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# THE MINIMUM ALGEBRAIC CONNECTIVITY OF CATERPILLAR UNICYCLIC GRAPHS*

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**Abstract.** A caterpillar unicyclic graph is a unicyclic graph in which the removal of all pendant vertices makes it a cycle. In this paper, the unique caterpillar unicyclic graph with minimum algebraic connectivity among all caterpillar unicyclic graphs is determined.

**Key words.** Algebraic connectivity, Caterpillar unicyclic graph, Characteristic polynomial.

**AMS subject classifications.** 05C50.

**1. Introduction.** Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. Let $d(v_i)$ be the degree of the vertex $v_i \in V(G)$ $(i = 1, 2, \ldots, n)$, and $D = D(G) = diag(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The *Laplacian matrix* $L(G) = D(G) - A(G)$ is the difference between $D(G)$ and the adjacency matrix $A(G)$. It is easy to see that $L(G)$ is a positive semidefinite symmetric matrix with the smallest eigenvalue 0 and the corresponding eigenvector is the all ones column vector, which is denoted by $e$. Denote its eigenvalues by

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0,$$

which are always enumerated in non-increasing order and repeated according to their multiplicity. Fiedler [4] showed that the second smallest eigenvalue of $L(G)$ is 0 if and only if $G$ is disconnected. Thus, the second smallest eigenvalue of $L(G)$ is popularly known as the *algebraic connectivity* of $G$ and is usually denoted by $\alpha(G)$. Let $P_n$ and $C_n$ be the path and the cycle on $n$ vertices, respectively. It is a known fact that

$$\alpha(P_n) = 4\sin^2 \tfrac{\pi}{2n} \text{ and } \alpha(C_n) = 4\sin^2 \tfrac{\pi}{n}.$$

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Let $Y \in \mathbb{R}^n$ be a column vector. It will be convenient to assume that the entry $y_i$ of $Y$ is corresponding to the vertex $v_i$ of $G$. Such a $Y$ is sometimes called a *valuation* of the graph $G$ (see, [11]). In the following, $y_i$ will be written as $Y(v_i)$ for convenient. If $X$ is a unit eigenvector of $G$ corresponding to $\alpha(G)$, we commonly call it a *Fiedler vector* of $G$. It is obvious that $X^T e = 0$ and

$$\alpha(G) = X^T L(G) X = \sum_{v_i v_j \in E} (X(v_i) - X(v_j))^2 = \min_{\substack{Y \in \mathbb{R}^n \setminus \{0\} \\ Y^T e = 0}} \frac{Y^T L(G) Y}{Y^T Y}.$$

Furthermore, from $L(G)X = \alpha(G)X$, we also have the set of equations below, known as eigenvalue equations of $G$:

$$(d(v) - \alpha(G))X(v) = \sum_{u \in N(v)} X(u) \quad \text{for } v \in V(G),$$

where $N_G(v)$ (or $N(v)$ for short) denotes the set of vertices which are adjacent to $v$ in $G$.

A *caterpillar unicyclic graph* is a unicyclic graph in which the removal of all pendant vertices makes it a cycle. Let $C_g = v_1 v_2 \cdots v_g v_1$ be a cycle with length $g$, where $v_i v_{i+1} \in E(C_g)$ for $i = 1, 2, \ldots, g-1$ and $v_g v_1 \in E(C_g)$, and let $C^{n_1, n_2, \ldots, n_k}_{g; i_1, i_2, \ldots, i_k}$ be the caterpillar unicyclic graph obtained from $C_g$ by attaching $n_j$ pendant edges at $v_{i_j}$ $(1 \le i_1 < \cdots < i_k \le g)$, respectively. By symmetry, we may always assume that $i_1 = 1$. For example, $C^{n_1, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1}$ and $C^{n_1, n_2, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1, \lfloor \frac{g}{2} \rfloor + 2}$ (see Fig. 1.1) are two caterpillar unicyclic graphs which will be used in the next section. If $C^{n_1, n_2, \ldots, n_k}_{g; i_1, i_2, \ldots, i_k}$ has $n$ vertices, then it is easy to see that $n_1 + n_2 + \cdots + n_k = n - g$.

