Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 720-728, July 2011

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NEW REPRESENTATIONS FOR THE MOORE-PENROSE INVERSE*

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Abstract. In this paper, some new representations of the Moore-Penrose inverse of a complex $m \times n$ matrix of rank r in terms of $(s \times t)$-constrained submatrices with $m \geq s \geq r$, $n \geq t \geq r$ are presented.

Key words. Constrained submatrix, Moore-Penrose inverse.

AMS subject classifications. 15A15, 15A09.

1. Introduction. We adopt the following notation in this paper. The set of $m \times n$ matrices with complex (respectively, real) entries is denoted by $\mathbb{C}^{m\times n}$ (respectively, $\mathbb{R}^{m\times n}$). The subset of $\mathbb{C}^{m\times n}$ (respectively, $\mathbb{R}^{m\times n}$) consisting of the rank r matrices is denoted by $\mathbb{C}_r^{m\times n}$ (respectively, $\mathbb{R}_r^{m\times n}$). We assume throughout that $r > 0$. For positive integers $r \leq l \leq m$,

$$\Delta_{l,m} = \{\{i_1, \dots, i_l\} | 1 \leq i_1 < \cdots < i_l \leq m\},$$
$$\Delta_{r,\alpha} = \{\{i'_1, \dots, i'_r\} \subseteq \{i_1, \dots, i_l\} = \alpha \in \Delta_{l,m} | i_1 \leq i'_1 < \cdots < i'_r \leq i_l\}.$$

For $\alpha \in \Delta_{s,m}$, and $\beta \in \Delta_{t,n}$, $A_{\alpha,\beta}$ is the $s \times t$ submatrix of A consisting of the rows of A indexed by α and the columns of A indexed by β, and $A_{\alpha,*}$ (respectively, $A_{*,\beta}$) denotes $A_{\alpha,\{1,2,\dots,n\}}$ (respectively, $A_{\{1,2,\dots,m\},\beta}$). For $\alpha = \{i_1, \dots, i_s\} \in \Delta_{s,m}$, P_α is the $s \times m$ matrix with 1 in positions $(1, i_1), \dots, (s, i_s)$ and 0 elsewhere; for $\beta = \{j_1, \dots, j_t\} \in \Delta_{t,n}$, Q_β is the $n \times t$ matrix with 1 at positions $(j_1, 1), \dots, (j_t, t)$ and 0 elsewhere. It is obvious that $P_\alpha A = A_{\alpha,*}$ and $AQ_\beta = A_{*,\beta}$. The *conjugate transpose* of A is denoted by A^H, and it is clear that $A^H_{\alpha,\beta} = \left(A^H\right)_{\alpha,\beta}$. The *r-th compound matrix* of A is denoted by $C_r(A)$. Thus, by [6], if $\mathrm{rank}(A) = r$, then

$$\mathrm{Tr}\, C_r\left(AA^H\right) = \mathrm{Tr}\, C_r\left(A^H A\right) = \sum_{\alpha\in\Delta_{r,m},\beta\in\Delta_{r,n}} \det\left(A_{\alpha,\beta}\right) \det\left(A^H_{\beta,\alpha}\right).$$

*Received by the editors on August 23, 2010. Accepted for publication on July 2, 2011. Handling Editor: Ravindra B. Bapat.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 720-728, July 2011

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http://math.technion.ac.il/iic/ela

The *Moore-Penrose inverse* of $A \in \mathbb{C}^{m\times n}$ is defined as the unique $X \in \mathbb{C}^{n\times m}$ satisfying

$$(1.1)\quad (1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^H = AX,\ \ (4)\ (XA)^H = XA,$$

and is usually denoted by $X = A^\dagger$ (see [6]). If $A \in \mathbb{C}^{m\times n}$ and $\text{rank}(A) = r > 0$, then A has a full-rank factorization $A = FG$, and it is a standard fact that

$$A^\dagger = G^\dagger F^\dagger. \tag{1.2}$$

Also $A^\dagger_{\alpha,\beta} = (A_{\alpha,\beta})^\dagger$.

