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STRUCTURE OF NILPOTENT MATRICES OVER FIELDS*

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Abstract. A zero-nonzero pattern $\mathcal{A}$ is said to be potentially nilpotent over a field $\mathbb{F}$ if there exists a nilpotent matrix with entries in $\mathbb{F}$ having zero-nonzero pattern $\mathcal{A}$. We explore the construction of potentially nilpotent patterns over a field. We present classes of patterns which are potentially nilpotent over a field $\mathbb{F}$ if and only if the field $\mathbb{F}$ contains certain roots of unity. We then introduce some sparse patterns of order $n \geq 4$ which are spectrally arbitrary over $\mathbb{C}$ but not over $\mathbb{R}$. We also identify all irreducible patterns of order four which are potentially nilpotent over $\mathbb{R}$ or $\mathbb{C}$.

Key words. Nonzero pattern, Spectrum, Potentially nilpotent, Spectrally arbitrary, Nilpotent-Jacobian method.

AMS subject classifications. 15A18, 05C05, 05C50, 15B35.

1. Introduction. In this paper, we explore properties of the combinatorial structure of nilpotent matrices over various fields. In particular, we consider possible patterns of nonzero entries in nilpotent matrices. A (*zero-nonzero*) *pattern* $\mathcal{A}$ is a square matrix whose entries come from the set $\{*, 0\}$ where $*$ denotes a nonzero entry. Fix a field $\mathbb{F}$. We then set

$$Q(\mathcal{A}, \mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid a_{i,j} \neq 0 \Leftrightarrow \mathcal{A}_{i,j} = * \text{ for all } i, j\}.$$

The set $Q(\mathcal{A}, \mathbb{F})$ is often denoted $Q(\mathcal{A})$ when $\mathbb{F}$ is known. An element $A \in Q(\mathcal{A}, \mathbb{F})$ is called a *matrix realization* of $\mathcal{A}$. A pattern $\mathcal{A}$ is *potentially nilpotent* (PN) over $\mathbb{F}$ if there exists $A \in Q(\mathcal{A}, \mathbb{F})$ such that A is nilpotent, i.e., $A^k = 0$ for some integer $k \geq 1$.

Identifying potentially nilpotent patterns is partially motivated by the study of spectrally arbitrary patterns (see for example [1, 3, 11, 13]). An $n \times n$ pattern $\mathcal{A}$ is said to be a *spectrally arbitrary pattern over* $\mathbb{F}$ if for every degree n monic polynomial $q(x)$ with coefficients in $\mathbb{F}$, there is a realization of $\mathcal{A}$ whose characteristic polynomial equals $q(x)$. The Nilpotent-Jacobian Method (see [2]), one of the main techniques for determining whether a pattern $\mathcal{A}$ is spectrally arbitrary over $\mathbb{R}$ or $\mathbb{C}$, relies on finding a nilpotent realization of $\mathcal{A}$. The need for constructions of potentially nilpotent patterns

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was also identified in [7]. The paper [3] summarizes much of the current knowledge about potentially nilpotent patterns over $\mathbb{R}$, while the more recent paper [1] introduces some new constructions for patterns associated with trees.

Much of the research on spectrally arbitrary patterns and potentially nilpotent patterns has focused upon the situation that $\mathbb{F} = \mathbb{R}$. However, over the last couple of years, this hypothesis has been relaxed, and as a consequence, we have seen that questions about patterns are dependent on the properties of $\mathbb{F}$. The papers [10, 12, 14, 16] give examples of this approach. Our goal is to explore how the structure of a potentially nilpotent pattern over a field $\mathbb{F}$ relates to properties of the $\mathbb{F}$, and provide further examples of patterns that are potentially nilpotent over various fields, including $\mathbb{R}$ and $\mathbb{C}$.

We have structured our paper as follows. After we review necessary conditions for nilpotence in Section 2, in Section 3 we describe some new constructions of potentially nilpotent patterns. In Section 4 we identify some patterns which are potentially nilpotent over a field $\mathbb{F}$ precisely when $\mathbb{F}$ contains certain roots of unity. In Section 5, we show that some of the sparse patterns, including some patterns described in Section 4, are spectrally arbitrary over $\mathbb{C}$ but not $\mathbb{R}$, making use of work developed in Section 4. In our last section, we identify all the irreducible patterns of order four that are potentially nilpotent over $\mathbb{R}$ and $\mathbb{C}$. The results of this section can be seen as an extension of the work of Corpuz and McDonald [4], who identified some of the potentially nilpotent irreducible patterns of order four over $\mathbb{R}$ (namely those that are spectrally arbitrary) and Yielding [16], who considered the same problem over $\mathbb{C}$. Nilpotent patterns of order 3 are characterized over $\mathbb{R}$ by Eschenbach and Li [8] and over other fields by Vander Meulen and Van Tuyl [14]. The observations of Section 6 are the result of an exhaustive computer search. The output of our search is included in the appendix which contains a nilpotent realization for all irreducible patterns of order four over $\mathbb{R}$ or $\mathbb{C}$.

2. Necessary conditions. The property of being potentially nilpotent is invariant under transposition and permutation similarity. Two patterns are *equivalent* if one can be obtained from the other via transposition and/or permutation similarity. If a pattern $\mathcal{A}$ is reducible, that is, if $\mathcal{A}$ is equivalent to a block upper triangular matrix $\mathcal{B}$, then $\mathcal{A}$ is potentially nilpotent if and only if each of the diagonal blocks of $\mathcal{B}$ are potentially nilpotent. Thus, we focus our attention on irreducible patterns.

One way to analyze the combinatorial structure of a pattern is to associate a digraph with the pattern. In particular, the digraph $D(\mathcal{A})$ of an $n \times n$ pattern $\mathcal{A}$ is defined to be a digraph on the vertex set $V = \{1, \ldots, n\}$, whose edge set consists of

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the arcs (i,j) whenever $(\mathcal{A})_{i,j} \neq 0$. For example, the digraph of the pattern

$$\begin{bmatrix} 0 & 0 & 0 & * \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & * & 0 & 0 \end{bmatrix}$$

is the digraph in Figure 3.1. A *cycle* in a pattern $\mathcal{A}$ (or in a matrix $A \in Q(\mathcal{A})$) corresponds to a cycle in the digraph $D(\mathcal{A})$. For $k \geq 1$, we say a (simple) k*-cycle* of $\mathcal{A}$ is a nonzero product $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}a_{i_k,i_1}$ with k distinct indices $\{i_1, i_2, \ldots, i_k\}$. We call a 1-cycle a *loop.* A *composite* k*-cycle* is a product of disjoint (simple) cycles of $\mathcal{A}$ using nonzero entries from exactly k rows (and corresponding columns) of $\mathcal{A}$. A *walk* from i_1 to i_k is a nonzero product $a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_{k-1},i_k}$ where the indices are not necessarily distinct. A walk with distinct indices is called a *path* and is often denoted $i_1 \to i_2 \to \cdots \to i_k$.

The *sign* of a simple k-cycle is $(-1)^{k-1}$. The sign of a composite cycle is the product of the signs of its simple cycles. The next observation has been useful in analyzing the implications of the combinatorial structure of a matrix A. The characteristic polynomial of A is of the form

$$p_A(x) = x^n - E_1x^{n-1} + E_2x^{n-2} - \cdots + (-1)^nE_n \tag{2.1}$$

where E_k is the sum of all signed k-cycles (simple and composite). When A is nilpotent, then $p_A(x) = x^n$, a fact that will exploited throughout this paper.

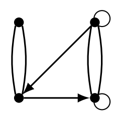
We end this section by highlighting two necessary conditions for a pattern to be potentially nilpotent, as originally observed in [8]:

Lemma 2.1. *Suppose $\mathcal{A}$ is potentially nilpotent. Then*

1. *whenever $D(\mathcal{A})$ has one k-cycle of length k, then it must have more than one k-cycle,*
2. *$D(\mathcal{A})$ cannot have a pair of vertices $\{i,j\}$ such that there is exactly one walk from i to j of length n.*

We refer to Lemma 2.1.1 as the *cycle condition* and Lemma 2.1.2 as the *walk condition.*

Example 2.2. Let $D(\mathcal{B})$ be the graph



Then $\mathcal{B}$ satisfies the cycle condition, but not the walk condition. Hence, $\mathcal{B}$ is not potentially nilpotent. In particular, $D(\mathcal{B})$ has exactly one walk of length 4 from

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vertex 1 to vertex 4 (label the vertices clockwise, starting from the top left vertex). It follows that if $A \in Q(\mathcal{B})$, then the entry in position $(1,4)$ of A^4 is nonzero.

3. Constructions. In this section, we introduce a construction for building potentially nilpotent patterns, thereby adding to the list of known constructions. We also introduce a way to add nonzero entries to a potentially nilpotent pattern that preserves the property of being potentially nilpotent over select fields.

Examples of constructions over $\mathbb{R}$ can be found in Eschenbach and Li [8]. They note that if $\mathcal{A}$ or $\mathcal{B}$ is potentially nilpotent, then $\mathcal{A} \otimes \mathcal{B}$ is potentially nilpotent, where $\otimes$ is the tensor product. Likewise, they note that potential nilpotence is closed under direct sums. Other constructions are explored in various papers (see for example [9, 11]), as well as the many papers which give constructions of spectrally arbitrary patterns. Eschenbach and Li [8] also consider the class of digraphs called C_r-cockades. One property of a C_r-cockade is that every simple cycle in the digraph is an r-cycle. Figure 3.1 is an example of a C_3-cockade. Eschenbach and Li prove that if $D(\mathcal{A})$ is an order n C_r-cockade with $n < 2r$, then $\mathcal{A}$ is potentially nilpotent over $\mathbb{R}$ if and only if $\mathcal{A}$ satisfies the cycle condition.

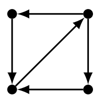

FIG. 3.1. *A potentially nilpotent C_3-cockade.*

A graph with at least one cycle is said to have a *center* vertex v if every simple cycle in the the graph includes the vertex v. The C_3-cockade in Figure 3.1 has two center vertices.

EXAMPLE 3.1. Figure 3.2 illustrates a construction of a digraph with a center vertex, having $m \geq 2$ (in the figure $m = 4$) cycles of length $a+b+c$ and m cycles of length $b+c+d$, with a, b, c and d nonzero. Such a digraph represents a potentially nilpotent pattern over $\mathbb{R}$. For example, one can obtain a nilpotent realization by assigning each arc the value 1 except for two arcs: one of the outside arcs corresponding to the ma arcs should be assigned value $-(m-1)$ and one of the outside arcs corresponding to the md arcs should be assigned value $-(m-1)$.

We describe a way to build potentially nilpotent patterns using patterns with a center vertex. We first introduce a definition. Given two graphs (or digraphs), say G and H, having one commonly labelled vertex v_m (e.g. $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $V(H) = \{v_m, v_{m+1}, \ldots, v_n\}$) we say the *merge* of G and H at v_m is the graph $G \cup H$ having vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edgeset (or arcset) $E(G) \cup E(H)$. We use the notation $A[i]$ to denote the submatrix of A obtained by deleting row i and column

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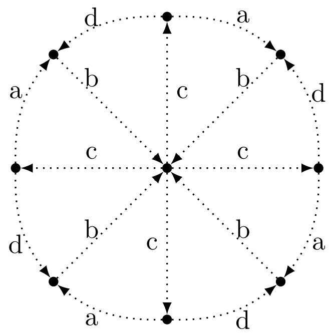

FIG. 3.2. *A graph with a center vertex and cycles of length $a+b+c$ and $b+c+d$.*

i of A.

THEOREM 3.2. *Let $\mathbb{F}$ be any field. Suppose $\mathcal{A}$ and $\mathcal{B}$ are potentially nilpotent patterns over $\mathbb{F}$ such that $D(\mathcal{A})$ has a center vertex v. Label any vertex of $D(\mathcal{B})$ as v. Then the pattern obtained by merging $D(\mathcal{A})$ with $D(\mathcal{B})$ at v is potentially nilpotent over $\mathbb{F}$.*

Proof. Let $A \in Q(\mathcal{A})$ and $B \in Q(\mathcal{B})$ be nilpotent matrices of order n and m, respectively. Note that a potentially nilpotent pattern with a center vertex will have no nonzero diagonal entries. If it did, it would have exactly one diagonal entry (at the center vertex), in which case the trace of the pattern would be nonzero. Assume that the n^{th} row and column of A corresponds to a center vertex. By (2.1), $p_{A[n]} = x^{n-1}$. The merge of A and B will result in the matrix M of the form

$$M = \begin{bmatrix} A[n] & \mathbf{v} & 0 \\ \mathbf{u}^T & b & \mathbf{y}^T \\ 0 & \mathbf{w} & B[1] \end{bmatrix}$$

for some $\mathbf{v}, \mathbf{u} \in \mathbb{F}^{n-1}$ $\mathbf{y}, \mathbf{w} \in \mathbb{F}^{m-1}$, and $b \in \mathbb{F}$. Observe that

$$\begin{aligned} p_M(x) &= \begin{vmatrix} xI - A[n] & -\mathbf{v} & 0 \\ -\mathbf{u}^T & x-b & -\mathbf{y}^T \\ 0 & -\mathbf{w} & xI - B[1] \end{vmatrix} \\ &= \begin{vmatrix} xI - A[n] & -\mathbf{v} & 0 \\ -\mathbf{u}^T & x & \mathbf{0}^T \\ 0 & -\mathbf{w} & xI - B[1] \end{vmatrix} + \begin{vmatrix} xI - A[n] & -\mathbf{v} & 0 \\ \mathbf{0}^T & x-b & -\mathbf{y}^T \\ 0 & -\mathbf{w} & xI - B[1] \end{vmatrix} \\ &\qquad - \begin{vmatrix} xI - A[n] & -\mathbf{v} & 0 \\ \mathbf{0}^T & x & \mathbf{0}^T \\ 0 & -\mathbf{w} & xI - B[1] \end{vmatrix} \\ &= p_A(x)p_{B[1]}(x) + p_{A[n]}(x)p_B(x) - xp_{A[n]}(x)p_{B[1]}(x) \end{aligned}$$

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$$= x^n p_{B[1]}(x) + x^{m+n-1} - x^n p_{B[1]}(x)$$
$$= x^{n+m-1}.$$

Thus, M is nilpotent. □

We say that $\mathcal{B}$ is a *superpattern* of $\mathcal{A}$ if $\mathcal{A}_{ij} \neq 0$ implies $\mathcal{B}_{ij} \neq 0$. In general, one cannot expect a superpattern of a potentially nilpotent pattern to be potentially nilpotent. For example, adding a nonzero diagonal entry to the pattern of a Jordan block produces a superpattern that is not potentially nilpotent. We conclude this section by observing a way of constructing a potentially nilpotent superpattern from a potentially nilpotent pattern that contains a directed path consisting of two or more center vertices.