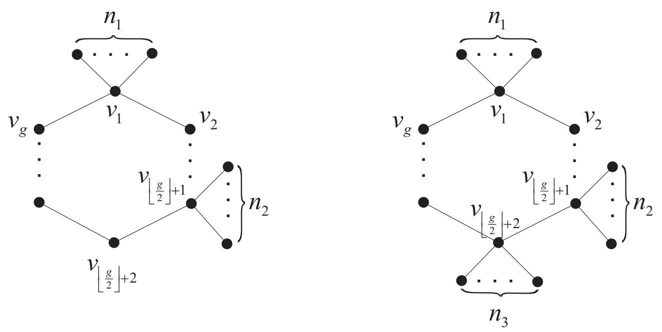

FIG. 1.1. *Two caterpillar unicyclic graphs* $C^{n_1, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1}$ *and* $C^{n_1, n_2, n_2}_{g; 1, \lfloor \frac{g}{2} \rfloor + 1, \lfloor \frac{g}{2} \rfloor + 2}$.

In [1], Fallat and Kirkland proved that for some choice of the parameters $n_1$, $n_2$, $\ldots$, $n_g$, the graph $C^{n_1, n_2, \ldots, n_g}_{g; 1, 2, \ldots, g}$ maximizes the algebraic connectivity over the

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class of unicyclic graphs with girth $g$. In particular, they proved that among all unicyclic graphs on $n$ vertices with girth 3, the graph $C_{3;1}^{n-3}$ has the maximum algebraic connectivity. In [2], Fallat, Kirkland and Pati proved that the graph $C_{4;1}^{n-4}$ has the maximum algebraic connectivity among all unicyclic graphs on $n$ vertices with girth 4. Furthermore, they proved that there is an $N$ such that for each $n > N$, the graph $C_{g;1}^{n-g}$ has the maximum algebraic connectivity among all unicyclic graphs on $n$ vertices with girth $g$. On the other hand, when $g$ is large relative to $n$, they showed that this graph does not maximize the algebraic connectivity. For the minimum algebraic connectivity, Guo [7] proved that the graph $C_{n,g}$ has the minimum algebraic connectivity among all connected graphs with girth $g$, where $C_{n,g}$ is called the lollipop graph, which is obtained by appending a $g$-cycle $C_g$ to a pendant vertex of a path on $n-g$ vertices. This confirmes the conjecture proposed by Fallat and Kirkland (see [1], [3]).

In this paper, we prove that the graph $C_{g;1,\lfloor\frac{g}{2}\rfloor+1}^{\lceil\frac{n-g}{2}\rceil,\lfloor\frac{n-g}{2}\rfloor}$ has the minimum algebraic connectivity among all caterpillar unicyclic graphs on $n$ vertices with girth $g$.

Throughout this paper, we shall denote by $\Phi(B) = \Phi(B;x) = \det(xI - B)$ the characteristic polynomial of the square matrix $B$. In particular, if $B = L(G)$, we write $\Phi(L(G))$ by $\Phi(G;x)$ or simply by $\Phi(G)$ and call $\Phi(G)$ the *Laplacian characteristic polynomial* of $G$.

**2. Lemmas and results.** Let $G$ be a graph and let $G' = G + e$ be the graph obtained from $G$ by inserting a new edge $e$ into $G$. The following lemma follows from Courant-Weyl inequalities (see [9]).

Lemma 2.1. *The Laplacian eigenvalues of $G$ and $G'$ interlace, that is,*

$$\mu_1(G') \geq \mu_1(G) \geq \mu_2(G') \geq \mu_2(G) \geq \cdots \geq \mu_n(G') = \mu_n(G) = 0.$$

By Lemma 2.1, we immediately have the following:

Corollary 2.2. *Let $G$ be a connected graph and $v$ be a pendant vertex of $G$. Then $\alpha(G) \leq \alpha(G - v)$.*

The following inequalities are known as Cauchy's inequalities and the whole theorem is also known as interlacing theorem [9].