For $A \in \mathbb{C}^{m\times m}$, the smallest nonnegative integer k such that $\text{rank}\left(A^{k+1}\right) = \text{rank}\left(A^k\right)$, is called the *index* of A, and is denoted by $\text{Ind}\,(A) = k$. If $\text{Ind}\,(A) = 1$, then there exists a unique matrix $X \in \mathbb{C}^{m\times m}$ satisfying

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (5)\ AX = XA. \tag{1.3}$$

Here $X = A^\sharp$ is called the *group inverse* of A. Furthermore, for a full-rank factorization $A = FG$ of A, $A^\sharp = F\,(GF)^{-2}\,G$ (see [6]).

For $A \in \mathbb{C}^{m\times n}_r$, in [7], Berg deduced the following representation of the Moore-Penrose inverse of A:

$$A^\dagger = \frac{1}{\text{Tr}\ C_r\left(A^H A\right)} \sum_{\alpha\in\Delta_{r,m},\beta\in\Delta_{r,n}} \text{Tr}\ C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right) Q_\beta A^\dagger_{\alpha,\beta} P_\alpha. \tag{1.4}$$

In [9], Bruening obtained the following representations of the Moore-Penrose inverse of A:

$$A^\dagger = \frac{1}{\text{Tr}\ C_r\left(A^H A\right)} \sum_{\alpha\in\Delta_{r,m}} \text{Tr}\ C_r\left(A^H_{*,\alpha}A_{\alpha,*}\right) A^\dagger_{\alpha,*} P_\alpha \tag{1.5}$$

and

$$A^\dagger = \frac{1}{\text{Tr}\ C_r\left(A^H A\right)} \sum_{\beta\in\Delta_{r,n}} \text{Tr}\ C_r\left(A^H_{\beta,*}A_{*,\beta}\right) Q_\beta A^\dagger_{*,\beta}. \tag{1.6}$$

In [10], for $A \in \mathbb{C}^{m\times m}_r$ with $\text{Ind}\,(A) = 1$, Cai and Chen deduced the following representation of the group inverse of A:

$$A^\sharp = \frac{\sum\limits_{\alpha\in\Delta_{r,m}\beta\in\Delta_{r,m}} \text{Tr}\ C_r\left(A_{\beta,\alpha}A_{\alpha,\beta}\right) Q_\beta A^\dagger_{\alpha,\beta} P_\alpha}{\text{Tr}\ C_r(A^2)}. \tag{1.7}$$

A detailed discussion of generalized inverses and their representations in terms of submatrices can be found in [2, 6, 10, 12, 14, 17, 18, 19, 20, 22, 23, 25, 26].

In this paper, we will deduce some new representations of the Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}_r$ in terms of $(s\times t)$-constrained submatrices, where $m \ge s \ge r$, $n \ge t \ge r$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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2. Preliminaries. In this section, we present some preliminary results.

LEMMA 2.1. [20] *Let* $A \in \mathbb{C}_r^{m\times n}$, $\alpha \in \Delta_{r,m}, \beta \in \Delta_{r,n}$ *with* $\operatorname{rank}(A_{\alpha,\beta}) = r$, *and let* P_α *and* Q_β *be* $r \times m$ *and* $n \times r$ *matrices, respectively. Then*

$$AQ_\beta A_{\alpha,\beta}^{-1} P_\alpha A = A. \tag{2.1}$$

Notice that $A = AQ_\beta \left(A_{\alpha,\beta}^{-1} P_\alpha A\right)$ is a full-rank decomposition of A and

$$A^\dagger = (P_\alpha A)^\dagger A_{\alpha,\beta} (AQ_\beta)^\dagger .$$

We now extend the above formula to more general forms.

LEMMA 2.2. *Let* $A \in \mathbb{C}_r^{m\times n}$, $\alpha \in \Delta_{s,m}, \beta \in \Delta_{t,n}, m \geq s \geq r, n \geq t \geq r$ *with* $\operatorname{rank}(P_\alpha A Q_\beta) = r$. *Then*

$$AQ_\beta (P_\alpha A Q_\beta)^\dagger P_\alpha A = A. \tag{2.2}$$

Proof. Let $A = FG$ be a full-rank decomposition of A. Because

$$r = \operatorname{rank}(FG) = \operatorname{rank}(P_\alpha FGQ_\beta) \leq \operatorname{rank}(GQ_\beta) \leq r,$$

we have that $\operatorname{rank}(GQ_\beta) = r$. Similarly, $\operatorname{rank}(P_\alpha F) = r$. Therefore $P_\alpha A Q_\beta = (P_\alpha F)(GQ_\beta)$ is a full-rank decomposition of $P_\alpha A Q_\beta$. By applying (1.2), we obtain

$$\begin{aligned} AQ_\beta (P_\alpha A Q_\beta)^\dagger P_\alpha A &= FGQ_\beta (P_\alpha FGQ_\beta)^\dagger P_\alpha FG \\ &= FGQ_\beta (GQ_\beta)^\dagger (P_\alpha F)^\dagger P_\alpha FG \\ &= A. \quad \square \end{aligned}$$

