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ON NONNEGATIVE OPERATORS AND FULLY CYCLIC PERIPHERAL SPECTRUM *

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Dedicated to Hans Schneider on the occasion of his seventieth birthday

Abstract. In this note the properties of the peripheral spectrum of a nonnegative linear operator A (for which the spectral radius is a pole of its resolvent) in a complex Banach lattice are studied. It is shown, e.g., that the peripheral spectrum of a natural quotient operator is always fully cyclic. We describe when the nonnegative eigenvectors corresponding to the spectral radius r span the kernel $N(r - A)$. Finally, we apply our results to the case of a nonnegative matrix, and show that they sharpen earlier results by B.-S. Tam [*Tamkang J. Math.* 21:65–70, 1990] on such matrices and full cyclicity of the peripheral spectrum.

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1. Introduction.

It follows from results of H.H. Schaefer [6, I.2.6 and V.4.6], that in the finite dimensional case $\mathbb{C}^n$ and in certain Banach function lattices a nonnegative operator A has a fully cyclic peripheral point spectrum iff for all $\lambda \in \mathbb{C}$ with $|\lambda| = r(A)$

$$x \in N(\lambda - A) \text{ implies } |x| \in N(r(A) - A);$$

here $r(A)$ denotes the spectral radius of A; see also B.-S. Tam [8, Lemma 2.1].

In this note we consider nonnegative operators A in a Banach lattice for which the spectral radius $r(A)$ is a pole of its resolvent. We give necessary and sufficient conditions that for a given $\lambda \in \mathbb{C}$ with $|\lambda| = r(A)$ the inclusion $\{|x| : x \in N(\lambda - A)\} \subset N(r(A) - A)$ holds; in particular, we give necessary and sufficient conditions that $N(r(A) - A)$ has a basis of nonnegative eigenvectors, and is a sublattice, respectively.

As examples show (see [8, Example 2.7]), the inclusion above is, even in the matrix case, not only a property of the spectrum and the associated directed graph of A. This will be very clear from Theorem 3.5. On the other hand, Theorem 4.2 in the matrix case and under the assumption that the nonnegative vectors in $N(r(A) - A)$ span this kernel will show that the property of $N(r(A) - A)$ being a sublattice can be characterized by properties of the reduced and of the singular graphs of A.

The main method of the investigation is the systematic application of an idea going back to Lotz and Schaefer (cf. [6]), and successfully developed by Greiner [2], [4]. The closed lattice ideals I_0 and I_1, defined below in terms of the Laurent expansion

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of the resolvent of A around the spectral radius, their quotient I_h and the restrictions of (or the induced operator by) A are the most important technical means of the study, and a number of the results presented here find their natural formulation in this terminology.

In Section 4 we compare the results of Section 3 on nonnegative operators in Banach lattices with those of B.-S. Tam [8] on nonnegative matrices. One of the main results here is Theorem 4.2, which shows that for a nonnegative square matrix A the kernel $N(r-A_1)$ for the restriction A_1 of A to the lattice ideal I_1 is a sublattice iff there does not exist a (strongly connected equivalence) class of A having access to two different distinguished basic classes (as expressed in the already standard graph theoretic terminology). The results of Section 4 sharpen the main results of Tam [8, Theorems 2.4 and 2.5] on matrices with fully cyclic peripheral spectrum.

2. Definitions and Preliminaries. In the following A denotes a nonnegative operator in a complex Banach lattice E. We assume that its spectral radius $r = r(A)$ is a pole of order p (≥ 1) of its resolvent $R(.,A)$, i.e., we have for some $\delta > 0$ and operators Q_k $(k = -p, -p+1, \ldots)$ the Laurent expansion

$$R(\lambda, A) = \sum_{k=-p}^{\infty} (\lambda - r)^k Q_k \qquad \text{if} \quad 0 < |\lambda - r| < \delta.$$

Note that Q_{-p} is nonnegative, since A is nonnegative. Following G. Greiner [2], [4, Chapters B III and C III], we define

$$I_1 = \{x \in E : (Q_{-p}|x| = \ldots = Q_{-3}|x| =) Q_{-2}|x| = 0\}$$

(hence $I_1 = E$ if $p = 1$), and

$$I_0 = \{x \in E : Q_{-1}|x| = 0\}.$$

Note that I_0 can be trivial. Then we have (see [4, p. 174 and p. 303]) that I_1 and I_0 are closed ideals of E, and they are invariant under A and Q_k $(k = -p, -p+1, \ldots)$. If A_1 and A_0 denote the restrictions of A to I_1 and I_0, respectively, then

$$\begin{array}{rcll} r(A_1) & = & r(A) & \text{is a pole of} \quad R(\cdot, A_1) \quad \text{of order } 1; \\ r(A_0) & < & r(A), & \text{here we set} \quad r(A_0) = -\infty \quad \text{if} \quad I_0 = \{0\}. \end{array}$$

