$$T_1T_2^{\dagger}T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix} \neq O,$$

that is $T_1^{11}(T_2^{11})^{-1}T_3^{11} \neq O$, which also implies that $T_1^{11} \neq O$, $T_2^{11} \neq O$ and $T_3^{11} \neq O$. Let $W_{22} = O$, then from (3.14), we obtain that the following equality:

$$\begin{aligned} &(T_1^{11}(T_2^{11})^{-1}T_3^{11})(T_1^{11}(T_2^{11})^{-1}T_3^{11})^{\dagger} \\ &= (T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11})(T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11})^{\dagger} \end{aligned}$$

holds for any W_{21}. This implies $T_1^{12} = O$. According to the fact: $R(T_1^*) \subseteq R(T_2^*) \Leftrightarrow T_1^{12} = O$, it follows that (1)⇒ (2). □

By Lemma 1.2 and Lemma 1.3, we know that for a bounded linear operator T_2, $X \in T_2\{1,4\}$ if and only $X^* \in T_2^*\{1,3\}$. So results for the operator product $T_1T_2^{(1,4)}T_3$ involving $\{1,4\}$-inverses of T_2 follow from the previous theorems in this section.

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THEOREM 3.4. *Let* $T_1 \in L(\mathbb{L},\ \mathbb{H})$, $T_2 \in L(\mathbb{L},\ \mathbb{K})$, *and* $T_3 \in L(\mathbb{H},\ \mathbb{K})$ *be such that* T_1, T_2, T_3 *have closed ranges. Suppose that* T_1, T_2, *and* T_3 *are not zero operators on Hilbert spaces. Then the following statements are equivalent:*

(1) *The equality* $T_1T_2^{(1,4)}T_3 = T_1T_2^{\dagger}T_3$ *holds for every* $T_2^{(1,4)} \in T_2\{1,4\}$;

(2) $R(T_3) \subseteq R(T_2)$.

COROLLARY 3.5. *Let* $T_1 \in L(\mathbb{L},\ \mathbb{H})$, $T_2 \in L(\mathbb{L},\ \mathbb{K})$, *and* $T_3 \in L(\mathbb{H},\ \mathbb{K})$ *be such that* T_1, T_2, T_3 *have closed ranges. Suppose that* T_1, T_2, *and* T_3 *are not zero operators on Hilbert spaces. Then the identity* $T_1T_2^{(1,4)}T_3 = O$ *holds for every* $T_2^{(1,4)} \in T_2\{1,4\}$ *if and only if* $R(T_3) \subseteq R(T_2)$ *and* $R(T_3) \subseteq N(T_1T_2^{\dagger})$.

THEOREM 3.6. *Let* $T_1 \in L(\mathbb{L},\ \mathbb{H})$, $T_2 \in L(\mathbb{L},\ \mathbb{K})$, *and* $T_3 \in L(\mathbb{H},\ \mathbb{K})$ *be such that* T_1, T_2, T_3 *have closed ranges. Suppose that* T_1, T_2, *and* T_3 *are not zero operators on Hilbert spaces and* $T_1T_2^{\dagger}T_3 \neq O$. *Then the following statements are equivalent.*

(1) $R(T_1T_2^{(1,4)}T_3)$ *is the same for every* $T_2^{(1,4)} \in B\{1,4\}$;

(2) $R(T_3) \subseteq R(T_2)$.

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