LEMMA 2.3. *Let* $A \in \mathbb{C}_r^{m\times n}$, $\alpha \in \Delta_{s,m}$, $\beta \in \Delta_{t,n}$, $m \geq s \geq r$, $n \geq t \geq r$ *with* $\operatorname{rank}(P_\alpha A Q_\beta) = r$. *Then*

$$A^\dagger = (P_\alpha A)^\dagger A_{\alpha,\beta} (AQ_\beta)^\dagger . \tag{2.3}$$

Proof. Let $A = FG$ be a full-rank decomposition of A. By applying Lemma 2.2 and (1.2), we obtain

$$\begin{aligned} (P_\alpha A)^\dagger (P_\alpha A Q_\beta)(AQ_\beta)^\dagger &= (P_\alpha FG)^\dagger (P_\alpha FGQ_\beta)(FGQ_\beta)^\dagger \\ &= G^\dagger (P_\alpha F)^\dagger (P_\alpha F)(GQ_\beta)(GQ_\beta)^\dagger F^\dagger \\ &= G^\dagger F^\dagger \\ &= A^\dagger. \quad \square \end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 22, pp. 720-728, July 2011

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3. Representations of the Moore-Penrose inverse. In this section, we derive some new representations of the Moore-Penrose inverse of A.

THEOREM 3.1. *Let $A \in \mathbb{C}_r^{m\times n}$, $m \geq s \geq r$ and $n \geq t \geq r$. Then $A^\dagger$ equals*

$$\binom{m-r}{s-r}^{-1}\binom{n-r}{t-r}^{-1}\frac{\sum_{\alpha\in\Delta_{s,m},\beta\in\Delta_{t,n}} \operatorname{Tr} C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)Q_\beta A^\dagger_{\alpha,\beta}P_\alpha}{\operatorname{Tr} C_r\left(A^H A\right)}. \tag{3.1}$$

Proof. For given $\alpha \in \Delta_{s,m}$ and $\beta \in \Delta_{t,n}$, let r' be the rank of the submatrix $A_{\alpha,\beta}$. By applying (1.4), we have

$$A^\dagger_{\alpha,\beta} = \sum_{\alpha'\in\Delta_{r',\alpha}} \frac{\operatorname{Tr} C_{r'}\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)}{\operatorname{Tr} C_{r'}\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)} A^\dagger_{\alpha',\beta}P_{\alpha'}. \tag{3.2}$$

Notice that if $r' < r$, $\alpha' \in \Delta_{r',\alpha}$, $\alpha \in \Delta_{s,m}$ and $\beta \in \Delta_{r,n}$, then $P_{\alpha'}P_\alpha = P_{\alpha'}$ and

$$\operatorname{Tr} C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right) = 0. \tag{3.3}$$

Therefore, applying (3.2) and simplifying by (3.3) gives

$$\begin{aligned}
&\operatorname{Tr} C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)Q_\beta A^\dagger_{\alpha,\beta}P_\alpha \\
&= \operatorname{Tr} C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)Q_\beta\left(\sum_{\alpha'\in\Delta_{r',\alpha}} \frac{\operatorname{Tr} C_{r'}\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)}{\operatorname{Tr} C_{r'}\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)} A^\dagger_{\alpha',\beta}P_{\alpha'}\right)P_\alpha \\
&= \sum_{\alpha'\in\Delta_{r',\alpha}} \frac{\operatorname{Tr} C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)}{\operatorname{Tr} C_{r'}\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)} \operatorname{Tr} C_{r'}\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^\dagger_{\alpha',\beta}P_{\alpha'} \\
&= \sum_{\alpha'\in\Delta_{r,\alpha}} \operatorname{Tr} C_r\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^\dagger_{\alpha',\beta}P_{\alpha'}. \qquad (3.4)
\end{aligned}$$