We first note that if a digraph D has exactly $k > 1$ center vertices, then the subgraph of D induced by these vertices will be a directed path on k vertices and D will have no directed cycles of length k or less. Note that in the statement of the following theorem, the digraph $D(\mathcal{A})$ might have more than m center vertices.

We say a field $\mathbb{F}$ *contains the* m^{th} *roots of unity* if the polynomial $x^m - 1$ factors into linear forms over $\mathbb{F}$. That is, $x^m - 1 = (x-1)(x-\zeta_1)(x-\zeta_2)\cdots(x-\zeta_{m-1})$ for some $\zeta_1, \dots, \zeta_{m-1} \in \mathbb{F}$. Note that if we let $\zeta_0 = 1$ and

$$S_k = \sum_{0 \le i_1 < \cdots < i_k \le m-1} \zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_k} \tag{3.1}$$

then $S_k = 0$ for $1 \le k \le m-1$ and $S_m = (-1)^{m-1}$.

THEOREM 3.3. *Let $m \ge 2$ and $\mathbb{F}$ be a field containing the m^{th} roots of unity. Let $\mathcal{A}$ be a potentially nilpotent pattern over $\mathbb{F}$ having a digraph with $m \ge 2$ center vertices $v_1, v_2, \dots, v_m$ corresponding to the first m rows of $\mathcal{A}$, such that $v_1 \to v_2 \to \cdots \to v_m$ is a directed path in the digraph of $D(\mathcal{A})$. Let $\mathcal{A}'$ be a superpattern of $\mathcal{A}$ obtained by introducing nonzero entries in positions $(1,1), (2,2), \dots, (m,m)$ and $(m,1)$. Then $\mathcal{A}'$ is potentially nilpotent over $\mathbb{F}$.*

Proof. Note that the digraph $D(\mathcal{A}')$ is formed from $D(\mathcal{A})$ by making an induced cycle on $v_1, v_2, \dots, v_m$ with loops at each vertex. Let $A \in Q(\mathcal{A})$ be a nilpotent matrix. Let $B \in Q(\mathcal{A}')$ be obtained from A by inserting z_i in position (i,i) for $1 \le i \le m$ and w in position $(m,1)$. Note that $D(\mathcal{A})$ has no cycles of length m or smaller. Since every cycle of $D(\mathcal{A})$ includes all the center vertices of $D(\mathcal{A})$, there can be no composite cycles of $D(\mathcal{A}')$ which contain both a cycle of $D(\mathcal{A})$ as well as an arc not in $D(\mathcal{A})$. It follows that

$$p_B(x) = p_A(x) - f_1 x^{n-1} + f_2 x^{n-2} + \cdots + (-1)^{m-1} f_{m-1} x^{n-m+1} + f_m x^{n-m}$$

where

$$f_1 = z_1 + z_2 + \cdots + z_m$$

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$$\begin{aligned} f_2 &= z_1z_2 + \cdots + z_{m-1}z_m \\ &\vdots \\ f_{m-1} &= z_1z_2\cdots z_{m-1} + \cdots + z_2z_3\cdots z_m \qquad \text{and} \\ f_m &= (-1)^m z_1z_2\cdots z_m - wa_{1,2}a_{2,3}\cdots a_{m-1,m}. \end{aligned}$$

Set $z_1, z_2, \cdots, z_m$ to be the m^{th} roots of unity and $w = -(a_{1,2}a_{2,3}\cdots a_{m-1,m})^{-1}$. Then $p_B(x) = p_A(x) = x^n$. □

EXAMPLE 3.4. Consider the pattern corresponding to the C_3-cockade in Figure 3.1. This graph has two center vertices. The field $\mathbb{R}$ has both second roots of unity, i.e., $x^2 - 1$ factors into linear forms over $\mathbb{R}$. If we add loops to the two center vertices, and add on additional edge to make the direct path connecting the two centers a cycle, we obtain the digraph in Figure 3.3. It follows from Theorem 3.3 that this

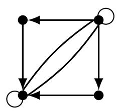

FIG. 3.3. *A potentially nilpotent superpattern of a C_3-cockade.*

superpattern of the C_3-cockade is potentially nilpotent over $\mathbb{R}$. In fact, the C_3 cockade is potentially nilpotent over every field (see the realization in the Appendix which only requires that $1 \in \mathbb{F}$). Thus, the superpattern represented in Figure 3.3 is potentially nilpotent over every field since $x^2 - 1$ factors into linear forms over every field.

4. Potentially nilpotent patterns over fields requiring roots of unity.

Let $\mathcal{A}_n$ be the pattern of order $n \geq 3$ whose digraph is an n-cycle with a loop at each vertex. The pattern $\mathcal{A}_n$ is equivalent to the $n \times n$ pattern

$$\mathcal{A}_n = \begin{bmatrix} * & 0 & \cdots & \cdots & 0 & * \\ * & * & 0 & & & 0 \\ 0 & * & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & * & * & 0 \\ 0 & \cdots & \cdots & 0 & * & * \end{bmatrix}.$$

As observed in [14], $\mathcal{A}_n$ is potentially nilpotent over a field $\mathbb{F}$ if and only if $x^n - 1$ factors into linear forms over $\mathbb{F}$. An important theorem from [14] that helped demonstrate this observation was the Loop Theorem:

THEOREM 4.1 ("Loop Theorem"). *Suppose $\mathcal{A}$ has $m \geq 2$ nonzero entries on the diagonal, and suppose that $D(\mathcal{A})$ has no simple k-cycles with $2 \leq k \leq m-1$. If $\mathcal{A}$ is*

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PN over $\mathbb{F}$, then $x^m - 1$ factors into m linear forms over $\mathbb{F}$ (and the diagonal entries of any nilpotent realization of $\mathcal{A}$ will be the roots of $x^m - 1$).

In this section, we describe some further patterns which are potentially nilpotent over a field $\mathbb{F}$ characterized by the availability of roots of unity in $\mathbb{F}$. In particular, we start by asking under what conditions the Loop Theorem has a converse.

QUESTION 4.2. Suppose $\mathcal{A}$ is irreducible and $\mathcal{A}$ satisfies the cycle conditions. Suppose further that $\mathcal{A}$ has exactly $m \geq 2$ nonzero entries on the diagonal and $D(\mathcal{A})$ has no k-cycles with $2 \leq k \leq m-1$. If $x^m - 1$ factors into linear forms over $\mathbb{F}$, does it follow that $\mathcal{A}$ is potentially nilpotent?

Considering $\mathcal{A}_n$, we see the answer is 'yes' when $m = n$, since, up to isomorphism $D(\mathcal{A}_n)$ is the only digraph with $m = n$ satisfying the cycle conditions. When $m = n-1$, then there are three classes of graphs to consider: we will show that the answer is 'yes' for first case, that the answer is also 'yes' for the second case if we add the condition that $\mathbb{F} \neq \mathbb{Z}_2$ and that the answer for the third class is 'yes' under the extra hypothesis that not all the roots of $x^m - 1$ are equal in $\mathbb{F}$.

LEMMA 4.3. *Suppose $\mathcal{A}$ is irreducible of order $n \geq 4$, $m = n-1$ and $\mathcal{A}$ satisfies the cycle conditions. Suppose further that $\mathcal{A}$ has exactly $m \geq 2$ nonzero entries on the diagonal and $D(\mathcal{A})$ has no k-cycles with $2 \leq k \leq m-1$. Then, ignoring loops, the digraph of $\mathcal{A}$ is one of the three digraphs in Figure 4.1. More specifically, $D(\mathcal{A})$ is one of the following:*

- *G_1 with one of its $n-1$ loops at w.*
- *G_2 with any $n-1$ of the vertices having a loop.*
- *G_3 with a loop at each of x and y, as well as any $n-3$ other vertices.*

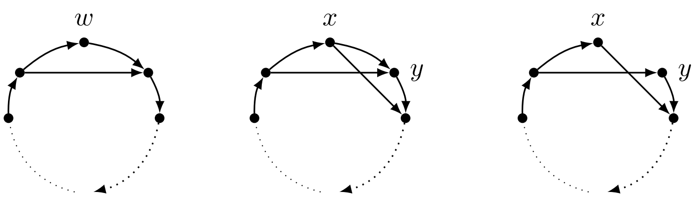

FIG. 4.1. *Digraphs G_1, G_2 and G_3.*

Proof. Suppose $\mathcal{A}$ has a simple n-cycle C. Then $\mathcal{A}$ must have a second (simple or composite) n-cycle in order to satisfy the cycle conditions. Thus, C must have a chord e. Since $\mathcal{A}$ has no k-cycle with $2 \leq k \leq n-2$, any chord of C must induce a 3-cycle in the underlying graph of $\mathcal{A}$. To avoid a 3-cycle in the digraph $D(\mathcal{A})$, e must be oriented to create an $(n-1)$-cycle in $\mathcal{A}$ as in G_1 in Figure 4.1. Thus, if C has exactly one chord, then $D(\mathcal{A})$, ignoring loops, must be G_1. In order to satisfy the

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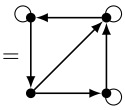
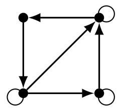
cycle conditions, in particular, since $\mathcal{A}$ has one n-cycle, one of the loops of $\mathcal{A}$ must be at vertex w in G_1 to obtain a composite n-cycle. The one vertex that is loopless in the digraph $D(\mathcal{A})$ can be any one of the remaining vertices of $D(\mathcal{A})$.

If C has two or more chords, then any two chords must cross (assuming C is drawn as a circle in the plane), otherwise $\mathcal{A}$ will have a k-cycle with $k < n-1$. Thus, if C has more than one chord, it must have exactly two chords. In this case, ignoring loops, $D(\mathcal{A})$ is G_2 in Figure 4.1. Any arrangement of the $n-1$ loops will result in a digraph satisfying the cycle conditions.

Suppose $\mathcal{A}$ has no simple n-cycle. Then since $\mathcal{A}$ has a composite $(n-1)$-cycle composed of loops, and $\mathcal{A}$ has no k-cycles with $k < n-1$, it follows that $\mathcal{A}$ must have an $(n-1)$-cycle $C = v_1 \to v_2 \to \cdots \to v_{n-1} \to v_1$. Let v_n be the vertex of $D(\mathcal{A})$ that is not on C. Since $\mathcal{A}$ is irreducible, there must be some i and j with $1 \le i, j \le n-1$ such that $v_i \to v_n$ and $v_n \to v_j$. Further, since $\mathcal{A}$ has no k-cycles with $k < n-1$, it follows that $j \equiv i + 2 \pmod{n-1}$. Thus, ignoring loops, $D(\mathcal{A})$ is G_3 in Figure 4.1. As for the placement of loops, there must be loops at both vertices x and y so that there is not exactly one (composite) n-cycle in $D(\mathcal{A})$. The placement of the remaining loops can be on any of the remaining vertices of $D(\mathcal{A})$. $\square$

The proof above indicates that, ignoring transposition, there will be $n-1$ non-equivalent patterns that satisfy the conditions of the lemma whose underlying graph is G_1, n nonequivalent patterns for G_2 and $n-2$ nonequivalent patterns for G_3. We will show that each of these graphs are potentially nilpotent (and in the next section that each of these graphs are spectrally arbitrary) over fields which contain roots of unity. We will introduce a definition of loop-equivalent along with Lemma 4.5 below in order to reduce the number of cases we need to consider.

Suppose D_1 is a digraph with a directed path $P = v_1 \to v_2 \to \cdots \to v_m$ such that, ignoring loops, the degree of each v_i is exactly 2 for $2 \le i \le m-1$ (for each vertex that is not an endpoint of P). We say a simple directed cycle C of D_1 is an *endpoint cycle of* P if an endpoint of P is on C but no arc of P is part of C. We say a vertex of P is *open* if it is either not an endpoint of P, or as an endpoint, if it is not contained in any endpoint k-cycle with $k > 1$. We say that a digraph D_2 is a *loop shift* of D_1 if D_2 is obtained from D_1 by moving a loop from an open vertex on P to another open vertex on P that did not have a loop. For example, if

$D_1 =$  and $D_2 =$ 

then D_2 is a loop shift of digraph D_1 (and vice versa).

LEMMA 4.4. *Suppose $D(\mathcal{A})$ is a loop shift of $D(\mathcal{B})$. If $A \in Q(\mathcal{A})$, then there*

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exists a matrix $B \in Q(\mathcal{B})$ *such that* $p_A(x) = p_B(x)$.

Proof. Suppose $D(\mathcal{A})$ is a loop shift of $D(\mathcal{B})$. Let $A \in Q(\mathcal{A})$. Let $B \in Q(\mathcal{B})$ be a matrix whose nonzero entries are the same as the corresponding nonzero entries in A. Let a be the nonzero diagonal position of A which has no match in B. Let b be the nonzero diagonal position of B which has no match in A. For every simple or composite k-cycle in $\mathcal{A}$ not using a, there is a corresponding k-cycle in $\mathcal{B}$ not using b. Likewise, for every simple or composite k-cycle in $\mathcal{A}$ using a, there is a corresponding k-cycle in $\mathcal{B}$ using b. Thus, if we set the entry of B in position b to have the same value as the entry of A in position a, then $E_k(A) = E_k(B)$ for $1 \leq k \leq n$. Hence, $p_A(x) = p_B(x)$. $\square$

If $D(\mathcal{A})$ can be obtained from $D(\mathcal{B})$ by a sequence of loop shifts, then we say that $\mathcal{A}$ is *loop equivalent* to $\mathcal{B}$. The next result is a consequence of Lemma 4.4.