Since $I_0 \subset\neq I_1$, the quotient space $I_h = I_1/I_0$ is well-defined and is a Banach lattice. Further, A_1 induces uniquely a nonnegative operator A_h in I_h such that $q_h A_1 = A_h q_h$, where q_h denotes the quotient map $I_1 \to I_1/I_0$. Then (see [4, p. 174 and p. 303]) $r(A_h) = r(A_1) = r(A) = r$, r is a pole of order 1 of $R(\cdot, A_h)$, and the residuum of $R(\cdot, A_h)$ at r (which is induced by Q_{-1} in I_h) is strictly positive in the sense that the zero element is the only nonnegative element in I_h which it maps to the zero element. Since r is a pole of order 1 of $R(\cdot, A_h)$, the associated residuum is a projection with kernel $R(r - A_h) =$ range of $r - A_h$. Therefore $R(r - A_h)$ cannot contain any nonzero element y of I_h for which either y or $-y$ is nonnegative.

Let J be the closed ideal of I_h generated by $N(r-A_h)$. J is A_h-invariant, since $N(r-A_h)$ is A_h-invariant. Let $A_i = A_h \mid J$ (i.e. the restriction of A_h to J). Further, let $I_j = I_h/J$ and $A_j = A_h/J$ (i.e. the operator induced by A_h in I_j). We set $r(A_j) = -\infty$ if $I_j = \{0\}$.

The important basic connections between different kernels and spectral radii of the operators defined above will be collected in the proposition below. The following useful lemma, which is not new, will be applied in the proof of the proposition and several times at other places of this note.

LEMMA 2.1. *Let T be a linear map in a vector space V and let M be a T-invariant linear submanifold of V. Let $T|M$ denote the restriction of T to M, let T/M denote the linear map induced by T in the quotient space V/M, and let $q_M : V \to V/M$ denote the quotient map. Then the following hold.*
(I) *If $T|M$ is bijective, then q_M maps the kernel $N(T)$ bijectively onto $N(T/M)$.*
(II) *If T/M is injective, then $N(T) = N(T \mid M)$.*

Proof. (I) Since $q_M T = (T/M) q_M$, the quotient map q_M maps $N(T)$ into $N(T/M)$. Let $z \in N(T/M)$. Take $v \in V$ such that $q_M(v) = z$, then $q_M(Tv) = T/M\, z = 0$, i.e., $Tv \in M$. Since $T|M$ is surjective, we have $Tv = (T|M)w$ for some $w \in M \subset V$. Thus $u = v - w \in N(T)$ and $q_M(u) = q_M(v) = z$. If $u \in N(T)$ satisfies $q_M(u) = 0$, then $u \in M$ and $(T|M)u = 0$. Since $T|M$ is injective, we obtain $u = 0$.
(II) Clearly we have $N(T|M) \subset N(T)$. Let $u \in N(T)$. Then $T/M\, q_M(u) = q_M(Tu) = 0$, and therefore $q_M(u) = 0$, since T/M is injective. Thus $u \in M \cap N(T) = N(T|M)$. □

PROPOSITION 2.2. *Let the assumptions and notations preceeding Lemma 2.1 hold. Then the following hold.*
(I) *$r(A) = r(A_1) = r(A_h) = r(A_i) > \max\{r(A_j), r(A_0)\}$, and $r(A)$ is a pole of order 1 of $R(\cdot, A_1), R(\cdot, A_h)$ and $R(\cdot, A_i)$, respectively.*
(II) *For all $\lambda \in \mathbb{C}$ with $|\lambda| = r(A) = r$ we have $N(\lambda - A) \supset N(\lambda - A_1)$, $N(\lambda - A_h) = N(\lambda - A_i)$, q_h maps $N(\lambda - A_1)$ bijectively onto $N(\lambda - A_h)$, $\dim N(r - A_1) \geq \dim N(\lambda - A_1) = \dim N(\lambda - A_h) = \dim N(\lambda - A_i)$, and $\{|z| : z \in N(\lambda - A_h)\} \subset N(r - A_h)$. It follows that $N(r - A_h)$ is a sublattice of I_h.*