Lemma 2.3. *Let $A$ be a Hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $B$ be a principal sub-matrix of $A$. Let $B$ has eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_m$ $(m \leq n)$. Then the inequalities $\lambda_{n-m+i} \leq \rho_i \leq \lambda_i$ hold for $i = 1, 2, \ldots, m$.*

Lemma 2.4. [5] *Let $G_1 = (V, E_1)$ be a graph on $n$ vertices and $G_2 = (V, E_2)$ be*

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*a graph obtained from $G_1$ by removing an edge and adding a new edge that was not there before. Then*

$$\mu_i(G_1) \geq \mu_{i+1}(G_2) \text{ and } \mu_i(G_2) \geq \mu_{i+1}(G_1) \text{ for } 1 \leq i \leq n-1.$$

LEMMA 2.5. [8] *Suppose that $g \geq 4$. Then $\alpha(C_{n,g}) > \alpha(C_{n,g-1})$.*

For $U \subseteq V(G)$, let $L_U(G)$ be the principal sub-matrix of $L(G)$ formed by deleting the rows and columns corresponding to all vertices in $U$. If $U = \{v\}$, then we simple write $L_U(G)$ as $L_v(G)$. Let $H_n$ be the matrix of order $n$ obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to two end vertices of $P_{n+2}$.

LEMMA 2.6. [7] *Set $\Phi(P_0) = 0, \Phi(H_0) = 1$. Then we have*

*(1)* $\Phi(P_n) = x\Phi(H_{n-1})$, $(n \geq 1)$;
*(2)* $\Phi(P_m)\Phi(P_n) - \Phi(P_{m-1})\Phi(P_{n+1}) = \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-2})\Phi(P_n)$, $(m \geq 2, n \geq 1)$.

COROLLARY 2.7. *For $m \geq k+1,\ n \geq 1$,*

$$\Phi(P_m)\Phi(P_n) - \Phi(P_{m-k})\Phi(P_{n+k}) = \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-k-1})\Phi(P_{n+k-1}).$$

*Proof.* From Lemma 2.6, we have

$$\begin{aligned}
&\Phi(P_m)\Phi(P_n) - \Phi(P_{m-k})\Phi(P_{n+k}) \\
&= \Phi(P_m)\Phi(P_n) - \Phi(P_{m-1})\Phi(P_{n+1}) + \Phi(P_{m-1})\Phi(P_{n+1}) \\
&\quad - \Phi(P_{m-2})\Phi(P_{n+2}) + \Phi(P_{m-2})\Phi(P_{n+2}) - \cdots - \Phi(P_{m-k+1})\Phi(P_{n+k-1}) \\
&\quad + \Phi(P_{m-k+1})\Phi(P_{n+k-1}) - \Phi(P_{m-k})\Phi(P_{n+k}) \\
&= \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-2})\Phi(P_n) + \Phi(P_{m-2})\Phi(P_n) - \Phi(P_{m-3})\Phi(P_{n+1}) \\
&\quad + \Phi(P_{m-3})\Phi(P_{n+1}) - \cdots - \Phi(P_{m-k})\Phi(P_{n+k-2}) \\
&\quad + \Phi(P_{m-k})\Phi(P_{n+k-2}) - \Phi(P_{m-k-1})\Phi(P_{n+k-1}) \\
&= \Phi(P_{m-1})\Phi(P_{n-1}) - \Phi(P_{m-k-1})\Phi(P_{n+k-1}). \quad \square
\end{aligned}$$

Suppose $G_1$ and $G_2$ are two disjoint graphs. $u \in V(G_1)$ and $v \in V(G_2)$. Let $G = G_1u : vG_2$ be the graph obtained by joining the vertex $u$ of the graph $G_1$ to the vertex $v$ of the graph $G_2$ with an edge.

LEMMA 2.8. [6] *Let $G_1$ and $G_2$ be two disjoint graphs. Then*

$$\Phi(G_1u : vG_2) = \Phi(G_1)\Phi(G_2) - \Phi(G_1)\Phi(L_v(G_2)) - \Phi(L_u(G_1))\Phi(G_2).$$

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By a similar argument as the proof of Lemma 2.8, which was shown in [6], we also have the following result.

Corollary 2.9. *Let $G_1$ and $G_2$ be two disjoint graphs. Suppose $w$ is a vertex of $G_1$ which is different from $u$. Then*

$$\Phi(L_w(G_1u : vG_2)) = \Phi(L_w(G_1))\Phi(G_2) - \Phi(L_w(G_1))\Phi(L_v(G_2)) - \Phi(L_{\{w,u\}}(G_1))\Phi(G_2).$$

Let $G$ and $H$ be two disjoint graphs with $|V(G)| = s \geq 2, |V(H)| = t \geq 2$. Let $u \in V(G)$ and $r \in V(H)$. Let $Gu \cdot rH$ be the graph obtained from $G$ and $H$ by identifying the two vertices $u$ and $r$ (suppose that the new vertex is still $u$). It is easy to see that $Gu \cdot rH$ has $n = s + t - 1$ vertices.