For each fixed $\alpha' \in \Delta_{r,\alpha}$, there are $\binom{m-r}{s-r}$ choices of s elements $i_1,\dots,i_s$ of α from $\{1,\dots,m\}$ such that $\alpha' \subseteq \alpha$. Therefore,

$$\begin{aligned}
&\sum_{\alpha\in\Delta_{s,m}}\left(\sum_{\alpha'\in\Delta_{r,\alpha}} \operatorname{Tr} C_r\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^\dagger_{\alpha',\beta}P_{\alpha'}\right) \\
&\qquad = \binom{m-r}{s-r}\sum_{\alpha'\in\Delta_{r,\alpha}} \operatorname{Tr} C_r\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^\dagger_{\alpha',\beta}P_{\alpha'}.
\end{aligned} \tag{3.5}$$

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A publication of the International Linear Algebra Society
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Similarly, for each fixed $\alpha' \in \Delta_{r,\alpha}$, by adopting the above formulas to $A^{\dagger}_{\alpha',\beta}$ and using the fact that $\mathrm{Tr}\, C_r(A_{\alpha',\beta'}A^H_{\beta',\alpha'}) = \mathrm{Tr}\, C_r(A^H_{\beta',\alpha'}A_{\alpha',\beta'})$, we have

$$\mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^{\dagger}_{\alpha',\beta}P_{\alpha'} = \sum_{\beta' \in \Delta_{r,\beta}} \mathrm{Tr}\, C_r\left(A^H_{\beta',\alpha'}A_{\alpha',\beta'}\right)Q_{\beta'}A^{\dagger}_{\alpha',\beta'}P_{\alpha'}, \tag{3.6}$$

and

$$\begin{aligned}&\sum_{\beta\in\Delta_{t,n}}\left(\sum_{\beta'\in\Delta_{r,\beta}}\mathrm{Tr}\, C_r\left(A^H_{\beta',\alpha'}A_{\alpha',\beta'}\right)Q_{\beta'}A^{\dagger}_{\alpha',\beta'}P_{\alpha'}\right)\\ &\qquad = \begin{pmatrix} n-r \\ t-r\end{pmatrix}\sum_{\beta'\in\Delta_{r,\beta}}\mathrm{Tr}\, C_r\left(A^H_{\beta',\alpha'}A_{\alpha',\beta'}\right)Q_{\beta'}A^{\dagger}_{\alpha',\beta'}P_{\alpha'}.\end{aligned} \tag{3.7}$$

Therefore, by combining the formulas in (3.4)-(3.7) and applying (1.4), we obtain

$$\begin{aligned}
&\begin{pmatrix} m-r \\ s-r\end{pmatrix}^{-1}\begin{pmatrix} n-r \\ t-r\end{pmatrix}^{-1}\frac{\sum_{\alpha\in\Delta_{s,m},\beta\in\Delta_{t,n}}\left(\mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)Q_\beta A^{\dagger}_{\alpha,\beta}P_\alpha\right)}{\mathrm{Tr}\, C_r\left(A^HA\right)}\\
&=\begin{pmatrix} m-r \\ s-r\end{pmatrix}^{-1}\begin{pmatrix} n-r \\ t-r\end{pmatrix}^{-1}\frac{\sum_{\alpha\in\Delta_{s,m},\beta\in\Delta_{t,n}}\left(\sum_{\alpha'\in\Delta_{r,\alpha}}\mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^{\dagger}_{\alpha',\beta}P_{\alpha'}\right)}{\mathrm{Tr}\, C_r\left(A^HA\right)},\\
&\hspace{28em}\text{(by (3.4))}\\
&=\begin{pmatrix} n-r \\ t-r\end{pmatrix}^{-1}\frac{\sum_{\alpha'\in\Delta_{r,\alpha}}\left(\sum_{\beta\in\Delta_{t,n}}\mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha'}A_{\alpha',\beta}\right)Q_\beta A^{\dagger}_{\alpha',\beta}P_{\alpha'}\right)}{\mathrm{Tr}\, C_r\left(A^HA\right)}, \quad \text{(by (3.5))}\\
&=\begin{pmatrix} n-r \\ t-r\end{pmatrix}^{-1}\frac{\sum_{\alpha'\in\Delta_{r,\alpha}}\left(\sum_{\beta\in\Delta_{t,n}}\sum_{\beta'\in\Delta_{r,\beta}}\mathrm{Tr}\, C_r\left(A^H_{\beta',\alpha'}A_{\alpha',\beta'}\right)Q_{\beta'}A^{\dagger}_{\alpha',\beta'}P_{\alpha'}\right)}{\mathrm{Tr}\, C_r\left(A^HA\right)}, \quad \text{(by (3.6))}\\
&=\frac{1}{\mathrm{Tr}\, C_r\left(A^HA\right)}\left(\sum_{\alpha'\in\Delta_{r,m},\beta'\in\Delta_{r,n}}\mathrm{Tr}\, C_r\left(A^H_{\beta',\alpha'}A_{\alpha',\beta'}\right)Q_{\beta'}A^{\dagger}_{\alpha',\beta'}P_{\alpha'}\right), \quad \text{(by (3.7))}\\
&=A^{\dagger},
\end{aligned}$$