Lemma 4.5. *If* $\mathcal{A}$ *is loop equivalent to a potentially nilpotent pattern* $\mathcal{B}$*, then* $\mathcal{A}$ *is potentially nilpotent.*

Theorem 4.6. *Suppose* $n \geq 4$ *and the digraph* G_1 *in Figure 4.1 has* $m = n-1$ *loops, including a loop at* w*. If* $D(\mathcal{A}) = G_1$*, then* $\mathcal{A}$ *is potentially nilpotent over* $\mathbb{F}$ *if and only if* $x^m - 1$ *splits into linear factors.*

Proof. Assume that $x^{(n-1)} - 1 = (x-1)(x-\zeta_1)(x-\zeta_2)\cdots(x-\zeta_{n-2})$. Suppose $D(\mathcal{A}) = G_1$. Note that $\mathcal{A}$ is loop equivalent to the nonzero pattern of the matrix

$$A = \begin{bmatrix} \zeta_{n-2} & 0 & \cdots & & & \cdots & 0 & -1 \\ 1 & 1 & 0 & & & & \ddots & 0 \\ 1 & 1 & \zeta_1 & 0 & & & & \vdots \\ 0 & 0 & 1 & \zeta_2 & \ddots & & & \\ \vdots & \ddots & 0 & 1 & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \zeta_{n-4} & 0 & 0 \\ \vdots & & & & 0 & 1 & \zeta_{n-3} & 0 \\ 0 & \cdots & & & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{4.1}$$

Then, using the notation of (3.1) with $m = n-1$,

$$p_A = x^n - S_1 x^{n-1} + S_2 x^{n-2} + \cdots + (-1)^{m-1} S_{m-1} x^2 + (1 + (-1)^m S_m)x = x^n$$

and so A is nilpotent. Since A is nilpotent, Lemma 4.5 implies that $\mathcal{A}$ is potentially nilpotent. The converse direction of the theorem follows from Theorem 4.1. $\square$

Theorem 4.7. *Suppose* $n \geq 4$ *and the digraph* G_2 *in Figure 4.1 has* $m = n-1$ *loops. If* $D(\mathcal{A}) = G_2$*, then* $\mathcal{A}$ *is potentially nilpotent over* $\mathbb{F}$ *if and only if* $x^m - 1$

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splits into linear factors and $\mathbb{F} \neq \mathbb{Z}_2$*.*

Proof. We will consider two cases.

Case 1: Suppose $D(\mathcal{A})$ is G_2 with loops at x and y and $n-3$ of the remaining vertices. Suppose $\mathcal{A}$ is potentially nilpotent. Let A be a nilpotent realization of $\mathcal{A}$. By Theorem 4.1, we know that $x^m - 1$ splits into linear forms and the diagonal entries of A are the roots $1, \zeta_1, \ldots, \zeta_{n-2}$ of $x^m - 1$. Via scaling, we can assume the $(2,2)$ entry of A is 1. By signature similarity, and loop equivalence, we can assume A is of the form

$$\begin{bmatrix} \zeta_{n-2} & 0 & \cdots & & & \cdots & 0 & c \\ 1 & 1 & 0 & & & & \ddots & 0 \\ a & 1 & \zeta_1 & 0 & & & & \vdots \\ 0 & b & 1 & \zeta_2 & \ddots & & & \\ \vdots & \ddots & 0 & 1 & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \zeta_{n-4} & 0 & 0 \\ \vdots & & & & 0 & 1 & \zeta_{n-3} & 0 \\ 0 & \cdots & & & \cdots & 0 & 1 & 0 \end{bmatrix}$$

where a, b, and c are nonzero. Then, using the notation of (3.1) with $m = n-1$,

$$p_A = x^n - S_1 x^{n-1} + \cdots + (-1)^{m-1} S_{m-1} x^2 + (-c(a+b) + (-1)^m S_m)x + c(a + b\zeta_1 - 1).$$

Since $E_n(A)$ has three nonzero summands, and $E_n(A) = 0$ (because A is nilpotent), it follows that $\mathbb{F} \neq \mathbb{Z}_2$. Conversely, if we set $a = 1 - b\zeta_1$, $c = -(a+b)^{-1}$, and choose b such that $a \neq 0$ and $(a+b) \neq 0$, then $p_A(x) = x^n$ and A is nilpotent. Hence, Lemma 4.5 implies that $\mathcal{A}$ is potentially nilpotent.

Case 2: Suppose that $D(\mathcal{A})$ is G_2 with loops at all vertices except x or y. Observe that the pattern with a loop at x and not y is equivalent, via transposition, to the pattern with a loop at y and not x. (One way to see this is to observe that the graph of the transpose of the pattern is obtained from the original by reversing all the arcs.) Thus, we need only consider the case where $\mathcal{A}$ has a loop at y and not at x.

Suppose $\mathcal{A}$ is potentially nilpotent. Let A be a nilpotent realization of $\mathcal{A}$. By Theorem 4.1, we know that $x^m - 1$ splits into linear forms and the diagonal entries of A are the roots $1, \zeta_1, \ldots, \zeta_{n-2}$ of $x^m - 1$. Via scaling, we can assume the $(n-1, n-1)$ entry of A is 1. By signature similarity, and loop equivalence, we can assume A is of

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the form

$$\begin{bmatrix} \zeta_1 & 0 & \cdots & \cdots & 0 & a & 1 \\ 1 & \zeta_2 & 0 & & & \ddots & 0 \\ 0 & 1 & \zeta_3 & \ddots & & & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \zeta_{n-2} & 0 & 0 \\ \vdots & & & 0 & 1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & b & c & 0 \end{bmatrix}$$

where a, b, and c are nonzero. Using the notation of (3.1) with $m = n-1$,

$$p_A = x^n - S_1 x^{n-1} + \cdots + (-1)^{m-1} S_{m-1} x^2 + (-(a+b) + (-1)^m S_m)x + (b-c).$$

Since $E_{n-1}(A)$ has three nonzero summands, and $E_{n-1}(A) = 0$ (because A is nilpotent), it follows that $\mathbb{F} \neq \mathbb{Z}_2$. Conversely, if we choose $a \neq -1$, and set $c = b = -a-1$, then A is nilpotent and Lemma 4.5 implies $\mathcal{A}$ is potentially nilpotent. □

THEOREM 4.8. *Suppose $n \geq 4$ and digraph G_3 in Figure 4.1 has $m = n-1$ loops, including one on x and one on y. If $D(\mathcal{A}) = G_3$, then $\mathcal{A}$ is potentially nilpotent over $\mathbb{F}$ if and only if $x^m - 1$ splits into linear factors, not all equal, over $\mathbb{F}$.*

Proof. Suppose $\mathcal{A}$ is potentially nilpotent. Let A be a nilpotent realization of $\mathcal{A}$. By Theorem 4.1, we know that the diagonal entries of A are the roots $1, \zeta_1, \ldots, \zeta_{n-2}$ of $x^m - 1$. Via scaling, we can assume the $(2,2)$ entry of A is 1. By signature similarity, and loop equivalence, we can assume A is of the form

$$\begin{bmatrix} \zeta_{n-2} & 0 & \cdots & & & \cdots & 0 & 1 \\ a & 1 & 0 & & & & \ddots & 0 \\ b & 0 & \zeta_1 & 0 & & & & \vdots \\ 0 & 1 & 1 & \zeta_2 & \ddots & & & \\ 0 & 0 & 0 & 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \zeta_{n-4} & 0 & 0 \\ \vdots & & & \ddots & 0 & 1 & \zeta_{n-3} & 0 \\ 0 & \cdots & & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{4.2}$$

where a and b are nonzero entries. Then, using the notation of (3.1) with $m = n-1$,

$$p_A = x^n - S_1 x^{n-1} + \cdots + (-1)^{m-1} S_{m-1} x^2 + (-(a+b) + (-1)^m S_m)x + (b + a\zeta_1).$$

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Note that if $\zeta_1 = 1$, then $E_n(A) = 0$ implies $E_{n-1}(A) \neq 0$. Thus, since A is nilpotent, $\zeta_1 \neq 1$. Therefore, not all the roots of $x^m - 1$ are equal. Conversely, assuming that not all the roots of $x^m - 1$ are equal, we can arrange the diagonal entries so that $\zeta_1 \neq 1$. In this case, set $b = -a\zeta_1$ and $a = -(1-\zeta_1)^{-1}$ to obtain $p_A(x) = x^n$ and A nilpotent. Thus, by Lemma 4.5, $\mathcal{A}$ is potentially nilpotent. ☐

The Loop Theorem does not have a converse when $m = n-2$. In particular, there are some $n \times n$ patterns that satisfy the cycle conditions (and the path conditions), have $m = n-2$ loops, and have no k-cycles with $2 \leq k \leq m-1$, but which fail to be potentially nilpotent over *every* field. In particular, let $n \geq 6$ and consider an $n \times n$ pattern $\mathcal{A}$ having the digraph in Figure 4.2 with loops at $n-2$ vertices including vertices $1, 2, 3$ and n. Let $A \in Q(\mathcal{A})$. By signature similarity we can assume $a_{i,i+1} = 1$ for $i = 1, \ldots, n-1$. Suppose further that $a_{2,2} = b$, $a_{n-2,n} = a$, $a_{n,1} = t$, and $a_{1,3} = c$, with b, a, t and c nonzero. We will observe that if $E_n(A) = 0$, then $E_{n-1}(A) \neq 0$, and hence A cannot be nilpotent. Specifically, by reflecting on the cycle structure of the pattern, we note that $E_n(A) = (-1)^{n-1}t + (-1)^{n-2}bct$ and $E_{n-1}(A) = (-1)^{n-2}(at + ct) + (-1)^{n-3}abct$. If $E_n(A) = 0$, then $b = c^{-1}$, in which case $E_{n-1}(A) = (-1)^{n-2}ct \neq 0$. Therefore, even though the pattern $\mathcal{A}$, which has $m = n-2$ loops, satisfies the cycle conditions, $\mathcal{A}$ is not potentially nilpotent over *any* field.

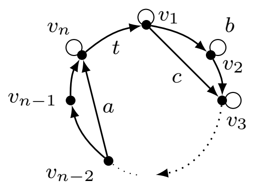

FIG. 4.2. *A digraph with no k-cycles,* $2 \leq k \leq n-3$.

5. Spectrally arbitrary patterns over $\mathbb{C}$. In this section, we describe some sparse $n \times n$ patterns which are spectrally arbitrary over $\mathbb{C}$ but are not spectrally arbitrary over $\mathbb{R}$. In [16], Yielding showed that $\mathcal{A}_3$ and $\mathcal{A}_4$ are spectrally arbitrary over $\mathbb{C}$ but not over $\mathbb{R}$, and demonstrated that if a pattern has a sufficient number of nonzero entries, it is necessarily spectrally arbitrary over $\mathbb{C}$. Below, we show that the sparse pattern $\mathcal{A}_n$ is spectrally arbitrary over $\mathbb{C}$ for all $n \geq 3$ and that the sparse potentially nilpotent patterns in Section 4 corresponding to the graphs G_1, G_2 and G_3 are spectrally arbitrary over $\mathbb{C}$. All superpatterns of the these patterns are also spectrally arbitrary over $\mathbb{C}$.

The main technique for determining if a pattern is spectrally arbitrary is known as the Nilpotent-Jacobian method (see Britz et al. [2, Lemma 2.1]). Introduced by Drew et al. [6] for real matrices, it was observed to be valid over $\mathbb{C}$ in [16]. Starting

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with a nilpotent realization N, the technique involves showing that an associated Jacobian matrix is non-singular. Making use of work of Pereira [13], Bergsma et al. [1] have shown that determining that the associated Jacobian matrix is nonsingular is equivalent to finding an independent set of n polynomials in $\operatorname{adj}(xI - N)^T$, in n positions corresponding to nonzero positions of N. These observations give us the following theorem.

THEOREM 5.1. *Suppose N is a nilpotent realization of an $n \times n$ pattern $\mathcal{A}$. If n of the entries in $\operatorname{adj}(xI - N)^T$, corresponding to n nonzero positions of $\mathcal{A}$, form a linearly independent set of polynomials, then every superpattern of $\mathcal{A}$ is spectrally arbitrary.*

As noted in Section 4, the pattern $\mathcal{A}_n$ is known to be potentially nilpotent over $\mathbb{F}$ if and only if $\mathbb{F}$ contains the n^{th} roots of unity. Suppose $x^n - 1 = (x - \zeta_1)(x - \zeta_2)\cdots(x - \zeta_n)$, where $\zeta_n = 1$ in $\mathbb{F}$. It was observed in [14] that a particular nilpotent realization of $\mathcal{A}_n$ is the matrix N with

$$N_{i,j} = \begin{cases} \zeta_i & \text{if } i = j,\ 1 \le i \le n \\ 1 & \text{if } i = j+1 \text{ and } 1 \le j \le n-1 \\ -1 & \text{if } i = 1 \text{ and } j = n \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 5.2. *For $n \geq 3$, every superpattern of $\mathcal{A}_n$ is spectrally arbitrary over $\mathbb{C}$.*

Proof. Observe that the n diagonal entries of $\operatorname{adj}(xI - N)^T$ form a linearly independent set. In particular, note that $[\operatorname{adj}(xI - N)]_{k,k} = \prod_{i \neq k}(x - \zeta_i)$ for $1 \le k \le n$. Further

$$P = \left\{ \prod_{i\neq 1}(x - \zeta_i), \prod_{i \neq 2}(x - \zeta_i), \dots, \prod_{i \neq n}(x - \zeta_i) \right\}$$

is a set of n linearly independent polynomials because P consists of scaled Lagrange polynomials (and $\zeta_i \neq \zeta_j$ if $i \neq j$, $1 \le i < j \le n$). The result follows from Theorem 5.1. $\square$

THEOREM 5.3. *Suppose $n \geq 4$ and $\mathcal{A}$ has digraph G_1 in Figure 4.1 with $m = n-1$ loops, including a loop at w. Then every superpattern of $\mathcal{A}$ is spectrally arbitrary over $\mathbb{C}$.*

Proof. Let N be the matrix of (4.1) with $x^{n-1} - 1 = (x - \zeta_0)(x - \zeta_1)\cdots(x - \zeta_{n-2})$ and $\zeta_0 = 1$. Let $W = \operatorname{adj}(xI - N)^T$ and $P = \{W_{1,1,}, W_{2,2}, \dots, W_{n-1,n-1}, W_{2,1}\}$. Note

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that

$$W_{1,1} = x\prod_{i=0}^{n-3}(x-\zeta_i), \qquad W_{2,2} = x\prod_{i=1}^{n-2}(x-\zeta_i)+1, \quad \text{and} \qquad W_{k,k} = x\prod_{\substack{i=0\\ i\neq k-2}}^{n-2}(x-\zeta_i)$$

for $3 \le k \le n-1$. Further, $W_{2,1} = -1$. Thus,

$$\operatorname{span}(P) = \operatorname{span}\left(\left\{x\prod_{i\neq 0}(x-\zeta_i), x\prod_{i\neq 1}(x-\zeta_i), \dots, x\prod_{i\neq n-2}(x-\zeta_i), 1\right\}\right).$$

Therefore $\dim(\operatorname{span}(P)) = n$ and P is a linearly independent set. The result follows from Theorem 5.1 and Theorem 4.6. ▯

Note that the patterns described in Theorem 4.7 (corresponding to the graph G_2) are superpatterns of those in Theorem 4.6 (corresponding to the graph G_1). Therefore these patterns too are spectrally arbitrary over $\mathbb{C}$. Finally we consider the patterns associated with the graph G_3.