Lemma 2.10. *Suppose $u$ and $v$ are two distinct vertices of $G$. Suppose $X$ is a Fiedler vector of $Gu \cdot rH$. If $(X(v) - X(u)) \sum_{\substack{w \in V(H) \\ w \neq r}} X(w) \geq 0$, then $\alpha(Gu \cdot rH) \geq \alpha(Gv \cdot rH)$. Moreover, the inequality is strict if $X(u) \neq X(v)$.*

*Proof.* Let $Y$ be a valuation of $Gv \cdot rH$ defined by

$$Y(w) = \begin{cases} X(w) - \frac{(t-1)(X(v)-X(u))}{n}, & w \in V(G); \\ X(w) + \frac{s(X(v)-X(u))}{n}, & w \in V(H), w \neq r. \end{cases}$$

It is easy to see that $Y^T e = 0$, $Y^T L(Gv \cdot rH)Y = X^T L(Gu \cdot rH)X = \alpha(Gu \cdot rH)$ and

$$\begin{aligned} Y^T Y &= X^T X - 2 \sum_{w \in V(G)} \frac{(t-1)(X(v) - X(u))}{n} X(w) + \frac{s(t-1)^2(X(v) - X(u))^2}{n^2} \\ &\quad + 2 \sum_{\substack{w \in V(H) \\ w \neq r}} \frac{s(X(v) - X(u))}{n} X(w) + \frac{(t-1)s^2(X(v) - X(u))^2}{n^2} \\ &= 1 + 2(X(v) - X(u)) \sum_{\substack{w \in V(H) \\ w \neq r}} X(w) + \frac{s(t-1)(X(v) - X(u))^2}{n} \\ &\geq 1. \end{aligned}$$

Clearly, the inequality is strict if $X(v) \neq X(u)$.

Thus, we have

$$\alpha(Gu \cdot rH) = X^T L(Gu \cdot rH)X \geq \frac{Y^T L(Gv \cdot rH)Y}{Y^T Y} \geq \alpha(Gv \cdot rH),$$

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and the inequality is strict if $X(v) \neq X(u)$. □

From Lemma 2.10, we immediately have the following.

COROLLARY 2.11. *Let $u, v$ be two vertices of a connected graph $G$ and there exist $s$ pendant edges $uu_1, uu_2, \ldots, uu_s$ at $u$. Suppose $X$ is a Fiedler vector of $G$. Let $G' = G - uu_1 - uu_2 - \cdots - uu_s + vu_1 + vu_2 + \cdots + vu_s$. If $X(v) \geq X(u) \geq 0$, then $\alpha(G) \geq \alpha(G')$. Moreover, the inequality is strict if $X(v) \neq X(u)$.*

LEMMA 2.12. *Let $C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}$ be the caterpillar unicyclic graph defined in Section 1. Then for $n \geq g+1$,*

$$\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}) < \alpha(P_{g-\lfloor\frac{g}{2}\rfloor}).$$

*Proof.* From Corollary 2.2 and Lemma 2.5, we have

$$\begin{aligned}\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}) \leq \alpha(C_{g+1,g}) < \alpha(C_{g+1}) &= 4\sin^2\frac{\pi}{g+1} \\ &\leq 4\sin^2\frac{\pi}{2(g-\lfloor\frac{g}{2}\rfloor)} = \alpha(P_{g-\lfloor\frac{g}{2}\rfloor}).\end{aligned}$$ □

REMARK 1. Since $\alpha(P_n)$ is a decreasing function on $n$, $\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}) < \alpha(P_j)$ for $j \leq g - \lfloor\frac{g}{2}\rfloor$.