Proof. (I) As seen above, the residuum $Q_{h,-1}$ of $R(\cdot, A_h)$ at r is strictly positive. Then its restriction $Q_{i,-1}$ to J is also strictly positive (note that $\{0\} \neq N(r - A_h) \subset J$). This implies the three equalities. Since $R(Q_{h,-1}) = N(r - A_h) \subset J$, it follows that $R(\cdot, A_j)$ is holomorphic at r. Therefore $r(A_j) < r(A)$, since A_j is a nonnegative operator in the Banach lattice I_j [3, Proposition 4.1.1.i]. Note that we have defined $r(A_j) = -\infty$ if $I_j = \{0\}$.
(II) The first inclusion is evident. For the second statement we apply Lemma 2.1 (II) with $V = I_h$, $M = J$ and $T = \lambda - A_h$. Note that $T/M = \lambda - A_j$ is injective, since $r(A_j) < r(A) = |\lambda|$. The third statement follows from Lemma 2.1 (I) if we set $V = I_1$, $M = I_0$ and $T = \lambda - A_1$. Note that $T|M = \lambda - A_0$ is bijective, since $r(A_0) < r(A) = |\lambda|$. The equality $\dim N(\lambda - A_1) = \dim N(\lambda - A_h) = \dim N(\lambda - A_i)$ is now evident. We shall prove $\dim N(r - A_1) \geq \dim N(r - A_i)$, which needs a proof only if $m = \dim N(r - A_1) < \infty$. By [4, C-III, Lemma 3.13], the ideal J is the mutually orthogonal sum of m A_i-invariant ideals J_k $(k = 1, \ldots, m)$, the restrictions

$A_{ik} = A_i|J_k$ are irreducible, and $r(A_{ik}) = r(A)$ is a pole of $R(\cdot, A_{ik})$ $(k = 1, \ldots, m)$. The eigenspaces $N(r - A_{ik})$ are one-dimensional; see [6, V. § 5]. By [6, Corollary to Theorem V.5.4], $\dim N(\lambda - A_{ik}) \leq 1$. A_i is the direct sum of the restrictions A_{ik}. Then $N(\lambda - A_i)$ is the direct sum of the $N(\lambda - A_{ik})$ $(k = 1, \ldots, m)$. Therefore $\dim N(\lambda - A_i) \leq m = \dim N(r - A_1)$. For the proof of the next statement take $z \in N(\lambda - A_h)$. Let $y = (r - A_h)\,|z|$. Then $y \leq r|z| - |A_h z| \leq |(\lambda - A_h)z| = 0$. By a remark preceding Lemma 2.1, then $y = 0$. □

3. Results and Proofs. We shall always assume that A is a nonnegative linear operator in a Banach lattice E, and its spectral radius is a pole of its resolvent.

We shall freely use the concepts and notations of Section 2. The next lemma is crucial for the main results of this note.

LEMMA 3.1. *Let $|\lambda| = r(A) = r$. For each $z \in N(\lambda - A_h)$ there exists a unique nonnegative w in $N(r - A_1)$ such that $q_h(w) = |z|$; for the unique $u \in N(\lambda - A_1)$ satisfying $q_h(u) = z$ it follows that $q_h(w) = q_h(|u|)$ and $w \geq |u|$.*

Proof. Let $z \in N(\lambda - A_h)$. By Proposition 2.2(II), there exists a unique $u \in N(\lambda - A_1)$ with $q_h(u) = z$. Proposition 2.2(II) implies $|z| \in N(r - A_h)$. Since q_h is a lattice homomorphism, we get $q_h((r - A_1)|u|) = (r - A_h)q_h(|u|) = (r - A_h)|z| = 0$, i.e. $(r - A_1)|u| \in I_0$. Since $(r - A_1)|u| \leq |(\lambda - A_1)u| = 0$, there exists a unique nonnegative u_0 in I_0 such that $(r - A_0)u_0 = -(r - A_1)|u|$ (notice that $r(A_0) < r(A) = r$, thus $r - A_0$ has a nonnegative inverse). Then $w = |u| + u_0$ is a vector we are looking for. If $v \in I_1$ satisfies $q_h(v) = q_h(|u|)$, then $|u| - v \in I_0$. If further $v \in N(r - A_1)$, then $|u| + u_0 - v \in I_0 \cap N(r - A_1) = N(r - A_0)$. But $r - A_0$ is injective, therefore $v = |u| + u_0 = w$, i.e. w is unique as stated. □

PROPOSITION 3.2. *Under our general assumptions the following hold.*

(I) *$N(r - A) \cap E_+ \subset N(r - A_1)$.*

(II) *$\mathrm{span}\,(N(r - A) \cap E_+) = N(r - A_1)$.*

(III) *the nonnegative eigenvectors of A corresponding to r span the eigenspace $N(r - A)$ iff $N(r - A) \subset I_1$ iff $N(r - A) = N(r - A_1)$.*

(IV) *if $N(r - A_1)$ is finite dimensional, then $N(r - A_1)$ has a basis of nonnegative eigenvectors of A_1 corresponding to r.*

Proof. (I) $u \in N(r - A)$ is equivalent to $R(\lambda, A)u = (\lambda - r)^{-1}u$ if $0 < |\lambda - r| < \delta$ for some $\delta > 0$. Therefore $u \in N(r - A) \cap E_+$ implies $u \in \{x \in E : Q_{-2}|x| = 0\} = I_1$.