THEOREM 5.4. *Suppose $n \ge 4$ and $\mathcal{A}$ has digraph G_3 in Figure 4.1 with $m = n-1$ loops, including a loop at w. Then every superpattern of $\mathcal{A}$ is spectrally arbitrary over $\mathbb{C}$.*

Proof. Let N be the matrix of (4.2) where $b = -a\zeta_1$ and $a = \frac{-1}{1-\zeta_1}$. For convenience, we let $\zeta_0 = 1$. Let $W = \operatorname{adj}(xI - N)^T$ and consider the set of $n+1$ polynomials

$$P = \{W_{1,1,}, W_{2,2}, \dots, W_{n-1,n-1}, W_{3,1}, W_{1,n}\}.$$

Note that

$$W_{1,1} = x\prod_{i=0}^{n-3}(x-\zeta_i), \qquad W_{2,2} = x\prod_{i=1}^{n-2}(x-\zeta_i)+b, \qquad W_{3,3} = x\prod_{\substack{i=0\\ i\neq 1}}^{n-2}(x-\zeta_i)+a$$

and, for $4 \le k \le n-1$,

$$W_{k,k} = x\prod_{\substack{i=0\\ i\neq k-2}}^{n-2}(x-\zeta_i).$$

Further, $W_{3,1} = x-1$ and $W_{1,n} = a(x-\zeta_1)+b(x-1) = -x$. It follows that $\operatorname{span}(P) = \operatorname{span}(\{W_{1,1,}, W_{2,2}-b, W_{3,3}-a, W_{4,4}, \dots, W_{n-1,n-1}, x, 1\})$. In fact, $\dim(\operatorname{span}(P)) = n$ since

$$\operatorname{span}(P) = \operatorname{span}\left(\left\{x\prod_{i\neq 0}(x-\zeta_i), x\prod_{i\neq 1}(x-\zeta_i), \dots, x\prod_{i\neq n-2}(x-\zeta_i), 1\right\}\right).$$

Thus, P contains a set of n linearly independent polynomials. The result now follows from Theorem 5.1 and Theorem 4.6. ▯

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6. Potentially nilpotent patterns of order 4. The goal of this section is to describe which irreducible patterns of order four are potentially nilpotent over $\mathbb{R}$ or $\mathbb{C}$. This section builds upon the work of Corpuz and McDonald [4], who identified the irreducible patterns of order four that are spectrally arbitrary over $\mathbb{R}$, and Yielding [16], who considered the same problem over $\mathbb{C}$ (see also [12]).

To investigate the question of which patterns are potentially nilpotent, we initiated a computer search to find all 4×4 irreducible patterns that satisfied both the cycle and walk conditions. The Appendix of this paper contains the matrices found in this computer search.

We quickly describe how the Appendix is divided. We have grouped together the irreducible patterns $\mathcal{A}$ that satisfy both the cycle and walk conditions according to their graph $D(\mathcal{A})$, ignoring loops. Each row of the table then corresponds to a matrix realization of $\mathcal{A}$ that is nilpotent. The entries of the matrices belong to either $\mathbb{R}$ or $\mathbb{C}$ (except in some special cases, which is explained below, where entries are denoted by $*$) and complex entries are only present if needed (see Observation 6.2). The last column contains an R or a C if the corresponding pattern was already known to be spectrally arbitrary over $\mathbb{R}$ or $\mathbb{C}$ (and hence, potentially nilpotent) by the work of either [4] or [15]. The last column contains an N if the pattern is potentially nilpotent, but not spectrally arbitrary over $\mathbb{R}$ or $\mathbb{C}$.

This exhaustive computer search enables us to observe a number of interesting results. First, we have identified all the 4×4 irreducible patterns which satisfy both the walk and cycle conditions, but which are not potentially nilpotent over any field. There are only three patterns that have this property; these patterns are represented in the tables with $*$'s in the matrix.

OBSERVATION 6.1. Suppose $\mathcal{A}$ is an irreducible 4×4 pattern that satisfies both the cycle and walk conditions. Then there is no field over which $\mathcal{A}$ is potentially nilpotent if and only if $D(\mathcal{A})$ is in Figure 6.1.

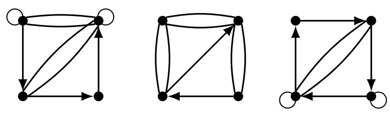

FIG. 6.1. *Not potentially nilpotent over any field.*

In particular, if $\mathcal{A}$ is an irreducible order four pattern satisfying the cycle and walk conditions, and if $D(\mathcal{A})$ is not in Figure 6.1, then $\mathcal{A}$, along with a nilpotent realization over $\mathbb{R}$ or $\mathbb{C}$, appears in the table in Appendix. It would suffice to show that none of the patterns with a digraph in Figure 6.1 can be potentially nilpotent. Let us consider the first graph $D(\mathcal{A})$ in Figure 6.1. If this pattern were nilpotent, we

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would be able to find nonzero elements $a, b, c, d, e, f, g, h, i$ in some field $\mathbb{F}$ such that the matrix

$$A = \begin{bmatrix} a & b & 0 & c \\ d & e & 0 & f \\ 0 & g & 0 & 0 \\ 0 & h & i & 0 \end{bmatrix}$$

is nilpotent. The characteristic polynomial of A is given by

$$p_A(x) = x^4 + (-a-e)x^3 + (-bd+ae-fh)x^2 + (-cdh-fgi+afh)x + (-cdgi+afgi).$$

Since A is nilpotent, $p_A(x) = x^4$, so the coefficient of x^k is zero for $k = 0, \ldots, 3$. In particular, $gi(-cd+af) = 0$, and because $gi \neq 0$, $cd = af$. But if $cd = af$, this means

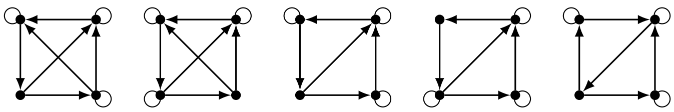
$$-cdh - fgi + afh = -fgi = 0,$$

i.e., $f = 0$ or $g = 0$ or $i = 0$, which is not permissible. The arguments for the other two patterns in Figure 6.1 are similar; we omit the details.

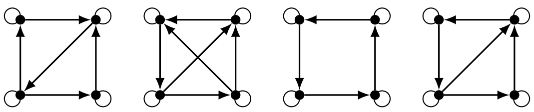
A number of the realizations in the Appendix contain entries in $\mathbb{C}$. As the next result demonstrates, these matrices are precisely the matrices that are potentially nilpotent over $\mathbb{C}$, but not over $\mathbb{R}$.

Observation 6.2. Let $\mathcal{A}$ be a 4×4 irreducible pattern that satisfies both the walk and cycle conditions. Then $\mathcal{A}$ is potentially nilpotent over $\mathbb{C}$ but not over $\mathbb{R}$ if and only if $D(\mathcal{A})$ is one of the following graphs:

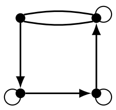
(i)



(ii)



(iii)



In particular, if $D(\mathcal{A})$ is not one of the above graphs, then a nilpotent realization for $\mathcal{A}$ over $\mathbb{R}$ is given in the Appendix. To see the converse, we first note that each pattern given above is potentially nilpotent over $\mathbb{C}$ as realized in the Appendix. (In fact, each of the patterns in Observation 6.2 are spectrally arbitrary over $\mathbb{C}$.) It therefore would suffice to show that the patterns are not potentially nilpotent over $\mathbb{R}$.

We actually have stronger results for the patterns of Observation 6.2(i). By Theorems 4.6 and 4.7, the first four patterns of (i) are potentially nilpotent over a

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field $\mathbb{F}$ if and only if $x^3 - 1$ factors into linear forms over $\mathbb{F}$. Further, by Theorem 4.8, the last pattern represented in (i) is potentially nilpotent over a field $\mathbb{F}$ if and only if $x^3 - 1$ factors into linear forms, not all equal, over $\mathbb{F}$.

The patterns with digraphs in Observation 6.2(ii) all fail to be nilpotent over $\mathbb{R}$ since they do not contain a 2-cycle (e.g., see [5, Lemma 3.2] or Theorem 4.1).

The pattern $\mathcal{A}$ with digraph in Observation 6.2(iii) is an oddity since it does not fit under either (i) or (ii). Suppose $A \in Q(\mathcal{A})$ is a nilpotent matrix. By scaling, we can assume one of the diagonal entries of A is 1 and by signature similarity (see for example [2, Lemma 2.3]) we can assume that A has the form

$$A = \begin{bmatrix} 0 & a & 0 & b \\ 1 & c & 0 & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

with characteristic polynomial

$$p_A(x) = x^4 + (-1 - d - c)x^3 + (d + c + dc - a)x^2 + (-dc + a + da)x + (-b - da).$$

Since A is nilpotent, $p_A(x) = x^4$. It follows from the coefficient of x^3 that $c = -1 - d$, and from the coefficient of x^2 that $a = d + c + dc$. Thus, the coefficient of x is $-1 - d - d^2 - d^3$. This implies that $d \in \{-1, i, -i\}$. But $d \neq -1$ otherwise $c = 0$. Therefore, $d \notin \mathbb{R}$.

By work of Corpuz and McDonald [4], we already know a large class of potentially nilpotent irreducible patterns of order four over $\mathbb{R}$, namely, those that are spectrally arbitrary. Therefore, to classify all potentially nilpotent irreducible patterns of order four over $\mathbb{R}$, we only need to determine which patterns are not covered in [4]. The desired patterns correspond to all entries in the Appendix with an N or $\mathbb{C}$ in the last column. We summarize our findings as follows:

Observation 6.3. Suppose that $\mathcal{A}$ is an irreducible 4×4 pattern that satisfies both the cycle and walk conditions, and furthermore, suppose $\mathcal{A}$ is not spectrally arbitrary over $\mathbb{R}$. Then $\mathcal{A}$ is potentially nilpotent over $\mathbb{R}$ if and only if $D(\mathcal{A})$ is in Figure 6.2 or Figure 6.3.

It is interesting to note that for many of the patterns of Observation 6.3 it is quite easy to see that the pattern is not spectrally arbitrary. Indeed, for a pattern $\mathcal{A}$ to be spectrally arbitrary, then $D(\mathcal{A})$ must have at least two loops. The majority of the digraphs in Figure 6.2 fail to have this property.

Finally, we can identify the irreducible patterns of order four which are potentially nilpotent over $\mathbb{C}$ by using the fact that Yielding [16] has already identified all the spectrally arbitrary patterns over $\mathbb{C}$.

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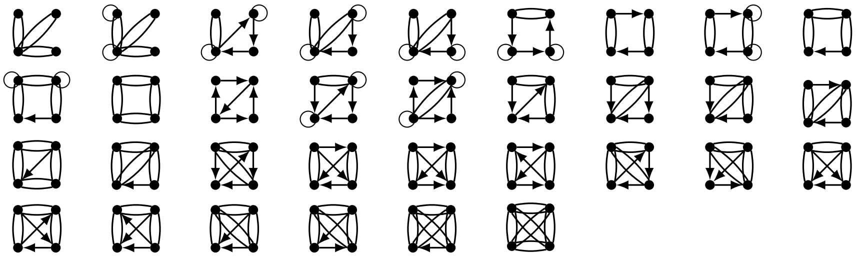

FIG. 6.2. *Potentially nilpotent over* $\mathbb{R}$*; not spectrally arbitrary over* $\mathbb{R}$ *or* $\mathbb{C}$*.*

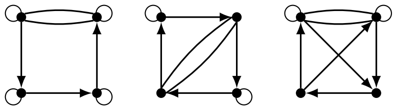

FIG. 6.3. *Potentially nilpotent over* $\mathbb{R}$*; spectrally arbitrary over* $\mathbb{C}$ *but not* $\mathbb{R}$

OBSERVATION 6.4. Suppose that $\mathcal{A}$ is an irreducible 4×4 pattern that satisfies both the cycle and walk conditions, and furthermore, suppose $\mathcal{A}$ is not spectrally arbitrary over $\mathbb{C}$. Then $\mathcal{A}$ is potentially nilpotent over $\mathbb{C}$ if and only if $D(\mathcal{A})$ is one of the graphs of Figure 6.2.

We do not have any examples of patterns $\mathcal{A}$ for which $\mathcal{A}$ is potentially nilpotent over $\mathbb{C}$ but not over $\mathbb{R}$, and $\mathcal{A}$ is not spectrally arbitrary over $\mathbb{C}$.

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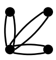
Appendix: Potentially nilpotent patterns of order 4 over $\mathbb{C}$ and $\mathbb{R}$. This appendix lists the order 4 irreducible patterns, up to equivalence, that satisfy both the cycle condition and the walk condition. For each digraph in the first column, there are a number of nonequivalent patterns having that digraph when ignoring loops: each row beside the digraph represents a pattern with a placement of loops that satisfies the walk and cycle conditions. The rows with asterisks represent irreducible patterns that satisfy the walk and cycle conditions but are not potentially nilpotent in any field. Those with numbers indicate a nilpotent realization over $\mathbb{R}$ *except* those that contain $i \in \mathbb{C}$. In the latter case, the patterns are potentially nilpotent only over $\mathbb{C}$, but not $\mathbb{R}$. The last column contains an R if the corresponding pattern is known to be spectrally arbitrary over $\mathbb{R}$ by [4, Theorem 1.2]. The last column contains a C if the corresponding pattern is known to be spectrally arbitrary over $\mathbb{C}$ by [15, Theorem 3.5.1]. Rows that contain an N correspond to patterns that are potentially nilpotent (over $\mathbb{R}$ or $\mathbb{C}$), but not spectrally arbitrary over $\mathbb{C}$.

