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# INFINITE PRODUCTS AND PARACONTRACTING MATRICES *

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**Abstract.** In [*Linear Algebra Appl.*, 161:227–263, 1992] the LCP-property of a finite set $\Sigma$ of square complex matrices was introduced and studied. A set $\Sigma$ is an LCP-set if all left infinite products formed from matrices in $\Sigma$ are convergent. It was shown earlier in [*Linear Algebra Appl.*, 130:65–82, 1990] that a set $\Sigma$ paracontracting with respect to a fixed norm is an LCP-set. Here a converse statement is proved: If $\Sigma$ is an LCP-set with a continuous limit function then there exists a norm such that all matrices in $\Sigma$ are paracontracting with respect to this norm. In addition the stronger property of $\ell$-paracontractivity is introduced. It is shown that common $\ell$-paracontractivity of a set of matrices has a simple characterization. It turns out that in the above mentioned converse statement the norm can be chosen such that all matrices are $\ell$-paracontracting. It is shown that for $\Sigma$ consisting of two projectors the LCP-property is equivalent to $\ell$-paracontractivity, even without requiring continuity.

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## 1. Introduction.

In the investigation of chaotic iteration procedures for consistent linear systems, matrices which are paracontracting with respect to some vector norm play an important role. It was shown in [3], that if $A_1, \ldots, A_m$ are finitely many $k \times k$ complex matrices which are paracontracting with respect to the same norm, then for any sequence $d_i,\ 1 \leq d_i \leq m,\ i = 1, 2, \ldots$ and any $x_0$ the sequence

$$x_i = A_{d_i} x_{i-1} \quad i = 1, 2, \ldots \tag{1}$$

is convergent. In particular $A^{(d)} = \lim_{i\to\infty} A_{d_i} \ldots A_{d_1}$ exists for all sequences $\{d_i\}_{i=1}^{\infty} = d$. Hence those sets are examples of sets of matrices all infinite products of which converge. Such sets have been studied in [2]. Following [2], we call them LCP–sets.

In this note we investigate the question of necessity. As our main result we show that under the additional assumption that the mapping

$$d = \{d_i\}_{i=1}^{\infty} \to A^{(d)} = \lim_{i\to\infty} A_{d_i} A_{d_{i-1}} \ldots A_{d_1} \tag{2}$$

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is continuous (which is equivalent to the set of fixed points of $A_i$ being the same for all $1 \leq i \leq m$), an LCP–set is necessarily paracontracting with respect to some norm. In this sense paracontractivity is equivalent to the LCP–property. We show, in addition, that continuity implies even the stronger property of $\ell$-paracontractiveness.

In the final section, we consider the case $m = 2$. We show that for $\Sigma$ consisting of two projectors the LCP-property is equivalent to $\ell$-paracontractivity, even without continuity.

**2. Notations and known results.** Let $|| \; ||$ denote a vector norm in $\mathbb{C}^k$. A $k \times k$ matrix $P$ is *paracontracting* with respect to $|| \; ||$, if for all $x$

$$Px \neq x \Leftrightarrow ||Px|| < ||x||.$$

We denote by $\mathcal{N}(|| \; ||)$ the set of all $k \times k$ matrices paracontracting w.r.t. $|| \; ||$. We call $P$ *$\ell$-paracontracting* w.r.t. $|| \; ||$, if there exists $\gamma > 0$ such that

$$||Px|| \leq ||x|| \Leftrightarrow \gamma ||Px \Leftrightarrow x||$$

holds for all $x \in \mathbb{C}^k$ and denote this set of matrices by $\mathcal{N}_\gamma(|| \; ||)$. Obviously

$$\mathcal{N}_\gamma(|| \; ||) \subset \mathcal{N}(|| \; ||). \tag{3}$$

The example of an orthogonal projection $P, P \neq I, P \neq 0$ which is paracontracting w.r.t. the Euclidean vector norm but never $\ell$-paracontracting shows that in (3) equality does not hold in general.