(II) Let $u \in N(r - A_1)$. Then its real and its imaginary parts belong to $N(r - A_1)$. Therefore we assume w.l.o.g. that u is real. Choose w as in Lemma 3.1. Then $u = \frac{1}{2}(w + u) - \frac{1}{2}(w - u)$, $w \pm u \in N(r - A_1)$ and $w \pm u \in E_+$.

(III) The first part follows from $N(r - A_1) = N(r - A) \cap I_1$ and (II), the second part ist clear.

(IV) follows simply from (II). □

REMARK 3.3. In connection with (III) and (IV) recall that U.G. Rothblum [5] proved that in the case $\dim E < \infty$ the generalized eigenspace $N((r - A)^p)$ (where p is the order of the pole r) always has a basis of nonnegative vectors. However, [1, Example 18] shows that in the general case ($\dim E = \infty$) the corresponding statement can be false even if $p = 2$ and $\dim N((r - A)^2) = 2$.

THEOREM 3.4. *Let the general assumptions hold. The following assertions are*

equivalent.
(I) $N(r-A_1)$ *is a sublattice of* E*;*
(II) *If* z_1 *and* z_2 *are disjoint vectors in* $N(r-A_h)$*, then the unique nonnegative* w_1 *and* w_2 *with* $w_i \in N(r-A_1)$ *and* $q_h(w_i) = |z_i|$ *for* $i = 1,2$ *(see Lemma 3.1) are disjoint.*

Proof. (I) $\Rightarrow$ (II) Let z_1 and z_2 be disjoint vectors in $N(r-A_h)$. For $i=1,2$ choose w_i as in Lemma 3.1. Then $q_h(|\alpha_1 w_1 + \alpha_2 w_2|) = |q_h(\alpha_1 w_1 + \alpha_2 w_2)| = |\alpha_1 z_1 + \alpha_2 z_2| = |\alpha_1|\,|z_1| + |\alpha_2|\,|z_2|$ for all complex α_1 and α_2, since z_1 and z_2 are disjoint [3, Theorem 1.1.1.vi], By Lemma 3.1, we get $w_1 + w_2 = |\alpha_1 w_1 + \alpha_2 w_2|$ for all α_1 and α_2 with modulus 1, since $w_1 + w_2$ is nonnegative and $|\alpha_1 w_1 + \alpha_2 w_2|$ belongs to $N(r-A_1)$; here we have used that, by (I), $N(r-A_1)$ is a sublattice. This is equivalent to the disjointness of w_1 and w_2.
(II) $\Rightarrow$ (I) Let $u \in N(r-A_1)$. Then $z = q_h(u) \in N(r-A_h)$. Since $N(r-A_h)$ is a sublattice of I_h (see Proposition 2.2(II)), z^+ and z^- (the positive and the negative parts of z, respectively) belong to $N(r-A_h)$. Applying Lemma 3.1, we choose nonnegative w_1 and w_2 in $N(r-A_1)$ such that $q_h(w_1) = z^+$ and $q_h(w_2) = z^-$. Then $q_h(u) = z = q_h(w_1 - w_2)$. Since u and $w_1 - w_2$ belong to $N(r-A_1)$, we obtain $u = w_1 - w_2$, by Proposition 2.2(II). Now z^+ and z^- are disjoint [3, Theorem 1.1.1.iv], therefore w_1 and w_2 are disjoint, by assumption. But then $|u| = w_1 + w_2 \in N(r-A_1)$. □

For the rest of this section we assume that I_0 is a projection band in E. Note that this holds in any Banach lattice with order continuous norm [3, Corollary 2.4.4.xii, Theorem 2.4.2.ii), and Theorem 1.2.9.], and a fortiori in $\mathbb{C}^\ell$. Then I_0 is also a projection band in I_1, and we can identify I_h with the kernel of the band projection of I_1 onto I_0, i.e. $I_1 = I_h \oplus I_0$, where $\oplus$ means disjoint sum in the lattice sense. Since I_0 is A_1-invariant, we can represent A_1 with respect to this direct sum as a triangular operator matrix as follows,

$$A_1 = \begin{bmatrix} A_h & 0 \\ A_{0h} & A_0 \end{bmatrix}.$$

THEOREM 3.5. *In addition to the general assumptions we assume that* I_0 *is a projection band in* E*. Let* $u \in N(\lambda - A_1)$ *with* $|\lambda| = r(A)$ *and* $u = u_h + u_0$*, where* $u_h \in I_h$ *and* $u_0 \in I_0$*. The following assertions are equivalent.*
(I) $|u| \in N(r-A_1)$*;*
(II) $|(\lambda - A_0)u_0| = (r-A_0)|u_0|$ *and* $|A_{0h}u_h| = A_{0h}|u_h|$*.*

Proof. $u \in N(\lambda - A_1)$ is equivalent to $(\lambda - A_h)u_h = 0$ and $(\lambda - A_0)u_0 = A_{0h}u_h$. The first equality implies $(r-A_h)|u_h| = 0$, by Proposition 2.2(II). Therefore $|u| \in N(r-A_1)$ is equivalent to $(r-A_0)|u_0| = A_{0h}|u_h|$. We also obtain

$$A_{0h}|u_h| \geq |A_{0h}u_h| = |(\lambda - A_0)u_0| \geq (r-A_0)|u_0|.$$

From this inequality it follows immediately that (I) and (II) are equivalent. □

The second equality in (II) means that $A_{0h} : I_h \to I_0$ is a lattice homomorphism on $N(\lambda - A_h)$. However, even if (I) holds, A_{0h} need not be a lattice homomorphism of *all* of I_h, as the following example shows.