For graphs labeled with an A (resp., B), the vertices are ordered clockwise 1234 (resp., 1324) starting at the top left vertex.

1 A		[0	0	0	1;	0	0	0	1;	0	0	0	−2;	1	1	1	0]	N
		[1	0	0	−1;	0	0	0	1;	0	0	0	1;	1	−1	1	−1]	N
		[1	0	0	$\frac{1}{4}$;	0	−3	0	$-\frac{81}{20}$;	0	0	2	$-\frac{16}{5}$;	1	1	1	0]	R
		[1	0	0	$\frac{1}{3}$;	0	−2	0	$-\frac{4}{3}$;	0	0	2	−4;	1	1	1	−1]	R

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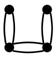
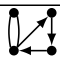
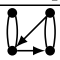
2		[1	0	0	a_1;	0	−1	a_1	0;	0	1	0	a_2;	1	0	1	0]	R
A		[0	0	0	−2;	0	−2	−2	0;	0	1	1	1;	1	0	1	1]	R
		[1	0	0	−1;	0	1	−1	0;	0	1	0	−1;	1	0	1	−2]	R
		[1	0	0	$-\frac{1}{2}$;	0	−1	$-\frac{1}{2}$	0;	0	1	$\frac{1}{4}$	$-\frac{1}{16}$;	1	0	1	$-\frac{1}{4}$]	R

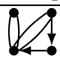
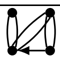
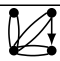
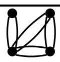
where $a_1 = 1+\sqrt{2}$ and $a_2 = -3-2\sqrt{2}$

3		[0	0	0	1;	0	1	-1	0;	0	0	0	1;	−1	1	0	−1]	N
A		[1	0	0	1;	0	a_1	1	0;	0	0	a_2	1;	a_3	a_4	0	0]	R
		[1	0	0	−9;	0	$\frac{4}{3}$	$-\frac{256}{27}$	0;	0	0	$\frac{4}{3}$	1;	1	1	0	$-\frac{11}{3}$]	R

where $a_1 = -\frac{1}{2}+\frac{1}{2}\sqrt{3+2\sqrt{5}}$, $a_2 = -\frac{1}{2}-\frac{1}{2}\sqrt{3+2\sqrt{5}}$, $a_3 = -\frac{3}{2}-\frac{\sqrt{5}}{2}$, $a_4 = 2+\sqrt{5}$

4		[1	0	0	a_2;	0	0	a_5	a_6;	0	1	−1	0;	1	0	1	0]	R
A		[0	0	0	−4;	0	1	1	−8;	0	1	1	0;	1	0	1	−2]	R
		[1	0	0	a_1;	0	0	$-2\sqrt{2}$	a_6;	0	1	−2	0;	1	0	1	1]	R
		[1	0	0	a_2;	0	1	a_3	a_4;	0	1	−2	0;	1	0	1	0]	R
		[2	0	0	−1;	0	−1	−4	−9;	0	1	−2	0;	1	0	1	1]	R

where $a_1 = -3+2\sqrt{2}$, $a_2 = -\frac{3}{2}+\frac{\sqrt{5}}{2}$, $a_3 = -\frac{3}{2}-\frac{\sqrt{5}}{2}$, $a_4 = 2-\sqrt{5}$, $a_5 = \frac{1}{2}-\frac{\sqrt{5}}{2}$, $a_6 = 8-8\sqrt{2}$

5		[0	0	0	1;	0	1	1	−2;	0	0	0	1;	1	1	0	−1]	N
A		[1	0	0	$-\frac{1}{2}$;	0	0	$\frac{1}{2}$	$-\frac{1}{2}$;	0	0	−1	1;	1	1	0	0]	R
		[0	0	0	1;	0	0	−1	1;	0	0	1	1;	−2	1	0	−1]	N
		[1	0	0	1;	0	0	−16	−8;	0	0	2	1;	1	1	0	−3]	R
		[1	0	0	$\frac{1}{4}$;	0	−3	−16	$-\frac{29}{4}$;	0	0	2	1;	1	1	0	0]	R
		[1	0	0	$-\frac{1}{4}$;	0	−1	$\frac{1}{2}$	$-\frac{7}{4}$;	0	0	−1	1;	1	1	0	1]	R
6		[0	0	0	−1;	0	1	−1	1;	0	1	−1	1;	1	1	0	0]	R
A		[1	0	0	$-\frac{1}{2}$;	0	0	−1	$\frac{1}{2}$;	0	1	0	$-\frac{1}{2}$;	1	1	0	−1]	R
		[1	0	0	−1;	0	0	1	−1;	0	1	−1	2;	1	1	0	0]	R
		[1	0	0	−1;	0	−1	1	−1;	0	1	0	1;	1	1	0	0]	R
		[0	0	0	$-\frac{3}{8}$;	0	1	$-\frac{1}{2}$	$\frac{1}{8}$;	0	1	$-\frac{1}{2}$	$\frac{1}{2}$;	1	1	0	$-\frac{1}{2}$]	R
		[−2	0	0	1;	0	0	1	1;	0	−2	1	1;	−2	1	0	1]	R
		[1	0	0	−1;	0	1	−1	−1;	0	1	0	−1;	1	1	0	−2]	R
		[−2	0	0	1;	0	1	1	1;	0	1	1	1;	−2	−2	0	0]	R
		[1	0	0	1;	0	−1	1	−4;	0	1	1	−8;	1	1	0	−1]	R
7		[1	0	0	$-\frac{1}{2}$;	0	0	$\frac{1}{2}$	$\frac{1}{2}$;	0	0	−1	−1;	1	1	1	0]	R
A		[1	0	0	$-\frac{1}{3}$;	0	0	$-\frac{2}{3}$	$-\frac{2}{3}$;	0	0	−2	−2;	1	1	1	1]	R
		[1	0	0	$\frac{1}{4}$;	0	2	$-\frac{61}{4}$	$-\frac{25}{4}$;	0	0	−3	−1;	1	1	1	0]	R
		[1	0	0	$-\frac{1}{4}$;	0	−1	$-\frac{1}{2}$	$-\frac{3}{4}$;	0	0	−1	−1;	1	1	1	1]	R
8		[0	0	0	−2;	0	1	−1	1;	0	1	−1	1;	1	1	1	0]	R
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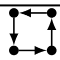
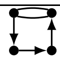
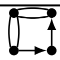
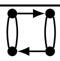
		[1	0	0	$-\frac{1}{3}$;	0	0	1	$\frac{2}{3}$;	0	−1	−1	$-\frac{1}{3}$;	1	1	1	0]	R
		[0	0	0	$-\frac{9}{8}$;	0	1	−2	$\frac{1}{4}$;	0	1	−2	$-\frac{1}{8}$;	1	1	1	1]	R
		[1	0	0	−1;	0	0	1	$-\frac{2}{3}$;	0	−1	1	$-\frac{1}{3}$;	1	1	1	−2]	R
		[1	0	0	$\frac{1}{2}$;	0	1	2	$\frac{7}{4}$;	0	1	−2	$-\frac{29}{4}$;	1	1	1	0]	R
		[1	0	0	1;	0	1	1	−2;	0	1	−1	−2;	1	1	1	−1]	R
9		[1	0	0	−1;	1	i	0	0;	0	1	$-i$	0;	0	0	1	−1]	C
A																		
10		[0	0	−1	−1;	0	1	1	0;	1	0	0	0;	0	1	0	−1]	N
B		[0	0	$-i$	−1;	0	1	1	0;	1	0	$-1-i$	0;	0	1	0	i]	C
		[1	0	−2	−1;	0	1	1	0;	1	0	−1	0;	0	1	0	−1]	C
11		[0	0	$-\frac{4}{3}$	$-\frac{5}{3}$;	0	1	$\frac{3}{5}$	0;	1	0	1	0;	1	1	0	−2]	R
B		[1	0	$\frac{1}{4}$	−1;	0	$-\frac{1}{2}$	$\frac{1}{16}$	0;	1	0	0	0;	1	1	0	$-\frac{1}{2}$]	R
		[1	0	−1	−1;	0	1	1	0;	1	0	−1	0;	1	1	0	−1]	R
12		[0	0	1	−1;	0	0	1	−1;	0	1	0	0;	1	0	0	0]	N
B		[0	0	$\frac{1}{4}$	$-\frac{1}{2}$;	0	0	$-\frac{1}{2}$	1;	0	1	1	0;	1	0	0	−1]	R
		[0	0	$\frac{1}{4}$	$-\frac{1}{2}$;	0	1	$-\frac{1}{2}$	1;	0	1	0	0;	1	0	0	−1]	R
		[0	0	−1	1;	0	1	−2	1;	0	1	−1	0;	1	0	0	0]	N
		[0	0	4	−2;	0	1	−1	1;	0	1	1	0;	1	0	0	−2]	R
		[1	0	−4	−3;	0	1	1	1;	0	1	−1	0;	1	0	0	−1]	R

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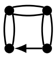
13		[0	0	1	1;	0	0	1	1;	1	1	0	0;	−2	0	0	0]	N
B		[0	0	$\frac{1}{2}$	$-\frac{1}{2}$;	0	0	−1	1;	1	1	1	0;	1	0	0	−1]	R
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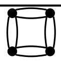
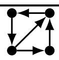
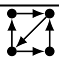
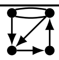
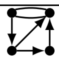
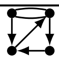
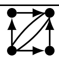
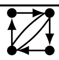
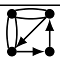
		[1	0	1	−1;	0	0	−1	1;	1	1	−1	0;	1	0	0	0]	N
		[0	0	$-\frac{1}{3}$	$-\frac{2}{3}$;	0	1	−2	−1;	1	1	−2	0;	1	0	0	1]	R
		[1	0	2	$-\frac{3}{2}$;	0	0	$-\frac{7}{2}$	$-\frac{7}{8}$;	1	1	1	0;	1	0	0	−2]	R
		[1	0	−4	1;	0	−1	1	1;	1	1	−1	0;	1	0	0	1]	R
14		[0	0	−1	−1;	0	0	1	1;	1	1	0	0;	1	1	0	0]	N
B		[0	0	$\frac{1}{2}$	$\frac{1}{2}$;	0	0	−1	−1;	1	1	1	0;	1	1	0	−1]	R
		[0	0	2	−1;	0	1	−1	1;	1	1	0	0;	1	−1	0	−1]	R
		[0	0	−1	−2;	0	1	$-\frac{1}{3}$	$\frac{1}{3}$;	1	1	1	0;	1	1	0	−2]	R
		[1	0	1	−1;	0	2	−1	−4;	1	1	−1	0;	1	1	0	−2]	R
15		[1	0	0	a_1;	1	$-\overline{a_1}$	0	0;	0	1	$-a_1$	0;	0	a_2	1	0]	C
A		[0	0	0	1;	a_1	1	0	0;	0	1	$-a_1$	0;	0	a_6	1	$-\overline{a_1}$]	C
		[1	0	0	a_3;	1	a_4	0	0;	0	1	$\overline{a_4}$	0;	0	a_5	1	1]	C

where $a_1 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$, $a_2 = -\frac{2}{1+\sqrt{3}i}$, $a_3 = 7 + 4\sqrt{2}i$, $a_4 = -1 + \sqrt{2}i$, $a_5 = -\frac{4}{7+4\sqrt{2}i}$, $a_6 = -\frac{1+\sqrt{3}i}{-1+\sqrt{3}i}$

16		[0	−1	0	0;	0	0	0	1;	0	1	0	0;	1	0	1	0]	N
A		[$-\overline{a_1}$	a_3	0	0;	0	1	0	a_2;	0	1	$-a_1$	0;	1	0	1	0]	C
		[i	−1	0	0;	0	1	0	$-\frac{i}{2}$;	0	1	$-i$	0;	1	0	1	−1]	C

where $a_1 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$, $a_2 = \frac{-3+\sqrt{3}i}{6}$, $a_3 = \frac{-1+\sqrt{3}i}{1+\sqrt{3}i}$