For a bounded set $\Sigma = \Sigma_1$ of complex $k \times k$ - matrices define $\Sigma_0 = \{I\}$ and for $n \geq 1$, $\Sigma_n = \{M_1 \; M_2 \ldots M_n : M_i \in \Sigma\}$, the set of all products of matrices in $\Sigma$ of length $n$. Let $\Sigma = \{A_1, \ldots, A_m\}$ be finite. For $d = (d_1, d_2, \ldots) \in \{1, \ldots, m\}^{\mathbb{N}}$, i.e. $1 \leq d_i \leq m$ for $i \in \mathbb{N}$ define $A^{(d)} = \lim_{n \to \infty} A_{d_n} A_{d_{n-1}} \ldots A_{d_1}$, if the limit exists. The set $\Sigma$ is an LCP–set (left–convergent–product), if for all $d \in \{1, \ldots, m\}^{\mathbb{N}}$ the limit $A^{(d)}$ exists. The function $d \to A^{(d)}$ mapping $\{1, \ldots, m\}^{\mathbb{N}}$ into the space of $k \times k$ - matrices is called the *limit function*.

We note in passing that in [2] also the right–convergent–product property (RCP) was introduced. For convenience we restrict our considerations to the left convergence case.

Introducing in $\{1, \ldots, m\}^{\mathbb{N}}$ the metric

$$\mathrm{dist}(d, d') = m^{-r} \qquad r \text{ smallest index such that } d_r \neq d'_r,$$

we define the concept of a *continuous limit function* in the standard way.

The set $\Sigma$ is *product bounded*, if there exists $\Delta > 0$ such that

$$||A|| \leq \Delta \quad \text{for all} \quad A \in \Sigma_n, \; n = 1, 2, \ldots$$

Here $||\ ||$ denotes any matrix norm. Obviously this concept is independent of the norm. G. Schechtman has proved that LCP–sets are product bounded (see [1, Theorem I]). We have the following result.

LEMMA 2.1. *For a set $\Sigma$ of $k \times k$ - matrices the following are equivalent.*
*(i) The set $\Sigma$ is product bounded.*
*(ii) There exists a vector norm $||\ ||$ such that $||Ax|| \leq ||x||$ for all $A \in \Sigma,\ x \in \mathbb{C}^k$.*
*(iii) There exists a multiplicative matrix norm $||\ ||$ such that $||A|| \leq 1$ for all $A \in \Sigma$.*

*Proof.* As $(ii) \Longrightarrow (iii)$ (the operator norm is multiplicative) and $(iii) \Longrightarrow (i)$ are obvious, only $(i) \Longrightarrow (ii)$ has to be shown.
For some vector norm $\nu$ define the norm

$$||x|| = \sup_{n \geq 0}\{\sup_{A \in \Sigma_n} \nu(Ax)\}$$

which is finite by $(i)$. Then $||Ax|| \leq ||x||$ for all $A \in \Sigma$. □

We remark that this result could also be derived from [5]. For a given matrix norm $||\ ||$ and bounded $\Sigma$ let $\widehat{\rho}_n = \widehat{\rho}_n(\Sigma) = \max\{||A||, A \in \Sigma_n\}$ and let $\widehat{\rho} = \widehat{\rho}(\Sigma) = \lim_{n\to\infty} \widehat{\rho}_n^{1/n}$. The quantity $\widehat{\rho}$ is called the joint spectral radius of $\Sigma$. It was introduced in [5] for general bounded sets in a normed algebra. In [5] and in [2] the limit is replaced by lim sup, however, it is implicitly shown in [2] (see there (3.12)), that the limit exists.

We give here a characterization of $\widehat{\rho}(\Sigma)$, which can be found essentially in [5]. Hence the proof, which is also an easy consequence of the previous Lemma, is omitted.

LEMMA 2.2. *For any bounded set $\Sigma$ of $k \times k$ - matrices*

$$\widehat{\rho}(\Sigma) = \inf_{\nu\ operator\ norm} \sup_{A \in \Sigma} \nu(A). \tag{4}$$

The following result is just a restatement of the Theorem in [3].

THEOREM 2.3. *Let $\Sigma \subset \mathcal{N}(||\ ||)$ for some vector norm $||\ ||$, $\Sigma$ finite. Then $\Sigma$ has the LCP–property.*

We finish this section by pointing out that if in addition $\Sigma \subset \mathcal{N}_\gamma(||\ ||)$ for some positive $\gamma$, then the proof of Theorem 2.3 is very simple. This is outlined below. It is a consequence of the following characterization of $\ell$-paracontractivity of the set $\Sigma$.