Example 3.6. Let

$$A = A_1 = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ & & & \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_h & 0 \\ A_{0h} & A_0 \end{bmatrix}.$$

For $x_h = [-2, 2, 0]^\top \in I_h$ we get $A_{0h} x_h = 0$ and $A_{0h}\,|x_h| = 4$.

Since $r(A_0) < r(A) = r$, the first equality in (II) implies $|u_0| = (r - A_0)^{-1}|(\lambda - A_0)u_0| \geq |(r - A_0)^{-1}(\lambda - A_0)u_0|$. We cannot expect that we have here equality, even if statement (I) in Theorem 3.5 holds, as is shown by the following example.

Example 3.7. Let

$$A = A_1 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ & & & \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} A_h & O \\ I & A_0 \end{bmatrix}.$$

The peripheral eigenvalues of the matrix A are 2 and -2. We have $u \in N(\pm 2 - A)$ iff $u = [\alpha, \pm\alpha, \pm\alpha, \alpha]^\top$, $\alpha \in \mathbb{C}$. Therefore statement (I) of Theorem 3.5 holds for $\lambda = \pm 2$. Let $u = [u_h^\top, u_0^\top]^\top \in N(-2 - A)$. Then $-u_0 = u_h = (-2 - A_0)u_0$, $(2 - A_0)^{-1}(-2 - A_0)u_0 = -\frac{1}{3}u_0$, but $(2 - A_0)^{-1}|(-2 - A_0)u_0| = |u_0|$.

In the next section we will use the following result.

Proposition 3.8. *In addition to the general assumptions we assume that I_0 is a projection band in E and that $N(r - A_1) = N(r - A_h) \oplus \{0\}$. For $\lambda \in \mathbb{C}$ with $|\lambda| = r(A) = r$ the following assertions are equivalent.*
(I) $N(\lambda - A) \subset I_1$;
(II) $\{|u| : u \in N(\lambda - A)\} \subset N(r - A)$.

Proof. (I) $\Rightarrow$ (II): From (I) we obtain $N(\lambda - A) = N(\lambda - A_1)$. Let $u \in N(\lambda - A)$ and for $z = q_h(u)$ choose w as in Lemma 3.1. Then $w \in N(r - A_1) = N(r - A_h) \oplus \{0\} \subset I_h$, and $w \geq |u|$. This implies $u \in I_h$ and then $|u| = w$, since $q_h(|u|) = q_h(w)$, where q_h is now (identified with) the band projection of I_1 onto I_h.
(II) $\Rightarrow$ (I) follows from Proposition 3.2(I); notice that we use for this implication only the general assumptions. ◻

4. The matrix case. In the last section of this note we want to compare the results in Section 3 with those of B.-S. Tam on nonnegative matrices [8, Theorems 2.4 and 2.5]. We shall use the graph theoretic concepts defined in [7, § 2], and [8, § 1].

In the complex Banach lattice $\mathbb{C}^\ell$ each ideal is of the form $I_\alpha = \{x \in \mathbb{C}^\ell : x_i \text{ (= i-th component of } x) = 0 \text{ if } i \notin \alpha\}$ for some $\alpha \subset \{1, \ldots, \ell\}$; see [6, p. 2]. Let α and β be nonempty subsets of $\{1, \ldots, \ell\}$. For $x \in \mathbb{C}^\ell$ we denote by x_α the subvector of x with indices from α. For an $\ell \times \ell$-matrix A we denote by $A_{\alpha\beta}$ the submatrix of A with row indices from α and column indices from β. We write A_α instead of $A_{\alpha\alpha}$. In the next lemma we collect some facts on the connection between the ideals defined in Section 2 and the ideals I_α for a class α of A (i.e., α is a strongly connected component in the directed graph associated with A [8, § 1].