that is,

$$A^{\dagger} = \begin{pmatrix} m-r \\ s-r\end{pmatrix}^{-1}\begin{pmatrix} n-r \\ t-r\end{pmatrix}^{-1}\frac{\sum_{\alpha\in\Delta_{s,m},\beta\in\Delta_{t,n}}\mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)Q_\beta A^{\dagger}_{\alpha,\beta}P_\alpha}{\mathrm{Tr}\, C_r\left(A^HA\right)}. \qquad \square$$

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A publication of the International Linear Algebra Society
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EXAMPLE 3.2. Take

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix} \in \mathbb{R}_3^{5\times 7},$$

$s = 4$ and $t = 6$. Then $\mathrm{Tr}\, C_r\left(A^H A\right) = 1632$, $\begin{pmatrix} m-r \\ s-r \end{pmatrix}\begin{pmatrix} n-r \\ t-r \end{pmatrix} = 8$ and after some calculations, we have

$$\sum_{\alpha\in\Delta_{s,m},\beta\in\Delta_{t,n}} \mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right) Q_\beta A^\dagger_{\alpha,\beta} P_\alpha$$
$$= 8\begin{bmatrix} 46 & 6 & 52 & 90 & 142 \\ 236 & -324 & -88 & 36 & -52 \\ -50 & -42 & -92 & 186 & 94 \\ 74 & 258 & 332 & -210 & 122 \\ 0 & 0 & 0 & 0 & 0 \\ 214 & -114 & 100 & -78 & 22 \\ -100 & -84 & -184 & 372 & 188 \end{bmatrix},$$

(see the appendix). Thus by (3.1),

$$A^\dagger = \frac{1}{1632}\begin{bmatrix} 46 & 6 & 52 & 90 & 142 \\ 236 & -324 & -88 & 36 & -52 \\ -50 & -42 & -92 & 186 & 94 \\ 74 & 258 & 332 & -210 & 122 \\ 0 & 0 & 0 & 0 & 0 \\ 214 & -114 & 100 & -78 & 22 \\ -100 & -84 & -184 & 372 & 188 \end{bmatrix}.$$

In a similar manner, we can derive the following formulas which are generalizations of the formulas in (1.5) and (1.6).

COROLLARY 3.3. *Let* $A \in \mathbb{C}^{m\times n}$ *with* $\mathrm{rank}\,(A) = r$*,* $\alpha \in \Delta_{s,m}$*,* $\beta \in \Delta_{t,n}$*,* $m \geq s \geq r$ *and* $n \geq t \geq r$*. Then*

$$A^\dagger = \begin{pmatrix} m-r \\ s-r \end{pmatrix}^{-1} \frac{\sum_{\alpha\in\Delta_{s,m}} \mathrm{Tr}\, C_r\left(A^H_{*,\alpha}A_{\alpha,*}\right) A^\dagger_{\alpha,*} P_\alpha}{\mathrm{Tr}\, C_r\left(A^H A\right)}$$
$$= \begin{pmatrix} n-r \\ t-r \end{pmatrix}^{-1} \frac{\sum_{\beta\in\Delta_{t,n}} \mathrm{Tr}\, C_r\left(A^H_{\beta,*}A_{*,\beta}\right) Q_\beta A^\dagger_{*,\beta}}{\mathrm{Tr}\, C_r\left(A^H A\right)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 720-728, July 2011