Let $\Sigma = \{A_i\}_{i \in I}$ be a set of matrices, not necessarily finite. Let $d = (d_1, \ldots, d_r) \in I^r$, $\nu$ a vector norm. Define

$$\nu_d(x) = \nu(x_r) + \sum_{k=1}^{r} \nu(x_k \Leftrightarrow x_{k-1}) \tag{5}$$

where the vectors $x_i$ are defined as in (1) and $x = x_0$. Then obviously, for any $i \in I$ and $d' = (i, d_1, \ldots, d_r)$

$$\nu_d(A_i x) = \nu_{d'}(x) \Leftrightarrow \nu(A_i x \Leftrightarrow x). \tag{6}$$

We define now

$$\nu_*(x) = \sup\{\nu_d(x) : d \text{ finite}\}. \tag{7}$$

This is a vector norm provided that $\nu_*(x) < \infty$ for all $x$.

THEOREM 2.4. *For a set of $k \times k$ - matrices $\{A_i\}_{i \in I}$ the following are equivalent.*

*(i) There exists a norm $\nu$ and a positive $\gamma$ such that*

$$A_i \in \mathcal{N}_\gamma(\nu) \qquad \textit{for all} \qquad i \in I.$$

*(ii) There exists a vector norm $\mu$ such that*

$$\mu_*(x) < \infty \qquad \textit{for all} \qquad x \in \mathbb{C}^k$$

*(iii) For all vector norms $\mu$*

$$\mu_*(x) < \infty \qquad \textit{for all} \qquad x \in \mathbb{C}^k$$

*Proof.* We show $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.
Assume that $(i)$ holds. Then from

$$\nu(A_i x \Leftrightarrow x) \leq \gamma^{-1}\{\nu(x) \Leftrightarrow \nu(A_i x)\} \qquad \forall i \in I, \forall x \tag{8}$$

we have, using the notation in (5), and assuming (w.l.o.g.) that $\gamma \leq 1$,

$$\begin{aligned} \nu_d(x) &\leq \nu(x_r) + \gamma^{-1} \sum_{k=1}^{r} (\nu(x_{k-1}) \Leftrightarrow \nu(x_k)) \\ &= \nu(x_r) + \gamma^{-1}\{\nu(x) \Leftrightarrow \nu(x_r)\} \leq \gamma^{-1}\nu(x). \end{aligned} \tag{9}$$

If $\mu$ is a fixed vector norm, then due to the compatibility of any two norms we have a constant $\kappa$ such that $\mu(x) \leq \kappa\nu(x)$ and hence also $\mu_d(x) \leq \kappa\nu_d(x)$. The inequality (9) gives that $\mu_*(x)$ exists, hence we have $(iii)$.
Obviously $(iii)$ implies $(ii)$.
Now we assume $(ii)$. From (6) we have

$$\mu_*(A_i x) \leq \mu_*(x) \Leftrightarrow \mu(A_i x \Leftrightarrow x) \leq \mu_*(x) \Leftrightarrow \gamma\mu_*(A_i x \Leftrightarrow x) \tag{10}$$

where we have chosen $\gamma$ such that $\mu(\xi) \geq \gamma\mu_*(\xi)$ for all $\xi$. Hence $(i)$ holds with $\nu = \mu_*$. $\square$

We indicate now the easy proof of the fact that a finite set $\Sigma = \{A_1, \ldots, A_m\} \subset \mathcal{N}_\gamma(\nu)$ has the LCP-property. It suffices to show that for any $x_0$ and any $d = (d_1, d_2, \ldots) \in \{1, ..., m\}^{\mathbb{N}}$ the sequence $\{x_i\}_{i=1}^{\infty}$ defined by (1) is convergent. By Theorem 2.4 we have $\nu_*(x_0) < \infty$, hence the sequence $\sum_{i=1}^{\infty} \nu(x_i \Leftrightarrow x_{i-1})$ is convergent. This implies that the sequence of the $x_i's$ is a Cauchy sequence.