LEMMA 4.1. *Let A be a nonnegative square matrix, and let α be a class of A.*
(I) *If I is an A-invariant ideal and $I_\alpha \cap I \neq \{0\}$, then $I_\alpha \subset I$.*
(II) *If $I_\alpha \subset I_0 \oplus I_j$, then α is nonbasic (i.e. $r(A_\alpha) < r(A)$).*
(III) *Let α be a basic class. Then α is distinguished iff $I_\alpha \subset I_1$ iff $I_\alpha \subset I_h$ iff $I_\alpha = J_k$ for some $k \in \{1, \ldots, m\}$.*
(IV) *If α has access to a distinguished basic class of A, then either α is this distinguished class of A or $I_\alpha \subset I_0$.*

Proof. (I) The ideal $I_\alpha \cap I$ is A_α-invariant. Since A_α is an irreducible matrix, $I_\alpha \cap I \neq \{0\}$ implies $I_\alpha \cap I = I_\alpha$.
(II) We have $r(A_\alpha) \leq \max\{r(A_0), r(A_j)\} < r(A)$.
(III) Let α be a distinguished basic class of A. Then there exists a nonnegative eigenvector x of A corresponding to r, such that x_α is strictly positive; see [7, Theorem 3.1]. By Proposition 3.2(I), we have that $x \in I_1$. Therefore $I_\alpha \subset I_1$. If α is a basic class of A with $I_\alpha \subset I_1$, then $I_\alpha \subset J \subset I_h$, by (II). Let α be a basic class of A with $I_\alpha \subset I_h$. Then $I_\alpha \subset J$, since $r(A_j) < r(A)$. A_i is the direct sum of the irreducible operators $A_{ik} = A_i|J_k$ ($k = 1, \ldots, m$), so we get $A_\alpha = A_{ik}$ for some $k \in \{1, \ldots, m\}$. Then $I_\alpha = J_k$ for this k. If $I_\alpha = J_k$ for some $k \in \{1, \ldots, m\}$, then $r(A_\alpha) = r(A)$ and A_α is irreducible. Therefore α is a strongly connected set in the directed graph associated with A. From the triangular structure of A_1 with respect to the direct sum $I_1 = I_j \oplus J_1 \oplus \ldots \oplus J_m \oplus I_0$ it follows that α is a strongly connected component in this graph.
(IV) Let α have access to a distinguished basic class β of A. Assume $\alpha \neq \beta$. Then α is nonbasic. Furthermore, there exists a nonnegative eigenvector x of A corresponding to r, such that x_α ist strictly positive; see [7, Theorem 3.1]. Proposition 3.2(I) implies $x \in I_1$. Therefore $x \in N(r - A_1)$. From Proposition 2.2(II) it follows that $N(r - A_1) \subset J \oplus I_0$. Therefore $I_\alpha \subset J \oplus I_0$. From the last part of the proof of (III) we see that $I_\alpha \cap J \neq \{0\}$ would imply $r(A_\alpha) = r(A)$. Hence we obtain $I_\alpha \subset I_0$, since α is nonbasic. ∎

B.-S. Tam [8, Lemma 2.1] showed that A has fully cyclic peripheral spectrum (see [6, Definition 2.5]), iff for all $\lambda \in \mathbb{C}$ with $|\lambda| = r(A)$

$$\{|u| : u \in N(\lambda - A)\} \subset N(r - A).$$

Thus, if A has a fully cyclic peripheral spectrum, then $N(r - A)$ is a sublattice of $\mathbb{C}^\ell$. By Proposition 3.2, the latter assertion implies statement (a) in [8, Theorems 2.4 and 2.5]. In the matrix case we have the following result as a supplement to Theorem 3.4.

THEOREM 4.2. *For a nonnegative square matrix the following assertions are equivalent.*
(I) *$N(r - A_1)$ is a sublattice, where $r = r(A)$,*
(II) *there does not exist a class of A which has access to two different distinguished basic classes.*

Proof. We prove that the assertion (II) is equivalent to Theorem 3.4(II).
(I) $\Rightarrow$ (II) Assume there exists a class α of A which has access to two different