LEMMA 2.13. [10] *Let $G$ be a connected graph with a cut vertex $v$. Then $\alpha(G) \leq 1$, the equality holds if and only if $v$ is adjacent to every vertex of $G$.*

LEMMA 2.14. *Let $C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}$ and $C^{n_1,n_2}_{g;1,i}$ $(2 \leq i \leq \lfloor\frac{g}{2}\rfloor)$ be the caterpillar unicyclic graphs defined in Section 1. Then for $n_1, n_2 \geq 1$,*

$$\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}) \leq \mu_{n-2}(C^{n_1,n_2}_{g;1,i}).$$

*Proof.* If $\mu_{n-2}(C^{n_1,n_2}_{g;1,i}) \geq 1$, then the result follows from Lemma 2.13. Thus, in the following, we assume that $\mu_{n-2}(C^{n_1,n_2}_{g;1,i}) < 1$. From Corollary 2.2 and Lemma 2.3, we have

$$\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}) \leq \alpha(C^{n_1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}) = \alpha(C^{n_1}_{g;1}) \leq \lambda_{g+n_1-2}(L_{v_i}(C^{n_1}_{g;1})) \leq \mu_{n-2}(C^{n_1,n_2}_{g;1,i}),$$

where $\lambda_{g+n_1-2}(L_{v_i}(C^{n_1}_{g;1}))$ denotes the second smallest eigenvalue of $L_{v_i}(C^{n_1}_{g;1})$. □

LEMMA 2.15. *For $2 \leq i \leq \lfloor\frac{g}{2}\rfloor$ and $n_1, n_2 \geq 1$, we have*

$$\alpha(C^{n_1,n_2}_{g;1,i}) > \alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1}).$$

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*Proof.* Applying Lemma 2.8 and Corollary 2.9 for $n_1$ times, we have

$$\begin{aligned}
&\Phi(C_{g;1,i}^{n_1,n_2}) - \Phi(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{n_1,n_2})\\
&= (x-1)\Phi(C_{g;1,i}^{n_1-1,n_2}) - x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,i}^{0,n_2})) - (x-1)\Phi(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{n_1-1,n_2})\\
&\quad + x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2}))\\
&\quad\vdots \qquad \vdots\\
&= (x-1)^{n_1}\Phi(C_{g;1,i}^{0,n_2}) - n_1x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,i}^{0,n_2})) - (x-1)^{n_1}\Phi(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2})\\
&\quad + n_1x(x-1)^{n_1-1}\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2})).
\end{aligned}$$

Note that $C_{g;1,i}^{0,n_2} = C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2} = C_{g;1}^{n_2}$. Then from the above equation, we have

$$\Phi(C_{g;1,i}^{n_1,n_2}) - \Phi(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{n_1,n_2}) = n_1x(x-1)^{n_1-1}[\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2})) - \Phi(L_{v_1}(C_{g;1,i}^{0,n_2}))] \tag{2.1}$$

Applying Corollary 2.9 again, we have

$$\begin{aligned}
&\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2})) - \Phi(L_{v_1}(C_{g;1,i}^{0,n_2}))\\
&= (x-1)\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2-1})) - x(x-1)^{n_2-1}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\
&\quad - (x-1)\Phi(L_{v_1}(C_{g;1,i}^{0,n_2-1})) + x(x-1)^{n_2-1}\Phi(H_{i-2})\Phi(H_{g-i})\\
&\qquad\qquad \vdots\\
&= (x-1)^{n_2}\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,0})) - n_2x(x-1)^{n_2-1}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\
&\quad - (x-1)^{n_2}\Phi(L_{v_1}(C_{g;1,i}^{0,0})) + n_2x(x-1)^{n_2-1}\Phi(H_{i-2})\Phi(H_{g-i})
\end{aligned}$$

Note that $L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,0}) = L_{v_1}(C_{g;1,i}^{0,0}) = L_{v_1}(C_g) = \Phi(H_{g-1})$. Thus, from the above equation, we have

$$\begin{aligned}
&\Phi(L_{v_1}(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{0,n_2})) - \Phi(L_{v_1}(C_{g;1,i}^{0,n_2}))\\
&= n_2x(x-1)^{n_2-1}[\Phi(H_{i-2})\Phi(H_{g-i}) - \Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})].
\end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), and from Lemma 2.6 and Corollary 2.7, we have