17		[1	0	−1	1;	0	0	1	0;	1	0	0	1;	0	1	0	−1]	R
B		[1	0	−1	1;	0	−1	1	0;	1	0	0	1;	0	1	0	0]	R
		[0	0	−3	$-\frac{6}{5}$;	0	1	5	0;	1	0	1	1;	0	1	0	−2]	R
		[1	0	−3	1;	0	0	−1	0;	1	0	−2	1;	0	1	0	1]	R
		[1	0	−3	$-\frac{1}{5}$;	0	1	5	0;	1	0	0	1;	0	1	0	−2]	R
		[1	0	−3	1;	0	1	−1	0;	1	0	−2	1;	0	1	0	0]	R
		[1	0	−2	$\frac{7}{4}$;	0	−1	4	0;	1	0	1	1;	0	1	0	−1]	R
18		[0	0	−1	−1;	0	1	1	0;	1	0	−1	0;	0	1	1	0]	R
B		[1	0	−1	1;	0	−1	−1	0;	1	0	0	0;	0	1	1	0]	R
		[0	0	−3	5;	0	1	$-\frac{1}{5}$	0;	1	0	1	0;	0	1	1	−2]	R
		[1	0	−3	5;	0	1	$-\frac{1}{5}$	0;	1	0	0	0;	0	1	1	−2]	R
		[1	0	−3	−1;	0	1	1	0;	1	0	−2	0;	0	1	1	0]	R
		[1	0	−2	4;	0	−1	$-\frac{1}{4}$	0;	1	0	1	0;	0	1	1	−1]	R
19		[0	0	1	1;	0	0	0	1;	−1	−1	1	0;	0	0	$-\frac{1}{2}$	−1]	N
B		[0	0	−1	1;	0	1	0	−1;	1	1	0	0;	0	0	1	−1]	R
		[1	0	−1	$\frac{1}{2}$;	0	−1	0	$\frac{1}{2}$;	1	1	0	0;	0	0	1	0]	R
		[0	0	−3	−3;	0	1	0	−1;	1	1	−2	0;	0	0	1	1]	R
		[1	0	−3	$-\frac{1}{3}$;	0	−2	0	$\frac{16}{3}$;	1	1	0	0;	0	0	1	1]	R
		[1	0	−3	$\frac{8}{3}$;	0	−2	0	$\frac{16}{3}$;	1	1	1	0;	0	0	1	0]	R
		[1	0	−2	$\frac{7}{2}$;	0	−1	0	$\frac{1}{2}$;	1	1	1	0;	0	0	1	−1]	R
20		[0	0	1	0;	0	0	−1	0;	0	0	−1	−1;	1	1	1	1]	N
B		[1	0	$\frac{1}{2}$	0;	0	−1	$-\frac{1}{2}$	0;	0	0	0	−1;	1	1	1	0]	R
		[1	0	$-\frac{1}{7}$	0;	0	2	$\frac{16}{7}$	0;	0	0	0	−7;	1	1	1	−3]	R
		[1	0	$\frac{1}{4}$	0;	0	−1	$-\frac{1}{4}$	0;	0	0	1	−2;	1	1	1	−1]	R
21		[0	0	*	0;	0	*	0	*;	0	*	0	*;	*	0	*	*]	*
B		[1	0	1	0;	0	−1	0	1;	0	1	0	−1;	1	0	1	0]	C
		[0	0	$-\frac{2}{3}$	0;	0	1	0	−3;	0	1	1	−3;	1	0	1	−2]	R
		[1	0	1	0;	0	1	0	−1;	0	1	0	−3;	1	0	1	−2]	R
		[−6	0	1	0;	0	2	0	1;	0	$\frac{4}{5}$	3	1;	180	0	−25	1]	R
22		[0	0	−2	1;	0	1	$-\sqrt{2}$	0;	1	0	0	a_1;	1	1	0	−1]	R
B		[1	0	−2	1;	0	0	4	0;	1	0	0	1;	1	1	0	−1]	R
		[1	0	a_2	$\sqrt{3}$;	0	−1	a_3	0;	1	0	0	1;	1	1	0	0]	R
		[0	0	a_4	$-\sqrt{3}$;	0	1	1	0;	1	0	1	a_5;	1	1	0	−2]	R
		[1	0	−1	−2;	0	1	$\frac{1}{2}$	0;	1	0	0	−2;	1	1	0	−2]	R
		[1	0	−1	−1;	0	1	1	0;	1	0	−1	1;	1	1	0	−1]	R

where $a_1 = -1 + \frac{\sqrt{2}}{2}$, $a_2 = -1 - \sqrt{3}$, $a_3 = 2 + \sqrt{3}$, $a_4 = -3 + \sqrt{3}$, $a_5 = -(-6 + 4\sqrt{3})/a_4$

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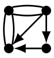
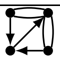
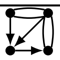
23		[0	0	$-\frac{1}{2}$	$-\frac{1}{2}$;	0	1	0	2;	1	1	0	1;	1	0	0	-1]	R
B		[1	0	1	-2;	0	-1	0	-1;	1	1	0	1;	1	0	0	0]	R
		[0	0	1	-4;	0	1	0	-1;	1	1	1	-7;	1	0	0	-2]	R
		[1	0	-1	-2;	0	1	0	1;	1	1	0	-1;	1	0	0	-2]	R
		[-1	0	1	1;	0	1	0	1;	-1	1	-1	2;	-1	0	0	1]	R
24		[0	0	1	1;	0	0	1	1;	-1	1	0	0;	0	0	1	0]	N
B		[0	0	1	2;	0	0	-2	-1;	1	1	1	0;	0	0	1	-1]	R
		[0	0	-2	2;	0	1	1	-3;	1	1	0	0;	0	0	1	-1]	R
		[0	0	-4	8;	0	1	1	-4;	1	1	1	0;	0	0	1	-2]	R
		[1	0	-1	$-\frac{1}{3}$;	0	-2	-2	$-\frac{2}{3}$;	1	1	0	0;	0	0	1	1]	R
		[1	0	-3	4;	0	2	-4	-8;	1	1	-3	0;	0	0	1	0]	R
		[1	0	-1	$\frac{3}{2}$;	0	-1	-1	$\frac{1}{2}$;	1	1	1	0;	0	0	1	-1]	R
25		[0	0	-2	-2;	0	1	1	0;	1	1	0	-1;	0	1	0	-1]	R
B		[1	0	1	$\frac{3}{2}$;	0	0	-2	0;	1	1	0	$\frac{1}{2}$;	0	1	0	-1]	R
		[1	0	-2	1;	0	0	1	0;	1	1	-1	1;	0	1	0	0]	R
		[1	0	-2	3;	0	-1	1	0;	1	1	0	3;	0	1	0	0]	R
		[1	0	-2	-5;	0	0	-1	0;	1	1	1	-7;	0	1	0	-2]	R
		[1	0	-1	$\frac{1}{2}$;	0	1	-2	0;	1	1	0	$-\frac{5}{2}$;	0	1	0	-2]	R
		[1	0	-4	-1;	0	1	1	0;	1	1	-2	-1;	0	1	0	0]	R
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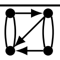
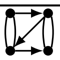
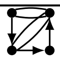
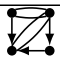
26		[0	0	-3	-2;	0	0	1	$\frac{2}{3}$;	0	1	1	1;	1	0	0	-1]	R
B		[0	0	-3	-2;	0	1	1	$\frac{5}{3}$;	0	1	0	1;	1	0	0	-1]	R
		[1	0	3	-2;	0	0	1	$-\frac{2}{3}$;	0	1	-1	1;	1	0	0	0]	R
		[1	0	-3	1;	0	-1	-2	$-\frac{1}{3}$;	0	1	0	1;	1	0	0	0]	R
		[0	0	-1	1;	0	1	1	$-\frac{1}{2}$;	0	-2	-2	1;	-1	0	0	1]	R
		[1	0	4	-2;	0	0	-1	$\frac{3}{4}$;	0	1	-2	1;	1	0	0	1]	R
		[1	0	1	-2;	0	1	-1	1;	0	1	0	1;	1	0	0	-2]	R
		[1	0	1	-1;	0	1	-2	1;	0	1	-2	1;	1	0	0	0]	R
		[1	0	8	-3;	0	1	-3	1;	0	1	1	1;	1	0	0	-3]	R
27		[0	0	$-\frac{2}{5}$	-2;	0	1	1	0;	0	1	0	-5;	1	1	0	-1]	R
B		[0	0	-1	-2;	0	1	1	0;	0	1	-1	-4;	1	1	0	0]	R
		[1	0	-10	1;	0	-1	-2	0;	0	1	0	$\frac{1}{4}$;	1	1	0	0]	R
		[0	0	-2	-2;	0	1	-1	0;	0	1	-2	-1;	1	1	0	1]	R
		[1	0	$-\frac{17}{29}$	-8;	0	2	1	0;	0	1	0	-29;	1	1	0	-3]	R
		[1	0	1	-1;	0	1	-1	0;	0	1	-1	1;	1	1	0	-1]	R
28		[0	0	$-\frac{4}{3}$	$\frac{2}{3}$;	0	1	-1	0;	1	0	0	$\frac{1}{3}$;	0	1	1	-1]	R
B		[0	0	-2	-2;	0	1	1	0;	1	0	-1	1;	0	1	1	0]	R
		[1	0	-2	1;	0	0	2	0;	1	0	0	1;	0	1	1	-1]	R
		[*	0	*	*;	0	0	*	0;	*	0	*	*;	0	*	*	0]	*
		[1	0	-2	1;	0	-1	1	0;	1	0	0	1;	0	1	1	0]	R
		[0	0	-8	-4;	0	1	-1	0;	1	0	-2	5;	0	1	1	1]	R
		[-2	0	1	2;	0	0	1	0;	-1	0	1	1;	0	1	-2	1]	R
		[1	0	-2	1;	0	1	-1	0;	1	0	0	-1;	0	1	1	-2]	R
		[1	0	-2	-1;	0	1	1	0;	1	0	-2	-1;	0	1	1	0]	R
		[1	0	-1	-1;	0	1	1	0;	1	0	-1	-1;	0	1	1	-1]	R
29		[0	0	1	1;	0	0	0	-1;	1	1	0	1;	0	0	-1	0]	N
B		[0	0	1	-2;	0	0	0	1;	1	1	1	-2;	0	0	1	-1]	R
		[0	0	1	-1;	0	1	0	-1;	1	1	0	-2;	0	0	1	-1]	R
		[1	0	-2	$\frac{3}{2}$;	0	-1	0	$\frac{1}{2}$;	1	1	0	1;	0	0	1	0]	R
		[0	0	-1	2;	0	1	0	-1;	1	1	1	-2;	0	0	1	-2]	R
		[1	0	1	-3;	0	2	0	-16;	1	1	0	-8;	0	0	1	-3]	R
		[1	0	-1	2;	0	2	0	-16;	1	1	-3	-6;	0	0	1	0]	R
		[1	0	-1	$-\frac{1}{2}$;	0	-1	0	$\frac{1}{2}$;	1	1	-1	-1;	0	0	1	1]	R

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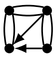
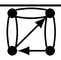
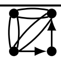
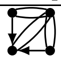
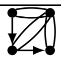
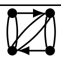
30		[0	0	1	−1;	0	0	−1	1;	1	1	1	−1;	1	0	0	−1]	R
B		[0	0	−1	−1;	0	1	1	4;	1	1	0	2;	1	0	0	−1]	R
		[0	0	−1	1;	0	1	−1	1;	1	1	−1	1;	1	0	0	0]	R
		[1	0	$-\frac{1}{2}$	$-\frac{3}{2}$;	0	0	1	1;	1	1	0	−1;	1	0	0	−1]	R
		[1	0	−1	1;	0	0	−1	1;	1	1	−1	2;	1	0	0	0]	R
		[1	0	1	−1;	0	−1	−1	2;	1	1	0	−1;	1	0	0	0]	R
		[0	0	2	−4;	0	1	−1	3;	1	1	1	−3;	1	0	0	−2]	R
		[−2	0	1	1;	0	0	1	−1;	−1	−1	1	−1;	−1	0	0	1]	R
		[1	0	1	−5;	0	1	1	−6;	1	1	0	−5;	1	0	0	−2]	R
		[1	0	−1	−3;	0	1	1	4;	1	1	−2	−5;	1	0	0	0]	R
		[−1	0	$\frac{1}{2}$	1;	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1;	−1	$\frac{1}{2}$	$\frac{1}{2}$	1;	$-\frac{1}{2}$	0	0	1]	R
31		[0	0	∗	∗ ;	0	0	∗	∗;	∗	∗	0	0;	∗	0	∗	0]	∗
B		[0	0	$-\frac{2}{3}$	1;	0	0	$-\frac{2}{3}$	1;	2	1	1	0;	1	0	1	−1]	R
		[0	0	1	−1;	0	1	−1	1;	1	1	0	0;	1	0	1	−1]	R
		[0	0	1	1;	0	1	1	2;	$-\frac{1}{2}$	$\frac{1}{2}$	−1	0;	−1	0	−1	0]	R
		[1	0	−1	−2;	0	0	$\frac{6}{5}$	$\frac{3}{5}$;	$\frac{1}{5}$	1	0	0;	1	0	1	−1]	R
		[1	0	−1	−6;	0	0	6	18;	1	1	−1	0;	1	0	1	0]	R
		[1	0	1	1;	0	−1	1	1;	−1	−1	0	0;	1	0	$\frac{1}{2}$	0]	R
		[0	0	−1	2;	0	1	−1	4;	4	1	1	0;	1	0	1	−2]	R
		[1	0	1	−2;	0	0	$-\frac{1}{2}$	$\frac{1}{2}$;	$-\frac{1}{2}$	1	1	0;	1	0	1	−2]	R
		[1	0	1	−3;	0	1	−1	2;	1	1	0	0;	1	0	1	−2]	R
		[1	0	1	−3;	0	1	−1	2;	1	1	−2	0;	1	0	−5	0]	R
		[1	0	−1	−1;	0	1	−2	−1;	−1	1	−1	0;	1	0	1	−1]	R
32		[0	0	1	1;	0	1	1	0;	$\frac{1}{2}$	0	0	$\frac{1}{2}$;	−1	−1	−1	−1]	R
B		[1	0	1	1;	0	0	−2	0;	−1	0	0	−1;	1	1	1	−1]	R
		[1	0	1	1;	0	−1	1	0;	−1	0	0	1;	$\frac{2}{9}$	$\frac{1}{3}$	$-\frac{2}{9}$	0]	R
		[0	0	a_1	a_2;	0	1	a_3	0;	−1	0	1	1;	1	1	1	−2]	R
		[−2	0	−1	1;	0	0	1	0;	−2	0	1	1;	−4	−4	−1	1]	R
		[−2	0	−2	−2;	0	1	−1	0;	1	0	0	1;	1	1	1	1]	R
		[1	0	1	$-\frac{5}{4}$;	0	$-\frac{1}{2}$	$-\frac{1}{4}$	0;	1	0	$\frac{1}{2}$	−1;	1	1	1	−1]	R
where $a_1 = 4 + \frac{2}{3}\sqrt{3}$, $a_2 = \frac{2}{3}\sqrt{3}$, $a_3 = 3 + 2\sqrt{3}$																		
33		[0	0	1	−1;	0	0	−2	1;	1	1	0	1;	0	0	1	0]	N
B		[0	0	1	1;	0	0	1	1;	−1	−1	−1	1;	0	0	1	1]	R
		[0	0	−1	1;	0	1	−1	1;	1	1	0	1;	0	0	1	−1]	R
		[1	0	−1	$\frac{1}{2}$;	0	−1	$\frac{1}{2}$	1;	1	1	0	$-\frac{1}{2}$;	0	0	1	0]	R
		[0	0	1	−2;	0	1	1	−4;	1	1	1	−5;	0	0	1	−2]	R
		[1	0	1	$-\frac{1}{3}$;	0	−2	−1	$\frac{7}{3}$;	1	1	0	−3;	0	0	1	1]	R
		[1	0	−1	2;	0	2	−3	−10;	1	1	−3	−3;	0	0	1	0]	R
		[1	0	−1	$\frac{3}{2}$;	0	−1	1	$\frac{1}{2}$;	1	1	1	−2;	0	0	1	−1]	R
34		[0	0	$\frac{2}{3}$	$-\frac{1}{3}$;	0	1	−1	0;	1	1	0	$-\frac{2}{3}$;	0	1	1	−1]	R
B		[0	0	1	1;	0	1	1	0;	1	−1	−1	−1;	0	1	1	0]	R
		[1	0	−1	1;	0	−1	−1	0;	1	1	0	1;	0	1	1	0]	R
		[0	0	−1	1;	0	1	$-\frac{1}{2}$	0;	1	1	1	$-\frac{3}{2}$;	0	2	1	−2]	R
		[1	0	−1	−1;	0	1	1	0;	1	1	0	−3;	0	1	1	−2]	R
		[1	0	−3	−1;	0	1	1	0;	1	1	−2	−1;	0	1	1	0]	R
		[1	0	−4	5;	0	−1	1	0;	1	−1	1	3;	0	1	1	−1]	R
35		[0	0	1	−1;	0	0	$\frac{1}{2}$	$-\frac{1}{2}$;	0	1	0	$\frac{1}{2}$;	1	0	1	0]	N
B		[0	0	1	−1;	0	0	1	−1;	0	1	1	−1;	1	0	1	−1]	R
		[0	0	1	−1;	0	1	1	−2;	0	1	0	−1;	1	0	1	−1]	R
		[0	0	−1	2;	0	1	1	−8;	0	1	−1	−4;	1	0	1	0]	R
		[1	0	−1	−1;	0	−1	$\frac{2}{3}$	1;	0	1	0	$-\frac{2}{3}$;	1	0	1	0]	R
		[0	0	−3	−3;	0	1	1	2;	0	1	−2	−1;	1	0	1	1]	R
		[1	0	$\sqrt{3}$	−1;	0	−2	−1	a_1;	0	1	0	−1;	1	0	1	1]	R
		[−1	0	−1	−2;	0	−1	−1	−1;	0	1	1	1;	1	0	1	1]	R
where $a_1 = 3 + \sqrt{3}$																		