Acknowledgments. The authors are grateful to Professors Ravindra B. Bapat, Bryan Shader and the referees for their valuable comments and suggestions which greatly improved this paper.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 720-728, July 2011

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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Appendix

In Example 3.1, we have

$$\sum_{\alpha\in\Delta_{s,m},\beta\in\Delta_{t,n}} \mathrm{Tr}\, C_r\left(A^H_{\beta,\alpha}A_{\alpha,\beta}\right)Q_\beta A^\dagger_{\alpha,\beta}P_\alpha$$

$$= \begin{bmatrix} 7 & 4 & 11 & 87 & 0\\ 26 & -40 & -14 & -6 & 0\\ -7 & -4 & -11 & 105 & 0\\ 15 & 36 & 51 & -33 & 0\\ 0 & 0 & 0 & 0 & 0\\ 27 & -12 & 15 & -21 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 35 & 5 & 40 & 48 & 0\\ 82 & -92 & -10 & -12 & 0\\ -7 & -1 & -8 & 63 & 0\\ 43 & 58 & 101 & -24 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -14 & -2 & -16 & 126 & 0 \end{bmatrix} + \begin{bmatrix} 35 & 20 & 55 & 87 & 0\\ 82 & -128 & -46 & -6 & 0\\ -7 & -4 & -11 & 105 & 0\\ 43 & 112 & 155 & -33 & 0\\ 0 & 0 & 0 & 0 & 0\\ 83 & -40 & 43 & -21 & 0\\ -14 & -8 & -22 & 210 & 0 \end{bmatrix} + \begin{bmatrix} 20 & 35 & 55 & 12 & 0\\ 10 & -56 & -46 & 6 & 0\\ -4 & -7 & -11 & 27 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 29 & 14 & 43 & -12 & 0\\ -8 & -14 & -22 & 54 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 28 & 16 & 44 & 87 & 0\\ 68 & -106 & -38 & -6 & 0\\ 0 & 0 & 0 & 0 & 0\\ 36 & 93 & 129 & -33 & 0\\ 0 & 0 & 0 & 0 & 0\\ 69 & -33 & 36 & -21 & 0\\ -14 & -8 & -22 & 210 & 0 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 15 & 27 & 0\\ 0 & 0 & 0 & 0 & 0\\ -3 & 0 & -3 & 33 & 0\\ -5 & 64 & 59 & -9 & 0\\ 0 & 0 & 0 & 0 & 0\\ 59 & -64 & -5 & -9 & 0\\ -6 & 0 & -6 & 66 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 60 & -90 & -30 & 0 & 0\\ 0 & 0 & 0 & 87 & 0\\ 40 & 85 & 125 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 65 & -25 & 40 & 0 & 0\\ 0 & 0 & 0 & 174 & 0 \end{bmatrix} + \begin{bmatrix} -22 & -25 & -47 & 0 & 87\\ 28 & -38 & -10 & 0 & -6\\ -42 & -39 & -81 & 0 & 105\\ 26 & 47 & 73 & 0 & -33\\ 0 & 0 & 0 & 0 & 0\\ 34 & -5 & 29 & 0 & -21\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 19 & -11 & 8 & 0 & 48\\ 86 & -88 & -2 & 0 & -12\\ -28 & -22 & -50 & 0 & 63\\ 51 & 66 & 117 & 0 & -24\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -56 & -44 & -100 & 0 & 126 \end{bmatrix} + \begin{bmatrix} 6 & -9 & -3 & 0 & 87\\ 84 & -126 & -42 & 0 & -6\\ -42 & -39 & -81 & 0 & 105\\ 54 & 123 & 177 & 0 & -33\\ 0 & 0 & 0 & 0 & 0\\ 90 & -33 & 57 & 0 & -21\\ -84 & -78 & -162 & 0 & 210 \end{bmatrix} + \begin{bmatrix} 16 & 31 & 47 & 0 & 12\\ 8 & -58 & -50 & 0 & 6\\ -13 & -16 & -29 & 0 & 27\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 33 & 18 & 51 & 0 & -12\\ -26 & -32 & -58 & 0 & 54 \end{bmatrix} + \begin{bmatrix} -1 & -13 & -14 & 0 & 87\\ 70 & -104 & -34 & 0 & -6\\ 0 & 0 & 0 & 0 & 0\\ 47 & 104 & 151 & 0 & -33\\ 0 & 0 & 0 & 0 & 0\\ 76 & -26 & 50 & 0 & -21\\ -84 & -78 & -162 & 0 & 210 \end{bmatrix}$$