$$\begin{aligned}
&\Phi(C_{g;1,i}^{n_1,n_2}) - \Phi(C_{g;1,\lfloor \frac{g}{2}\rfloor+1}^{n_1,n_2})\\
&= n_1n_2x^2(x-1)^{n_1+n_2-2}[\Phi(H_{i-2})\Phi(H_{g-i}) - \Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})]\\
&= n_1n_2(x-1)^{n_1+n_2-2}[\Phi(P_{i-1})\Phi(P_{g-i+1}) - \Phi(P_{\lfloor \frac{g}{2}\rfloor})\Phi(P_{g-\lfloor \frac{g}{2}\rfloor})]\\
&= -n_1n_2(x-1)^{n_1+n_2-2}\Phi(P_{\lfloor \frac{g}{2}\rfloor-i+1})\Phi(P_{g-\lfloor \frac{g}{2}\rfloor-i+1}).
\end{aligned}$$

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Let $\alpha = \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$. From Lemma 2.12, we have

$$\begin{aligned}&(-1)^{n_1+n_2+g-1}[\Phi(C^{n_1,n_2}_{g;1,i},\alpha) - \Phi(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1},\alpha)]\\ &= (-1)^{n_1+n_2+g}[n_1n_2(\alpha-1)^{n_1+n_2-2}\Phi(P_{\lfloor \frac{g}{2}\rfloor-i+1},\alpha)\Phi(P_{g-\lfloor \frac{g}{2}\rfloor-i+1},\alpha)].\end{aligned}$$

By Remark 1 and the fact $0 < \alpha < 1$, the above expression is positive. Note that $n_1+n_2+g = n$ is the order of the graph $C^{n_1,n_2}_{g;1,i}$. So $(-1)^{n-1}\Phi(C^{n_1,n_2}_{g;1,i},\alpha) > 0$. Thus, from Lemma 2.14, we have $\alpha(C^{n_1,n_2}_{g;1,i}) > \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$. ☐

LEMMA 2.16. *For $n_1 \geq n_2+2$, $\alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) > \alpha(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$.*

*Proof.* We separate the proof into two cases.

**Case 1.** $n_2 \geq 1$.

Applying Lemma 2.8 and Corollary 2.9 for several times, we have

$$\begin{aligned}&\Phi(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) - \Phi(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})\\
&= (x-1)\Phi(C^{n_1-1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) - x(x-1)^{n_2}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&\quad - (x-1)\Phi(C^{n_1-1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) + x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&= x(x-1)^{n_1-1}\Phi(L_{v_1}(C^{0,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})) - x(x-1)^{n_2}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&= x(x-1)^{n_1}\Phi(L_{v_1}(C^{0,n_2-1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})) - x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\
&\quad - x(x-1)^{n_2+1}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-2,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&\quad + x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\
&= x(x-1)^{n_1}\Phi(L_{v_1}(C^{0,n_2-1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})) - x(x-1)^{n_2+1}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{n_1-2,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&= x(x-1)^{n_1+n_2-1}\Phi(L_{v_1}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&\quad - n_2x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\
&\quad - x(x-1)^{n_1+n_2-1}\Phi(L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}))\\
&\quad + (n_1-1)x^2(x-1)^{n_1+n_2-2}\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1}).\end{aligned}$$

Note that $L_{v_1}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) = L_{v_{\lfloor \frac{g}{2}\rfloor+1}}(C^{0,0}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) = L_{v_1}(C_g)$. Then from Lemma 2.6 and the above equation we have

$$\begin{aligned}&\Phi(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1}) - \Phi(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})\\
&= x^2(x-1)^{n_1+n_2-2}(n_1-n_2-1)\Phi(H_{\lfloor \frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor \frac{g}{2}\rfloor-1})\\
&= (x-1)^{n_1+n_2-2}(n_1-n_2-1)\Phi(P_{\lfloor \frac{g}{2}\rfloor})\Phi(P_{g-\lfloor \frac{g}{2}\rfloor}).\end{aligned}$$

Let $\alpha = \alpha(C^{n_1,n_2}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$. From Lemma 2.4, we have $\alpha \leq \mu_{n-2}(C^{n_1-1,n_2+1}_{g;1,\lfloor \frac{g}{2}\rfloor+1})$.