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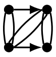
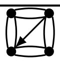
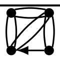
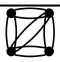
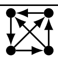
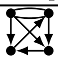
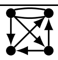
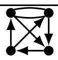
36		[0	0	1	1;	0	1	1	0;	0	-1	0	1;	-1	2	1	-1]	R
B		[0	0	1	1;	0	1	1	0;	0	-2	-1	1;	-1	3	2	0]	R
		[1	0	1	-1;	0	-1	-1	0;	0	1	0	1;	1	1	1	0]	R
		[0	0	2	-1;	0	1	-1	0;	0	1	1	-1;	1	3	1	-2]	R
		[1	0	1	-1;	0	1	-1	0;	0	1	0	-1;	1	1	1	-2]	R
		[1	0	3	-5;	0	-1	-1	0;	0	1	-1	4;	1	1	1	1]	R
37		[0	0	-1	-1;	0	0	1	1;	1	1	0	1;	1	1	0	0]	N
B		[0	0	1	1;	0	0	1	2;	-2	1	1	-1;	-2	1	0	-1]	R
		[0	0	1	-1;	0	1	1	-2;	1	1	0	-1;	1	1	0	-1]	R
		[1	0	1	-4;	0	-1	1	1;	1	1	0	$\frac{5}{2}$;	1	1	0	0]	R
		[0	0	-1	2;	0	1	1	-5;	1	1	1	-4;	1	1	0	-2]	R
		[1	0	-1	$\frac{2}{3}$;	0	-2	1	$-\frac{11}{3}$;	1	1	0	1;	1	1	0	1]	R
		[1	0	-1	-1;	0	-1	1	-1;	1	1	1	$-\frac{3}{2}$;	1	1	0	-1]	R
38		[0	0	1	-1;	0	0	$-\frac{1}{2}$	$\frac{1}{2}$;	1	1	0	$\frac{1}{2}$;	1	0	1	0]	N
B		[0	0	-2	2;	0	0	2	-2;	2	1	1	-1;	1	0	1	-1]	R
		[0	0	1	$-\frac{1}{2}$;	0	1	-2	2;	1	1	0	$\frac{1}{2}$;	1	0	1	-1]	R
		[0	0	1	1;	0	1	1	-4;	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	1;	-1	0	1	0]	R
		[1	0	-1	1;	0	0	1	-1;	1	-2	0	1;	1	0	1	-1]	R
		[1	0	2	-2;	0	0	$\frac{1}{2}$	-1;	$-\frac{3}{4}$	1	-1	2;	1	0	1	0]	R
		[1	0	1	-2;	0	-1	1	5;	1	1	0	-1;	1	0	1	0]	R
		[0	0	1	-2;	0	1	1	-3;	1	1	1	-3;	1	0	1	-2]	R
		[1	0	-1	-4;	0	0	-2	-2;	-1	1	1	2;	1	0	1	-2]	R
		[1	0	1	-2;	0	1	-1	1;	-1	1	0	1;	1	0	1	-2]	R
		[1	0	-1	-2;	0	1	1	3;	1	1	-2	-1;	1	0	1	0]	R
		[1	0	1	-1;	0	-1	1	2;	-1	1	1	-1;	1	0	1	-1]	R
39		[0	0	-1	-1;	0	0	1	-1;	1	1	1	1;	1	1	1	-1]	R
B		[0	0	1	-1;	0	1	-1	5;	$-\frac{1}{3}$	-2	0	$-\frac{20}{3}$;	1	1	1	-1]	R
		[1	0	$\frac{1}{2}$	-2;	0	-1	$-\frac{1}{2}$	1;	-1	1	0	1;	1	1	1	0]	R
		[0	0	-1	$-\frac{4}{3}$;	0	1	1	$\frac{2}{3}$;	-1	-1	-2	$-\frac{7}{3}$;	1	1	1	1]	R
		[1	0	1	$-\frac{2}{3}$;	0	-2	-1	$-\frac{10}{3}$;	1	1	0	1;	1	1	1	1]	R
		[1	0	$\frac{1}{3}$	-1;	0	1	$\frac{2}{3}$	-1;	1	-2	-1	1;	1	1	1	-1]	R
40		[1	0	0	-2;	1	a_1	0	0;	1	a_1	0	0;	0	1	$-\frac{1}{2}$	$\overline{a_1}$]	C
A		[1	0	0	$-\frac{1}{2}$;	1	a_1	0	0;	1	-1	$\overline{a_1}$	0;	0	1	1	0]	C
		[1	0	0	1;	1	$-i$	0	0;	i	1	i	0;	0	$-i$	1	-1]	C

where $a_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

41		[0	1	2	1;	-1	0	1	0;	0	0	1	1;	0	1	0	-1]	R
A		[0	1	-2	2;	-1	1	1	0;	0	0	-1	1;	0	-1	0	0]	R
		[1	1	-1	2;	-1	0	1	0;	0	0	0	1;	0	-1	0	-1]	R
		[1	1	-3	2;	-1	0	1	0;	0	0	-1	1;	0	-1	0	0]	R
		[1	1	-1	1;	-1	-1	1	0;	0	0	0	1;	0	1	0	0]	R
		[0	1	-6	2;	-3	1	1	0;	0	0	-2	1;	0	-1	0	1]	R
		[1	1	$-\frac{1}{3}$	2;	-3	0	1	0;	0	0	1	1;	0	-1	0	-2]	R
		[1	1	$-\frac{1}{3}$	3;	-3	1	1	0;	0	0	0	1;	0	-1	0	-2]	R
		[1	1	1	1;	-3	-2	6	0;	0	0	1	$\frac{1}{3}$;	0	1	0	0]	R
		[1	1	1	$\frac{1}{2}$;	-2	-1	1	0;	0	0	1	1;	0	$\frac{1}{2}$	0	-1]	R
42		[0	1	1	1;	-1	0	0	0;	0	1	1	1;	0	-1	0	-1]	R
A		[0	1	1	1;	-1	1	0	0;	0	$-\frac{1}{2}$	0	-1;	0	$-\frac{1}{2}$	0	-1]	R
		[0	1	1	1;	-1	1	0	0;	0	-2	-1	-1;	0	1	0	0]	R
		[0	1	1	1;	-3	1	0	0;	0	-2	-2	-2;	0	$\frac{1}{3}$	0	1]	R
		[1	1	1	1;	-3	-2	0	0;	0	1	0	$-\frac{3}{2}$;	0	$-\frac{2}{3}$	0	1]	R
		[1	1	1	1;	-2	1	0	0;	0	-1	-1	$-\frac{1}{2}$;	0	-1	0	-1]	R
43		[0	1	1	0;	-1	0	1	0;	0	0	1	1;	$-\frac{1}{2}$	$\frac{1}{2}$	0	-1]	R
A		[0	1	-1	0;	-1	1	1	0;	0	0	0	1;	$-\frac{1}{3}$	$\frac{2}{3}$	0	-1]	R
		[1	1	-1	0;	-1	-1	1	0;	0	0	0	1;	1	1	0	0]	C
		[0	1	1	0;	-3	1	1	0;	0	0	-2	1;	3	2	0	1]	R
		[1	1	-1	0;	-3	-2	1	0;	0	0	0	1;	2	1	0	1]	R
		[1	1	-2	0;	-2	1	1	0;	0	0	-1	1;	-1	2	0	-1]	R

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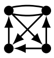
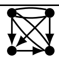
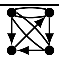
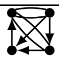
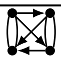
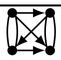
44		[0	−1	−1	0;	1	0	0	−2;	0	1	1	1;	1	0	0	−1]	R
A		[0	−1	−1	0;	1	1	0	$-\frac{3}{2}$;	0	1	0	$-\frac{1}{2}$;	1	0	0	−1]	R
		[1	−1	1	0;	1	0	0	−1;	0	1	0	−1;	1	0	0	−1]	R
		[1	−1	$-\frac{1}{2}$	0;	1	−1	0	−1;	0	1	0	1;	1	0	0	0]	R
		[0	−3	1	0;	1	1	0	$\frac{1}{8}$;	0	1	−2	$\frac{35}{8}$;	1	0	0	1]	R
		[1	−3	6	0;	1	0	0	−2;	0	1	−2	$-\frac{7}{6}$;	1	0	0	1]	R
		[1	−3	−2	0;	1	−2	0	−1;	0	1	0	1;	1	0	0	1]	R
		[1	−2	$\frac{11}{3}$	0;	1	1	0	1;	0	1	−1	$\frac{7}{11}$;	1	0	0	−1]	R
45		[0	−2	1	−3;	1	0	−1	2;	1	0	1	0;	0	0	1	−1]	R
A		[0	1	−2	−2;	1	1	1	−5;	1	0	0	0;	0	0	1	−1]	R
		[0	1	1	1;	−2	1	2	1;	1	0	−1	0;	0	0	1	0]	R
		[1	1	−1	1;	−2	0	1	$\frac{1}{2}$;	−1	0	0	0;	0	0	−2	−1]	R
		[1	1	1	1;	−2	−1	1	−1;	1	0	0	0;	0	0	−2	0]	R
		[0	1	1	1;	−1	1	1	−1;	−2	0	−2	0;	0	0	$-\frac{1}{2}$	1]	R
		[1	−2	−1	1;	1	0	−1	1;	1	0	−2	0;	0	0	1	1]	R
		[1	−1	−2	5;	1	1	−1	1;	1	0	0	0;	0	0	1	−2]	R
		[1	−1	−2	1;	1	1	−1	−1;	1	0	−2	0;	0	0	1	0]	R
		[1	−1	−1	1;	1	1	−1	−1;	1	0	−1	0;	0	0	1	−1]	R
46		[0	1	1	1;	1	0	1	0;	−1	0	0	1;	0	$\frac{1}{2}$	0	0]	N
A		[0	1	−2	1;	1	0	−1	0;	1	0	1	−2;	0	1	0	−1]	R
		[0	1	1	2;	-2	1	1	0;	1	0	0	1;	0	$-\frac{1}{3}$	0	−1]	R
		[0	1	−2	−2;	1	1	1	0;	1	0	−1	−2;	0	1	0	0]	R
		[1	1	1	1;	1	0	1	0;	−2	0	0	−4;	0	−1	0	−1]	R
		[1	−2	1	−6;	1	0	−2	0;	1	0	−1	−2;	0	1	0	0]	R
		[1	−2	1	−6;	1	−1	−1	0;	1	0	0	−3;	0	1	0	0]	R
		[0	−4	1	7;	1	1	1	0;	1	0	1	2;	0	1	0	−2]	R
		[1	−1	−2	2;	1	0	−2	0;	1	0	1	$-\frac{3}{2}$;	0	1	0	−2]	R
		[1	−1	−2	$\frac{11}{2}$;	1	1	−1	0;	1	0	0	$\frac{1}{2}$;	0	1	0	−2]	R
		[1	−1	−2	2;	1	1	−2	0;	1	0	−2	1;	0	1	0	0]	R
		[−1	1	1	1;	−1	−1	2	0;	−1	0	1	$\frac{1}{2}$;	0	2	0	1]	R
47		[0	1	−2	0;	1	0	1	−1;	1	0	1	−1;	1	0	0	−1]	R
A		[0	1	−2	0;	1	1	1	−4;	1	0	0	−1;	1	0	0	−1]	R
		[0	1	−2	0;	1	1	−5	6;	1	0	−1	2;	1	0	0	0]	R
		[1	1	−2	0;	1	0	1	−2;	1	0	0	−1;	1	0	0	−1]	R
		[1	1	−2	0;	1	0	1	$-\frac{2}{3}$;	1	0	−1	$\frac{2}{3}$;	1	0	0	0]	R
		[1	$-\frac{1}{2}$	$-\frac{1}{2}$	0;	1	−1	−1	−1;	1	0	0	1;	1	0	0	0]	R
		[0	1	1	0;	−1	−2	−2	1;	−2	0	1	1;	−1	0	0	1]	R
		[1	−4	1	0;	1	0	−1	$-\frac{4}{5}$;	1	0	1	$-\frac{16}{5}$;	1	0	0	−2]	R
		[−2	1	1	0;	−2	1	2	1;	−1	0	0	−1;	1	0	0	1]	R
		[−2	1	1	0;	−2	1	−1	1;	−1	0	1	$-\frac{1}{2}$;	−4	0	0	0]	R
		[1	−1	−1	0;	1	1	1	−2;	1	0	−1	−1;	1	0	0	−1]	R
48		[0	$\frac{1}{2}$	$-\frac{1}{2}$	1;	0	0	−1	1;	0	1	0	1;	1	0	0	0]	N
A		[0	1	2	2;	0	0	1	1;	0	1	1	1;	−1	0	0	−1]	R
		[0	1	1	1;	0	1	1	1;	0	1	−1	−1;	−2	0	0	0]	R
		[1	1	1	1;	0	0	−1	$-\frac{4}{3}$;	0	2	0	$\frac{4}{3}$;	1	0	0	−1]	R
		[0	1	1	1;	0	1	1	1;	0	−2	1	−5;	−1	0	0	−2]	R
		[1	1	1	1;	0	0	−1	$-\frac{1}{2}$;	0	2	1	$-\frac{5}{2}$;	−1	0	0	−2]	R
		[1	1	1	1;	0	1	1	−1;	0	−1	−1	1;	−1	0	0	−1]	R
49		[0	1	1	1;	0	0	1	1;	0	1	0	0;	−1	0	1	0]	N
A		[0	2	4	−2;	0	0	1	−1;	0	1	1	0;	1	0	1	−1]	R
		[0	2	−2	1;	0	1	−2	1;	0	1	0	0;	1	0	1	−1]	R
		[0	1	1	1;	0	1	1	1;	0	−2	−1	0;	1	0	$\frac{1}{2}$	0]	R
		[1	1	$-\frac{1}{4}$	1;	0	0	1	−2;	0	−2	−1	0;	1	0	$-\frac{1}{4}$	0]	R
		[0	1	$-\frac{1}{2}$	1;	0	−2	1	−2;	0	−1	1	0;	−2	0	$-\frac{1}{2}$	1]	R
		[1	1	$\frac{1}{2}$	1;	0	0	1	−1;	0	−1	−2	0;	−2	0	2	1]	R
		[−1	1	−2	1;	0	−1	1	1;	0	−1	1	0;	−1	0	−1	1]	R