$$+ \begin{bmatrix} 6 & -9 & -3 & 0 & 27\\ 0 & 0 & 0 & 0 & 0\\ -14 & -11 & -25 & 0 & 33\\ -2 & 67 & 65 & 0 & -9\\ 0 & 0 & 0 & 0 & 0\\ 62 & -61 & 1 & 0 & -9\\ -28 & -22 & -50 & 0 & 66 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 60 & -90 & -30 & 0 & 0\\ -29 & -29 & -58 & 0 & 87\\ 40 & 85 & 125 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 65 & -25 & 40 & 0 & 0\\ -58 & -58 & -116 & 0 & 174 \end{bmatrix} + \begin{bmatrix} -16 & -20 & 0 & 76 & 40\\ 32 & -56 & 0 & 8 & -16\\ -48 & -44 & 0 & 116 & 24\\ 48 & 76 & 0 & -84 & 40\\ 0 & 0 & 0 & 0 & 0\\ 48 & -4 & 0 & -36 & 8\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 44 & 4 & 0 & 8 & 56\\ 110 & -122 & 0 & -2 & -14\\ -33 & -25 & 0 & 71 & 13\\ 99 & 119 & 0 & -125 & 93\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -66 & -50 & 0 & 142 & 26 \end{bmatrix}$$

$$+ \begin{bmatrix} 36 & 16 & 0 & 32 & 84\\ 96 & -184 & 0 & 40 & -48\\ -48 & -44 & 0 & 116 & 24\\ 120 & 212 & 0 & -188 & 144\\ 0 & 0 & 0 & 0 & 0\\ 132 & -32 & 0 & -64 & 36\\ -96 & -88 & 0 & 232 & 48 \end{bmatrix} + \begin{bmatrix} 41 & 61 & 0 & -43 & 59\\ -4 & -92 & 0 & 52 & -44\\ -18 & -22 & 0 & 38 & -2\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 57 & 37 & 0 & -55 & 39\\ -36 & -44 & 0 & 76 & -4 \end{bmatrix} + \begin{bmatrix} 23 & 7 & 0 & 43 & 73\\ 80 & -152 & 0 & 32 & -40\\ 0 & 0 & 0 & 0 & 0\\ 102 & 178 & 0 & -162 & 118\\ 0 & 0 & 0 & 0 & 0\\ 111 & -25 & 0 & -57 & 29\\ -96 & -88 & 0 & 232 & 48 \end{bmatrix} + \begin{bmatrix} 16 & -4 & 0 & 12 & 24\\ 0 & 0 & 0 & 0 & 0\\ -16 & -12 & 0 & 36 & 8\\ 16 & 108 & 0 & -68 & 56\\ 0 & 0 & 0 & 0 & 0\\ 80 & -84 & 0 & -4 & -8\\ -32 & -24 & 0 & 72 & 16 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 70 & -130 & 0 & 30 & -30\\ -29 & -29 & 0 & 87 & 29\\ 95 & 155 & 0 & -125 & 125\\ 0 & 0 & 0 & 0 & 0\\ 100 & -20 & 0 & -40 & 40\\ -58 & -58 & 0 & 174 & 58 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -19 & 53 & 34\\ 66 & 0 & -34 & 14 & -20\\ -3 & 0 & -45 & 75 & 30\\ -21 & 0 & 69 & -51 & 18\\ 0 & 0 & 0 & 0 & 0\\ 39 & 0 & 9 & -15 & -6\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 30 & 0 & 14 & 17 & 31\\ 174 & 0 & -64 & 26 & -38\\ -6 & 0 & -27 & 45 & 18\\ -15 & 0 & 114 & -69 & 45\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -12 & 0 & -54 & 90 & 36 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 21 & 33 & 54\\ 210 & 0 & -114 & 54 & -60\\ -3 & 0 & -45 & 75 & 30\\ -69 & 0 & 189 & -111 & 78\\ 0 & 0 & 0 & 0 & 0\\ 123 & 0 & 9 & -15 & -6\\ -6 & 0 & -90 & 150 & 60 \end{bmatrix}$$