distinguished basic classes α_1 and α_2. Then α has to be a nonbasic class, since a basic class does not have access to a distinguished basic class. By Lemma 4.1(IV) this implies $I_\alpha \subset I_0$. From [7, 3.1(i)] it follows that there are two nonnegative eigenvectors w_1 and w_2 of A corresponding to r with $\alpha \subset \text{supp}(w_i)$, $\alpha_i \subset \text{supp}(w_i)$ and $\alpha_i = \text{supp}(q_h(w_i))$. By Proposition 3.2(II), $w_i \in N(r - A_1)$, $i = 1, 2$. Since α_1 and α_2 are disjoint sets, we get that $q_h(w_1)$ and $q_h(w_2)$ are disjoint nonnegative vectors in $N(r - A_h)$. But w_1 and w_2 are not disjoint, and this contradicts Theorem 3.4(II).
(II) $\Rightarrow$ (I) If z is in $N(r - A_h)$, then $|z|$ belongs to $N(r - A_h)$; see Proposition 2.2(II). Then $\text{supp}(z) = \text{supp}(|z|)$ is the union of some distinguished basic classes. From [7, 3.1(i) and (ii)] it follows that for the unique nonnegative $w \in N(r - A_1)$ with $q_h(w) = |z|$ its support is the union of all classes which have access to distinguished basic classes contained in $\text{supp}(z)$. Since w is a nonnegative eigenvector of A, the ideal generated by w is A-invariant; this ideal is I_σ, where $\sigma = \text{supp}(w)$. Now let z_1 and z_2 be disjoint vectors in $N(r - A_h)$, and assume that the unique nonnegative w_1 and w_2 in $N(r - A_1)$ with $q_h(w_i) = |z_i|$ for $i = 1, 2$ are not disjoint. Then there exists a class α of A with $I_\alpha \cap I_{\sigma_1} \cap I_{\sigma_2} \neq \{0\}$, where $\sigma_i = \text{supp}(w_i)$ for $i = 1, 2$. Since I_{σ_i} are A-invariant for $i = 1, 2$, we have $I_\alpha \subset I_{\sigma_1} \cap I_{\sigma_2}$. Then α has access to a distinguished basic class α_1 in σ_1 and to one α_2 in σ_2. Now $\alpha_i \subset \text{supp}(z_i)$. Therefore $I_{\alpha_i} \subset I_h$. By Lemma 4.1(III), the α_i are distinguished. Thus α has access to two different distinguished basic classes of A. This contradicts (II), so w_1 and w_2 are disjoint. Thus Theorem 3.4(II) holds. □

Theorem 4.2 and Proposition 3.2 show that statements (a) and (b) in [8, Theorem 2.4] together are equivalent to the statement that $N(r - A)$ is a sublattice of $\mathbb{C}^\ell$.

The next theorem will show, how conditions (c) in Theorems 2.4 and 2.5 of [8] are related to our results.

THEOREM 4.3. *Consider for a nonnegative square matrix A and a peripheral eigenvalue λ of A satisfying the following statements.*
(I) *If λ is an eigenvalue of A_α for some class α of A, then α is distinguished.*
(II) $N(\lambda - A) \subset I_1$.
(III) *If λ is an eigenvalue of A_α for some class α of A, then all initial classes of the family*

$$F(\alpha) = \{\gamma : \gamma \text{ is a class of } A,\ \lambda \text{ is an eigenvalue of } A_\gamma,\ \gamma >= \alpha\}$$

are distinguished.
(IV) *If λ is an eigenvalue of A_α for some class α of A, then λ is also an eigenvalue of A_β for some distinguished class β of A, which has access to α.*
(V) *λ is an eigenvalue of A_β for some distinguished class β of A.*
(VI) $N(\lambda - A) \cap I_1 \neq \{0\}$.
Then (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Leftrightarrow$ (IV) $\Rightarrow$ (V) $\Leftrightarrow$ (VI).

Proof. (I) $\Rightarrow$ (II) Let $0 \neq u \in N(\lambda - A)$. Then for all classes α of A, which are final in $\text{supp}(u)$, λ is an eigenvalue of A_α; see [8, Lemma 2.3]. Therefore, by assumption, all (final) classes in $\text{supp}(u)$ have access to a distinguished basic class of A. By Lemma 4.1(III) and (IV), we have $u \in I_1$.
(II) $\Rightarrow$ (III) Take an initial class β in $F(\alpha)$. Notice that $F(\alpha)$ is nonempty, since α

belongs to it. Then β is basic, since $|\lambda| = r(A)$. Let ω be the union of all classes of A, which have access to β and are different from β. Set $\delta = \{1,\ldots,\ell\}\backslash(\omega\cup\beta)$. Note that one or both of the sets ω and δ can be empty. Then the corresponding matrices in the decomposition below do not appear. With respect to the decomposition $\mathbb{C}^\ell = I_\delta \oplus I_\beta \oplus I_\omega$ the matrix A has the block triangular form

$$\begin{bmatrix} A_\delta & 0 & 0 \\ A_{\beta\delta} & A_\beta & 0 \\ A_{\omega\delta} & A_{\omega\beta} & A_\omega \end{bmatrix}.$$

By the choice of β, λ is not an eigenvalue of A_ω. Take an eigenvector u_β of A_β corresponding to λ. Define $u_\omega = (\lambda - A_\omega)^{-1}A_{\omega\beta}u_\beta$ and $u = [o^\top, u_\beta^\top, u_\omega^\top]^\top$. Then u is an eigenvector of A corresponding to λ. Now $u \in N(\lambda - A) \subset I_1$ by assumption. Then $o \neq [o^\top, u_\beta^\top, o^\top]^\top \in I_\beta \cap I_1$ implies $I_\beta \cap I_1 \neq \{0\}$. By Lemma 4.1(I), we have $I_\beta \subset I_1$. Since β is basic, we get from Lemma 4.1 (III) that it is distinguished.