**3. Main result.** It is tempting to conjecture that the converse statement of Theorem 2.3 also holds, namely that if $\Sigma$ is an LCP-set, then there exists a vector norm $|| \; ||$ such that $\Sigma \subset \mathcal{N}(|| \; ||)$. We were unable to decide this question in general. However, the converse is true if $\Sigma$ is an LCP-set with a continuous limit function. More precisely, the following theorem holds.

THEOREM 3.1. *Let $\Sigma = \{A_1, \ldots, A_m\}$ be a finite set of $k \times k$ - matrices and let the subspaces $M_i = N(I \Leftrightarrow A_i)$, $i = 1, \ldots, m$. Then the following are equivalent.*
*(i) The set $\Sigma$ has the LCP–property and $M_i = M_j$ for $i, j = 1, \ldots, m$.*
*(ii) The set $\Sigma$ has the LCP–property with continuous limit function.*
*(iii) There exists a vector norm $|| \; ||$ in $\mathbb{C}^k$ and a positive $\gamma$ such that $\Sigma \subset \mathcal{N}_\gamma(|| \; ||)$ and $M_i = M_j$ for $i, j = 1, \ldots, m$.*
*(iv) There exists a vector norm $|| \; ||$ in $\mathbb{C}^k$ such that $\Sigma \subset \mathcal{N}(|| \; ||)$ and $M_i = M_j$ for $i, j = 1, \ldots, m$.*

*Proof.* We will show $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$ .
To prove $(i) \Longrightarrow (ii)$, we are going to show that

$$||A^{(d)} \Leftrightarrow A^{(d')}|| \leq (2 + \Delta)||A_{(r)} \Leftrightarrow A^{(d)}||, \tag{11}$$

where $|| \; ||$ is a fixed operator norm, $(d)$, $(d') \in \{1, \ldots, m\}^{\mathbb{N}}$, $d_i = d'_i$ for $i \leq r$, and $\Delta$ is the bound in the definition of product boundedness. Here we use the fact that by [1], $\Sigma$ is product bounded. Also we use the notation

$$A_{(r)} = A_{d_r} A_{d_{r-1}} \ldots A_{d_1}, \;\; A'_{(s)} = A_{d'_s} \ldots A_{d'_1}.$$

Let $M_0 = N(I \Leftrightarrow A_i)$, $i = 1, \ldots, m$ the common pointwise invariant subspace of the matrices $A_i$. If $i \in \{1, \ldots, m\}$ occurs infinitely often in the sequence $d_1, d_2, \ldots$, then by the usual reasoning $A_i A^{(d)} = A^{(d)}$, and hence all columns of $A^{(d)}$ are in $M_0$. Hence $A_j A^{(d)} = A^{(d)}$ for all $A_j \in \Sigma$. This implies the relation

$$A'_{(r+s)} \Leftrightarrow A_{(r)} = (A_{d'_{r+s}} \ldots A_{d'_{r+1}} \Leftrightarrow I)(A_{(r)} \Leftrightarrow A^{(d)}) \qquad s > 0$$

and hence $||A'_{(r+s)} \Leftrightarrow A_{(r)}|| \leq (1 + \Delta)||A_{(r)} \Leftrightarrow A^{(d)}||$. Taking $s \to \infty$, we get

$$||A^{(d')} \Leftrightarrow A_{(r)}|| \leq (1 + \Delta)||A_{(r)} \Leftrightarrow A^{(d)}||,$$

from which (11) follows. This implies continuity: Given $\epsilon > 0$, as $A_{(r)} \to A^{(d)}$, there exists $r_0$ such that

$$||A_{(r_0)} \Leftrightarrow A^{(d)}|| \leq (2 + \Delta)^{-1}\epsilon.$$

Now, if $(d')$ is such that $\mathrm{dist}(d, d') \leq m^{-r_0-1}$, then $d_i = d'_i$ for $i \leq r_0$ and hence by (11)

$$||A^{(d')} \Leftrightarrow A^{(d)}|| \leq (2 + \Delta)||A_{(r_0)} \Leftrightarrow A^{(d)}|| \leq \epsilon.$$

We remark that although this step is not directly contained in [2], we have used tools and ideas from that paper.
Finally, we show $(ii) \Longrightarrow (iii)$. Assume that $(ii)$ holds. By Theorem 4.2 in [2] the subspaces $M_i$ are the same for $i = 1, \ldots, m$. By a similarity transformation, i.e.