$$+ \begin{bmatrix} -15 & 0 & 56 & -22 & 34\\ 66 & 0 & -70 & 38 & -32\\ 3 & 0 & -21 & 24 & 3\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 15 & 0 & 42 & -27 & 15\\ 6 & 0 & -42 & 48 & 6 \end{bmatrix} + \begin{bmatrix} 12 & 0 & 11 & 38 & 49\\ 174 & 0 & -94 & 44 & -50\\ 0 & 0 & 0 & 0 & 0\\ -57 & 0 & 159 & -96 & 63\\ 0 & 0 & 0 & 0 & 0\\ 102 & 0 & 9 & -15 & -6\\ -6 & 0 & -90 & 150 & 60 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 1 & 13 & 14\\ 0 & 0 & 0 & 0 & 0\\ -3 & 0 & -13 & 23 & 10\\ -69 & 0 & 85 & -47 & 38\\ 0 & 0 & 0 & 0 & 0\\ 123 & 0 & -43 & 17 & -26\\ -6 & 0 & -26 & 46 & 20 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 150 & 0 & -80 & 40 & -40\\ 0 & 0 & -29 & 58 & 29\\ -45 & 0 & 140 & -70 & 70\\ 0 & 0 & 0 & 0 & 0\\ 90 & 0 & 10 & -5 & 5\\ 0 & 0 & -58 & 116 & 58 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -3 & -17 & 52 & 35\\ 0 & -66 & 10 & -8 & 2\\ 0 & 3 & -47 & 76 & 29\\ 0 & 21 & 55 & -44 & 11\\ 0 & 0 & 0 & 0 & 0\\ 0 & -39 & 35 & -28 & 7\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -30 & 34 & 7 & 41\\ 0 & -174 & 52 & -32 & 20\\ 0 & 6 & -31 & 47 & 16\\ 0 & 15 & 104 & -64 & 40\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 12 & -62 & 94 & 32 \end{bmatrix} + \begin{bmatrix} 0 & -15 & 31 & 28 & 59\\ 0 & -210 & 26 & -16 & 10\\ 0 & 3 & -47 & 76 & 29\\ 0 & 69 & 143 & -88 & 55\\ 0 & 0 & 0 & 0 & 0\\ 0 & -123 & 91 & -56 & 35\\ 0 & 6 & -94 & 152 & 58 \end{bmatrix} + \begin{bmatrix} 0 & 15 & 46 & -17 & 29\\ 0 & -66 & -26 & 16 & -10\\ 0 & -3 & -19 & 23 & 4\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & -15 & 52 & -32 & 20\\ 0 & -6 & -38 & 46 & 8 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -12 & 19 & 34 & 53\\ 0 & -174 & 22 & -14 & 8\\ 0 & 0 & 0 & 0 & 0\\ 0 & 57 & 121 & -77 & 44\\ 0 & 0 & 0 & 0 & 0\\ 0 & -102 & 77 & -49 & 28\\ 0 & 6 & -94 & 152 & 58 \end{bmatrix} + \begin{bmatrix} 0 & -15 & 11 & 8 & 19\\ 0 & 0 & 0 & 0 & 0\\ 0 & 3 & -15 & 24 & 9\\ 0 & 69 & 39 & -24 & 15\\ 0 & 0 & 0 & 0 & 0\\ 0 & -123 & 39 & -24 & 15\\ 0 & 6 & -30 & 48 & 18 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -150 & 20 & -10 & 10\\ 0 & 0 & -29 & 58 & 29\\ 0 & 45 & 110 & -55 & 55\\ 0 & 0 & 0 & 0 & 0\\ 0 & -90 & 70 & -35 & 35\\ 0 & 0 & -58 & 116 & 58 \end{bmatrix}$$

$$= 8 \begin{bmatrix} 46 & 6 & 52 & 90 & 142\\ 236 & -324 & -88 & 36 & -52\\ -50 & -42 & -92 & 186 & 94\\ 74 & 258 & 332 & -210 & 122\\ 0 & 0 & 0 & 0 & 0\\ 214 & -114 & 100 & -78 & 22\\ -100 & -84 & -184 & 372 & 188 \end{bmatrix}.$$