(III) $\Rightarrow$ (IV) is clear.

(IV) $\Rightarrow$ (III) Take an initial class κ of A in $F(\alpha)$. Then λ is an eigenvalue of A_κ. By assumption, there exists a distinguished basic class β of A with $\beta >= \kappa$ and λ is an eigenvalue of A_β. Then $\beta >= \kappa >= \alpha$, thus $\beta \in F(\alpha)$. Therefore $\kappa = \beta$, since κ is initial in $F(\alpha)$.

(IV) $\Rightarrow$ (V) is clear, since each eigenvalue of A has to be an eigenvalue of A_α for at least one class α of A.

(V) $\Rightarrow$ (VI) For a distinguished basic class β let u_β be an eigenvector of A_β corresponding to λ. By Lemma 4.1(III), $A_\beta = A_{ik}$ for some $k \in \{1,\ldots,m\}$. Let $u_h = [u_1^\top,\ldots,u_m^\top]^\top$ with $u_k = u_\beta$ and $u_i = 0$ if $i \neq k$. The vector u_h is an eigenvector of A_h corresponding to λ, since $A_h = A_{i1}\oplus\ldots\oplus A_{im}$. Define $u_0 = (\lambda - A_0)^{-1}A_{0h}u_h$, and $u_1 = [u_h^\top, u_0^\top]^\top$. Then u_1 is an eigenvector of A_1 corresponding to λ. Thus $\{0\} \neq N(\lambda - A_1) = N(\lambda - A)\cap I_1$.

(VI) $\Rightarrow$ (V) Let $u \in I_1$ be an eigenvector of A corresponding to λ. If β is a final class of A in supp(u), then β is a basic class [8, Lemma 2.3]. Since $I_\beta \subset I_1$, β is distinguished (see Lemma 4.1(III). $\square$

The following examples will show that the converses of the three one-way implications in Theorem 4.3 are not true in general.

Example 4.4. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ a & b & 0 & 0 & 1 \\ c & d & 0 & 1 & 0 \end{bmatrix} \quad \text{with nonnegative numbers } a,b,c \text{ and } d.$$

Then $\alpha = \{1,2\}$, $\beta = \{3\}$ and $\gamma = \{4,5\}$ are the classes of A. The matrices A_α, A_γ and A have eigenvalues 1 and -1. The matrix A_β has the eigenvalue 1. Therefore all classes of A are basic. The classes β and γ are distinguished and α is not distinguished. Further, here $I_1 = \{x \in \mathbb{C}^5 : x_1 = x_2 = 0\} = I_\beta \oplus I_\gamma$ and $I_0 = \{0\}$. Now let

(1) $a = d = 1$ and $b = c = 0$. Then statement (I) is not true for $\lambda = -1$, but $N(-1-A) = \{x \in \mathbb{C}^5 : x_1 = x_2 = x_3 = 0,\ x_4 + x_5 = 0\} \subset I_1$. For $\lambda = r(A)$ statement (I) in Theorem 4.3 is equivalent to: "All basic classes of A are distinguished." This is equivalent to $p = 1$, where p is the order of the pole $r(A)$ of the resolvent, cf. Section 2. The example also shows that even for $\lambda = r(A)$ (II) does not imply (I), since $N(1-A) = \{x \in \mathbb{C}^5 : x_1 = x_2 = 0,\ x_4 - x_5 = 0\} \subset I_1$ and $p = 2$.

(2) $a = b = c = d = 1$. Then statement (II) is not true for $\lambda = -1$, since $N(-1-A) = \{x \in \mathbb{C}^5 : x_1 + x_2 = 0,\ x_2 + 2x_3 = 0,\ x_4 + x_5 = 0\}$. But statement (III) is true for $\lambda = -1$, since $F(\alpha) = \{\gamma, \alpha\}$ with $\gamma >\!\!-\; \alpha$ and $F(\gamma) = \{\gamma\}$.

(3) $a = b = c = d = 0$. Then statement (IV) is not true for $\lambda = -1$, since $\gamma >\neq \alpha$. But statement (V) is true, since -1 is an eigenvalue of A_γ.

Part (b) of [8, Theorem 2.5], which states that each distinguished basic class of A is initial (in the family of all classes of A), is equivalent to $N(r-A_1) = N(r-A_h) \oplus \{0\}$. This follows immediately from [7, 3.1(i) and (ii)]. Combining this observation with Proposition 3.8 we obtain the following result.

Corollary 4.5. *Let A be a nonnegative square matrix, such that each distinguished class of A is initial (in the family of all classes of A). Then A has a fully cyclic peripheral spectrum iff $N(\lambda - A) \subset I_1$ for all peripheral eigenvalues of A.*

This corollary is stronger than [8, Theorem 2.5] as Theorem 4.3 (I) $\Rightarrow$ (II) and Example 4.4(1) show.

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