$$\Sigma \to S^{-1}\Sigma S = \{S^{-1}A_iS: \ i = 1, \ldots, m\}$$

which does not change the properties involved, we can assume that $M_i$ is spanned by the first $r$ unit vectors $e_1, \ldots, e_r$, so that for $i = 1, \ldots, m$,

$$A_i = \begin{pmatrix} I_r & C_i \\ 0 & \widetilde{A}_i \end{pmatrix}.$$

Obviously $\widetilde{\Sigma} = \{\widetilde{A}_1, \ldots, \widetilde{A}_m\}$ has the LCP–property also and its limit function is identically zero. Otherwise if $\widetilde{A}^{(d)} \neq 0$, for some $d \in \{1, \ldots, m\}^{\mathbb{N}}$ we would have $\widetilde{A}_r\widetilde{A}^{(d)} = \widetilde{A}^{(d)}$ for at least one $r$ and $\widetilde{A}_r$ would have 1 as an eigenvalue. This contradicts our assumptions. But then, from Theorem 4.1 in [2], it follows that $\widehat{\rho}(\widetilde{\Sigma}) < 1$. We select some $q$ in $(\widehat{\rho}(\widetilde{\Sigma}), 1)$. By Lemma 2.2 we find a norm $|| \; ||$ on $\mathbb{C}^{k-r}$ such that

$$||\widetilde{A}_ix|| \leq q||x|| \quad \text{for all} \quad x \in \mathbb{C}^{k-r} \quad \text{and all} \quad i = 1, \ldots, m. \tag{12}$$

Denoting by $|| \; ||_2$ the Euclidean norm in $\mathbb{C}^r$, we introduce for any positive $\epsilon$ the following vector norm in $\mathbb{C}^k$:

$$\mu_\epsilon(x) = \mu_\epsilon\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \epsilon||x_1||_2 + ||x_2||.$$

Then we observe that

$$\begin{aligned} \mu_\epsilon(A_ix) &= \mu_\epsilon\begin{pmatrix} x_1 + C_ix_2 \\ \widetilde{A}_ix_2 \end{pmatrix} \\ &= \epsilon||x_1 + C_ix_2||_2 + ||\widetilde{A}_ix_2|| \\ &\leq \epsilon||x_1||_2 + (\epsilon||C_i|| + q)||x_2||, \end{aligned} \tag{13}$$

where $||C_i|| = \max\left\{\frac{||C_ix||_2}{||x||}, \ x \in \mathbb{C}^{k-r}\right\}$. Choose $\epsilon > 0$ such that $\tilde{q} = \max_i(\epsilon||C_i|| + q) < 1$ and let $\gamma = (1 \Leftrightarrow \tilde{q})/(1 + \tilde{q})$. Then we get after some manipulations using (12) and (13) the inequality

$$\mu_\epsilon(A_ix) \leq \mu_\epsilon(x) \Leftrightarrow \gamma\mu_\epsilon(A_ix \Leftrightarrow x).$$

Hence $\Sigma \subset \mathcal{N}_\gamma(\mu_\epsilon)$ and $(iii)$ is proved.
$(iii) \Longrightarrow (iv)$ is trivial, while $(iv) \Longrightarrow (i)$ is Theorem 2.3. $\square$

**4. Final remarks.** The conjecture at the beginning of the previous section remains unsolved even in the case $m = 2$. The following related result was proved in [6].

THEOREM 4.1. *For $\Sigma = \{A_1, A_2\}$ the following are equivalent.*
*(i) $\Sigma$ is an LCP-set.*
*(ii) (a) there exist a vector norm $|| \ ||$ such that*

$$||A_i x|| \leq ||x||, \qquad i = 1,2 \qquad \textit{for all} \qquad x \in \mathbb{C}^k,$$
$$||A_1 A_2 x|| = ||x|| \Longrightarrow A_1 x = A_2 x = x$$

*(b) For $i = 1, 2$ if $\lambda$ is an eigenvalue of $A_i$, $|\lambda| = 1$, then $\lambda = 1$.*
Notice that here we have finitely many conditions characterizing the LCP-property. Nevertheless *(ii)* seems not to imply paracontractivity of $\Sigma$.