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Thus, similar to the proof of Lemma 2.15, we have

$$\begin{aligned}&(-1)^{n_1+n_2+g-1}[\Phi(C^{n_1-1,n_2+1}_{g;1,\lfloor\frac{g}{2}\rfloor+1},\alpha)-\Phi(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1},\alpha)]\\ &=(-1)^{n_1+n_2+g-1}(\alpha-1)^{n_1+n_2-2}(n_1-n_2-1)\Phi(P_{\lfloor\frac{g}{2}\rfloor},\alpha)\Phi(P_{g-\lfloor\frac{g}{2}\rfloor},\alpha)<0.\end{aligned}$$

Then, we have $\alpha(C^{n_1,n_2}_{g;1,\lfloor\frac{g}{2}\rfloor+1})>\alpha(C^{n_1-1,n_2+1}_{g;1,\lfloor\frac{g}{2}\rfloor+1})$.

**Case 2.** $n_2=0$.

From Lemma 2.8 and Corollary 2.9, we have

$$\begin{aligned}&\Phi(C^{n_1-1,1}_{g;1,\lfloor\frac{g}{2}\rfloor+1})-\Phi(C^{n_1}_{g;1})\\ &=(x-1)\Phi(C^{n_1-1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1})-x\Phi(L_{v_{\lfloor\frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}))\\ &\quad-(x-1)\Phi(C^{n_1-1}_{g;1})+x(x-1)^{n_1-1}\Phi(L_{v_1}(C^0_{g;1}))\\ &=x(x-1)^{n_1-1}\Phi(L_{v_1}(C^0_{g;1}))-x\Phi(L_{v_{\lfloor\frac{g}{2}\rfloor+1}}(C^{n_1-1,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}))\\ &=x(x-1)^{n_1-1}\Phi(L_{v_1}(C^0_{g;1}))-x(x-1)^{n_1-1}\Phi(L_{v_{\lfloor\frac{g}{2}\rfloor+1}}(C^{0,0}_{g;1,\lfloor\frac{g}{2}\rfloor+1}))\\ &\quad+(n_1-1)x^2(x-1)^{n_1-2}\Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1})\\ &=(n_1-1)x^2(x-1)^{n_1-2}\Phi(H_{\lfloor\frac{g}{2}\rfloor-1})\Phi(H_{g-\lfloor\frac{g}{2}\rfloor-1}).\end{aligned}$$

By a similar argument as that of Case 1, the result follows. ◻

Now we give the main result of this paper.

Theorem 2.17. *Let $G$ be a caterpillar unicyclic graph on $n$ vertices with girth $g$. Then*

$$\alpha(G)\geq\alpha(C^{\lceil\frac{n-g}{2}\rceil,\lfloor\frac{n-g}{2}\rfloor}_{g;1,\lfloor\frac{g}{2}\rfloor+1}),$$

*and the equality holds if and only if $G=C^{\lceil\frac{n-g}{2}\rceil,\lfloor\frac{n-g}{2}\rfloor}_{g;1,\lfloor\frac{g}{2}\rfloor+1}$.*

*Proof.* Since $G$ is a caterpillar unicyclic graph on $n$ vertices with girth $g$, we may assume that

$$G=C^{n_1,n_2,\ldots,n_k}_{g;\ i_1,i_2,\ldots,i_k},\quad n_j\geq1 \text{ for } 1\leq j\leq k\leq g; 1\leq i_1<i_2<\cdots<i_k\leq g.$$

For $k=1$, the result follows from Case 2 of the proof of Lemma 2.16. For $k=2$, the result follows from Lemmas 2.15 and 2.16.

For $k=3$, let $X$ be a Fiedler vector of $G=C^{n_1,n_2,n_3}_{g;\ i_1,i_2,i_3}$. Since $-X$ is also a Fiedler vector of $G$, without loss of generality, we may assume that there are at least two of $X(v_{i_1})$, $X(v_{i_2})$ and $X(v_{i_3})$ being nonnegative. By renumbering, we may assume that $G\cong C^{n_1,n_2,n_3}_{g;\ 1,i,j}$ with $X(v_j)\geq X(v_i)\geq X(v_1)$ and $X(v_i)\geq0$. Suppose $X(v_j)>X(v_i)$. Then by Corollary 2.11, $\alpha(G)>\alpha(C^{n_1,n_2+n_3}_{g;\ 1,j})$. By renumbering of the vertices, we

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may assume that $j \leq \lfloor \frac{g}{2} \rfloor + 1$. By Lemmas 2.15 and 2.16 if necessary, we obtain the result.