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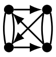
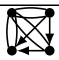
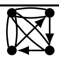
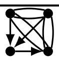
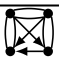
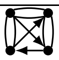
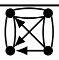
50		[0	1	0	1;	0	0	1	1;	$-\frac{2}{9}$	$\frac{2}{3}$	0	0;	$-\frac{2}{3}$	0	2	0]	N
A		[0	−2	0	−2;	0	0	1	−1;	1	1	1	0;	1	0	1	−1]	R
		[0	$\frac{2}{3}$	0	1;	0	1	−2	1;	5	1	−1	0;	1	0	1	0]	R
		[0	1	0	1;	0	1	1	1;	-2	1	1	0;	4	0	2	−2]	R
		[−1	1	0	1;	0	1	1	4;	4	−1	−1	0;	−1	0	1	1]	R
51		[0	1	1	1;	−1	0	−2	−1;	1	0	1	1;	−1	0	0	−1]	R
A		[0	−1	−1	1;	1	1	1	−2;	1	0	0	−1;	1	0	0	−1]	R
		[1	−1	−1	1;	1	0	−2	1;	1	0	0	−1;	1	0	0	−1]	R
		[1	1	1	1;	−2	0	2	2;	−1	0	−1	−1;	2	0	0	0]	R
		[0	−1	−1	−1;	1	1	1	−1;	1	0	1	−2;	1	0	0	−2]	R
		[−2	1	1	1;	−5	0	−2	−5;	1	0	1	1;	1	0	0	1]	R
		[1	−1	−1	−1;	1	1	−1	−1;	1	0	0	−2;	1	0	0	−2]	R
		[−1	1	1	1;	1	−1	−1	−1;	−2	0	1	1;	−1	0	0	1]	R
52		[0	1	1	1;	$-\frac{1}{5}$	0	−1	0;	$-\frac{2}{5}$	0	0	$\frac{1}{2}$;	$\frac{3}{5}$	1	0	0]	N
A		[0	1	−1	−1;	1	0	1	0;	1	0	1	1;	1	1	0	−1]	R
		[1	2	−2	−1;	1	0	1	0;	1	0	0	1;	1	1	0	−1]	R
		[0	−1	−1	−1;	1	1	−1	0;	1	0	1	−1;	1	−1	0	−2]	R
		[−2	1	1	1;	−1	0	1	0;	−1	0	1	1;	−1	1	0	1]	R
		[1	2	1	−1;	−1	1	2	0;	1	0	−2	$-\frac{1}{3}$;	1	1	0	−1]	R
53		[0	2	1	1;	1	0	$\frac{3}{2}$	1;	1	−2	0	0;	0	0	1	0]	N
A		[0	1	1	1;	−1	0	1	−1;	$-\frac{1}{2}$	$\frac{1}{2}$	1	0;	0	0	−1	−1]	R
		[0	1	1	1;	1	1	1	1;	−1	−1	0	0;	0	0	−1	−1]	R
		[0	1	−1	1;	1	1	1	2;	1	−1	−1	0;	0	0	4	0]	R
		[1	1	1	1;	−1	−1	−1	−1;	1	1	0	0;	0	0	1	0]	R
		[0	1	1	1;	−1	1	1	$\frac{4}{5}$;	−1	−1	1	0;	0	0	$-\frac{10}{3}$	−2]	R
		[1	1	1	1;	−1	1	1	2;	−1	−1	0	0;	0	0	−2	−2]	R
		[1	1	1	1;	−1	1	1	$\frac{1}{2}$;	−3	1	−2	0;	0	0	$-\frac{12}{5}$	0]	R
		[1	1	1	1;	−1	−1	−1	−1;	−2	−1	1	0;	0	0	−1	−1]	R
54		[0	−1	−1	2;	1	0	−1	$\frac{1}{2}$;	0	1	0	$-\frac{3}{2}$;	1	0	0	0]	N
A		[0	2	$\frac{2}{3}$	−1;	1	0	−2	$\frac{1}{2}$;	0	1	1	−1;	1	0	0	−1]	R
		[0	1	−1	−1;	1	1	−1	−2;	0	1	0	−2;	1	0	0	−1]	R
		[0	−1	1	−1;	1	1	1	1;	0	1	−1	1;	1	0	0	0]	R
		[1	−1	1	1;	1	0	−1	$-\frac{3}{2}$;	0	1	0	$-\frac{3}{2}$;	1	0	0	−1]	R
		[1	1	−1	−1;	1	0	−1	−1;	0	1	−1	−1;	1	0	0	0]	R
		[1	−1	$\frac{2}{3}$	−1;	1	−1	1	−2;	0	1	0	−1;	1	0	0	0]	R
		[0	−1	1	−1;	1	1	−1	−1;	0	1	1	1;	1	0	0	−2]	R
		[1	−1	−1	−1;	1	0	−1	−3;	0	1	1	−1;	1	0	0	−2]	R
		[−2	1	1	1;	−2	1	1	1;	0	1	0	$-\frac{1}{2}$;	−2	0	0	1]	R
		[1	−1	1	−1;	1	1	−1	1;	0	1	−2	4;	1	0	0	0]	R
		[1	−1	−1	−2;	1	−1	1	−2;	0	1	1	1;	1	0	0	−1]	R
55		[0	1	1	1;	2	0	1	0;	0	−1	0	1;	−1	1	0	0]	N
A		[0	−1	−1	−1;	1	0	1	0;	0	1	1	1;	1	1	0	−1]	R
		[0	1	−1	−1;	1	1	−1	0;	0	1	0	a_1;	1	a_2	0	−1]	R
		[0	2	$-\frac{5}{3}$	−1;	1	1	−2	0;	0	1	−1	−1;	1	−2	0	0]	R
		[1	−2	2	−1;	1	−1	2	0;	0	1	0	1;	1	−1	0	0]	R
		[0	−1	$\frac{1}{2}$	−1;	1	1	−1	0;	0	1	1	1;	1	−1	0	−2]	R
		[−2	1	−4	1;	1	1	1	0;	0	−2	0	2;	−2	−7	0	1]	R
		[−1	1	$\frac{1}{2}$	1;	2	1	1	0;	0	−2	−1	−2;	−2	$\frac{1}{2}$	0	1]	R

where $a_1 = -\frac{2}{1+\sqrt{3}}$, $a_2 = 1 + \sqrt{3}$

56		[0	1	0	1;	2	0	1	1;	1	−1	0	−1;	−1	0	0	0]	N
A		[0	−1	0	−1;	1	0	1	1;	1	1	1	2;	1	0	0	−1]	R
		[0	1	0	1;	−1	1	1	2;	1	−1	0	−2;	1	0	0	−1]	R
		[0	1	0	−1;	1	1	1	−2;	1	−1	−1	2;	1	0	0	0]	R
		[1	1	0	−1;	1	−1	−1	1;	1	1	0	−1;	1	0	0	0]	R
		[0	−1	0	−1;	1	1	1	−4;	1	−1	1	−6;	1	0	0	−2]	R
		[1	−1	0	−4;	1	1	−2	4;	1	−1	0	−4;	1	0	0	−2]	R
		[1	−2	0	−1;	1	1	−1	−1;	1	−1	−1	−1;	1	0	0	−1]	R

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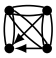
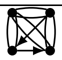
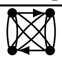
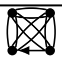
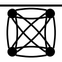
57		[0	-1	1	$-\frac{1}{2}$;	-1	0	-1	1;	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$;	-1	0	0	0]	N
A		[0	1	1	1;	-2	0	1	2;	-1	1	1	2;	1	0	0	-1]	R
		[0	1	-2	-2;	1	1	-2	-2;	1	-1	-1	$\frac{1}{2}$;	1	0	0	0]	R
		[1	-1	2	-1;	1	0	1	$-\frac{8}{5}$;	1	-1	0	$\frac{1}{5}$;	1	0	0	-1]	R
		[1	-1	2	-1;	1	0	-1	$\frac{4}{3}$;	1	1	-1	$-\frac{1}{3}$;	1	0	0	0]	R
		[0	-1	-1	-2;	1	1	1	-1;	1	1	1	1;	1	0	0	-2]	R
		[1	-1	-1	-2;	1	0	1	-2;	1	1	1	-2;	1	0	0	-2]	R
		[1	-1	-1	$-\frac{1}{2}$;	1	-2	1	$-\frac{5}{2}$;	1	$-\frac{1}{2}$	1	-1;	1	0	0	0]	R
		[1	-1	-1	-1;	1	1	-1	1;	1	-1	-1	-1;	1	0	0	-1]	R
58		[0	-1	1	-1;	1	0	1	-1;	1	1	0	0;	1	0	$\frac{1}{2}$	0]	N
A		[0	2	-1	-1;	1	0	1	-1;	1	-1	1	0;	1	0	1	-1]	R
		[0	-1	-1	2;	1	1	1	-4;	1	-1	-1	0;	1	0	$-\frac{2}{3}$	0]	R
		[-1	-2	1	-1;	1	0	1	-1;	1	1	0	0;	1	0	1	1]	R
		[1	-1	a_1	a_2;	1	0	$\frac{\sqrt{3}}{3}$	-1;	1	1	-1	0;	1	0	-1	0]	R
		[0	1	1	-1;	-1	-2	-2	1;	1	1	1	0;	1	0	1	1]	R
		[-2	1	1	-6;	1	0	1	1;	1	1	1	0;	1	0	1	1]	R
		[1	1	1	1;	$\frac{1}{11}$	1	$\frac{4}{11}$	$\frac{2}{11}$;	-1	-3	-2	0;	-1	0	-1	0]	R
		[1	-1	1	-1;	1	-1	1	$-\frac{1}{2}$;	1	-1	1	0;	1	0	1	-1]	R

where $a_1 = \frac{2\sqrt{3}}{3} - 1$, $a_2 = 1 - \sqrt{3}$

59		[0	1	1	1;	0	0	-2	1;	$\frac{1}{2}$	$-\frac{5}{24}$	1	-1;	-1	$-\frac{11}{12}$	0	-1]	R
A		[0	1	1	1;	0	1	1	-1;	-2	4	0	-4;	-1	2	0	-1]	R
		[0	1	1	1;	0	1	1	-1;	-3	5	-1	1;	-1	2	0	0]	R
		[0	1	1	1;	0	1	-1	1;	1	1	1	1;	-1	-2	0	-2]	R
		[1	1	1	1;	0	1	1	-1;	-1	1	-1	1;	-1	1	0	-1]	R
60		[0	2	-2	2;	1	0	1	-1;	1	-1	0	4;	1	1	0	0]	N
A		[0	1	2	-1;	-1	0	1	$-\frac{1}{2}$;	$-\frac{1}{10}$	$\frac{9}{20}$	1	1;	$\frac{1}{5}$	$\frac{1}{10}$	0	-1]	R
		[0	1	1	1;	-1	1	1	1;	2	-2	0	2;	-1	1	0	-1]	R
		[1	1	1	1;	-1	-1	1	-1;	1	1	0	1;	-1	1	0	0]	R
		[0	1	1	1;	-1	1	1	1;	$\frac{2}{3}$	1	1	1;	-3	$-\frac{2}{3}$	0	-2]	R
		[1	1	-2	$-\frac{1}{5}$;	1	-2	1	$-\frac{4}{5}$;	1	-1	0	$-\frac{3}{5}$;	1	1	0	1]	R
		[1	1	1	1;	-1	-1	-2	-3;	-1	-1	1	3;	1	1	0	-1]	R
61		[0	-2	$-\frac{4}{3}$	$\frac{4}{3}$;	1	0	$\frac{4}{3}$	$-\frac{4}{3}$;	1	1	0	2;	1	1	1	0]	N
A		[0	1	1	1;	-1	0	2	-6;	-2	1	1	5;	1	1	1	-1]	R
		[0	1	1	1;	-1	1	$\frac{1}{2}$	1;	-1	$-\frac{1}{2}$	1	1;	-1	$\frac{41}{8}$	$-\frac{39}{8}$	-2]	R
		[1	1	1	1;	-1	-1	-1	-1;	1	1	1	1;	-1	-1	-1	-1]	R