In the case of two projectors $P_i, i = 1, 2$, not necessarily orthogonal, the conjecture can be proved.

THEOREM 4.2. *Let $P_i$, $i = 1, 2$ be projectors, i.e. $P_i^2 = P_i$, $i = 1, 2$. Then the following are equivalent.*
*(i) $\{P_1, P_2\}$ is an LCP-set.*
*(ii) There exists a vector norm $|| \ ||$ and a positive $\gamma$ such that*

$$\{P_1, P_2\} \subset \mathcal{N}_\gamma(|| \ ||).$$

The proof is given after the following auxiliary result.

LEMMA 4.3. *Let $A, B$ be complex $k \times k$ -matrices such that*
*(i) $B$ is convergent, i.e. the powers of $B$ converge, and*
*(ii) $\lim_{n\to\infty} AB^n = 0$.*
*Then there exists $\alpha \in (0, 1)$ such that for any norm $|| \ ||$*

$$||AB^n|| \leq C\alpha^n \qquad \textit{for all} \qquad n \in \mathbb{N}.$$

*with $C > 0$ a constant depending on the norm.*

*Proof.* By eventually changing the basis accordingly, we have by $(i)$ that $B$ is of the form

$$B = \begin{pmatrix} I_r & 0 \\ 0 & B_0 \end{pmatrix}$$

with $\alpha = ||B_0|| < 1$ for a suitable norm. Here $r$ is the dimension of $N(I \Leftrightarrow B)$ and we assume $r > 0$. Otherwise nothing has to be proved. Partitioning $A = (A_1, A_2)$, where $A_1$ contains the first $r$ columns of $A$, we get $AB^n = (A_1, A_2 B_0^n)$, and we see from $(ii)$ that $A_1 = 0$. But then clearly

$$||AB^n|| = ||(0, A_2 B_0^n)|| \leq C\alpha^n$$

for a suitable $C$. $\square$

*Proof of Theorem 4.2.* Obviously we need only to show the implication $(i) \Longrightarrow (ii)$.
Let $|| \ ||$ denote a vector norm satisfying $||P_i x|| \leq ||x||, i = 1, 2, x \in \mathbb{C}^k$ (See Lemma 2.1, $(ii)$) and define for $n \geq 0$

$$\begin{aligned} a_n(x) &= ||(P_1 - I)(P_2 P_1)^n x|| \\ b_n(x) &= ||(P_2 - I)P_1(P_2 P_1)^n x|| \\ c_n(x) &= ||(P_2 - I)(P_1 P_2)^n x|| \\ d_n(x) &= ||(P_1 - I)P_2(P_1 P_2)^n x|| \end{aligned}$$

By $(i)$ the sequence

$$x_0 = x, x_{2i+1} = P_1 x_{2i}, x_{2i+2} = P_2 x_{2i+1}, i = 0, \ldots$$

is convergent, which gives that $a_n(x) = ||x_{2n+1} - x_{2n}|| \to 0$ and $b_n(x) = ||x_{2n+2} - x_{2n+1}|| \to 0$. The analogous result holds for $c_n$ and $d_n$. Similarly we prove that the matrices $P_1 P_2$ and $P_2 P_1$ are convergent. Hence by the previous Lemma $r_n(x) \leq C\alpha^n$ for suitable $C > 0, \alpha \in (0, 1)$ and $r = a, b, c, d$. This shows that the following expression

$$||x||_* = ||x|| + \max\left(\sum_{n=0}^{\infty}(a_n(x) + b_n(x)),\ \sum_{n=0}^{\infty}(c_n(x) + d_n(x))\right)$$

is finite, and it is easy to see that $||x||_* = 0$ if and only if $x = 0$. Hence it is a norm in $\mathbb{C}^k$. (This is essentially the same construction as in (7), but in this special case we can give a closed expression for the norm). By some simple manipulations we get

$$||P_1 x||_* \leq ||x||_* - a_0(x) = ||x||_* - ||P_1 x - x||$$

and the same result for $P_2$. As there is a $\gamma > 0$ satisfying $||x|| \geq \gamma ||x||_*$ we see that $\{P_1, P_2\} \subset \mathcal{N}_\gamma(|| \ ||_*)$. ∎

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