So now we assume that $X(v_j) = X(v_i)$. If the distance between $v_1$ and $v_i$ or the distance between $v_1$ and $v_j$ less than $\lfloor \frac{g}{2} \rfloor$, then (by renumbering the vertices if necessary) we may assume that $i \leq \lfloor \frac{g}{2} \rfloor$. Then by Corollary 2.11, Lemmas 2.15 and 2.16, we obtain that $\alpha(G) \geq \alpha(C_{g;\ 1,i}^{n_1,n_2+n_3}) > \alpha(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+1}^{n_1,n_2+n_3}) \geq \alpha(C_{g;1,\lfloor \frac{g}{2} \rfloor+1}^{\lfloor \frac{n-g}{2} \rfloor, \lceil \frac{n-g}{2} \rceil})$.

Thus, we have to deal with the case that the distance between $v_1$ to both $v_i$ and $v_j$ are $\lfloor \frac{g}{2} \rfloor$. Note that the necessary condition for the occurrence of this case is $g$ being odd. So now $G \cong C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2,n_3}$ with $X(v_{\lfloor \frac{g}{2} \rfloor+1}) = X(v_{\lfloor \frac{g}{2} \rfloor+2})$. Note that

$$\alpha(G) = X^T L(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2,n_3})X = X^T L(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3})X \geq \alpha(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3}).$$

Suppose that $\alpha(G) = \alpha(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3})$. From the above equation we can see that $X$ is also a Fiedler vector of $C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3}$. From the eigenvalue equations of $G$, we have

$$\begin{aligned} &\left(d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - \alpha(G)\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}) \\ &= \sum_{w \in N(v_{\lfloor \frac{g}{2} \rfloor+1})} X(w) = X(v_{\lfloor \frac{g}{2} \rfloor+1}) + X(v_{\lfloor \frac{g}{2} \rfloor}) + \sum_{\substack{w \in N(v_{\lfloor \frac{g}{2} \rfloor+1}) \\ w \neq v_{\lfloor \frac{g}{2} \rfloor}, v_{\lfloor \frac{g}{2} \rfloor+2}}} X(w). \end{aligned}$$

Note that for $w \in N(v_{\lfloor \frac{g}{2} \rfloor+1}) \setminus \{v_{\lfloor \frac{g}{2} \rfloor}, v_{\lfloor \frac{g}{2} \rfloor+2}\}$, $(1-\alpha(G))X(w) = X(v_{\lfloor \frac{g}{2} \rfloor+1})$. Thus, the above equation becomes

$$X(v_{\lfloor \frac{g}{2} \rfloor}) = \left(d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 1 - \alpha(G) - \frac{d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 2}{1-\alpha(G)}\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}). \tag{2.3}$$

Similarly, from the eigenvalue equations of $C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3}$, we have

$$\left(2 - \alpha(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3})\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}) = X(v_{\lfloor \frac{g}{2} \rfloor+2}) + X(v_{\lfloor \frac{g}{2} \rfloor}).$$

Then

$$X(v_{\lfloor \frac{g}{2} \rfloor}) = \left(1 - \alpha(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3})\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}). \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\left(d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 2 - \frac{d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) - 2}{1-\alpha(G)}\right) X(v_{\lfloor \frac{g}{2} \rfloor+1}) = 0.$$

Since $n_2 \geq 1$, $d_G(v_{\lfloor \frac{g}{2} \rfloor+1}) > 2$. Thus, we have $X(v_{\lfloor \frac{g}{2} \rfloor+1}) = 0$. From the eigenvalue equations of $G$, it is easy to see that $X = \mathbf{0}$. It yields a contradiction. So $\alpha(G) > \alpha(C_{g;\ 1,\lfloor \frac{g}{2} \rfloor+2}^{n_1,n_2+n_3})$.

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For $k \geq 4$, from Corollary 2.11, there exists some caterpillar unicyclic graph, say $C_{g;\,1,i,j}^{n_1,n_2,n_3}$ for $n_1, n_2, n_3 \geq 1$ and $1 < i < j$, such that $\alpha(G) \geq \alpha(C_{g;\,1,i,j}^{n_1,n_2,n_3})$. This case is referred to the case when $k = 3$.

Hence, the proof is completed. □

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