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EXTREMIZING ALGEBRAIC CONNECTIVITY SUBJECT TO GRAPH THEORETIC CONSTRAINTS*

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Dedicated to Hans Schneider on the occasion of his seventieth birthday.

Abstract. The main problem of interest is to investigate how the algebraic connectivity of a weighted connected graph behaves when the graph is perturbed by removing one or more connected components at a fixed vertex and replacing this collection by a single connected component. This analysis leads to exhibiting the unique (up to isomorphism) trees on n vertices with specified diameter that maximize and minimize the algebraic connectivity over all such trees. When the radius of a graph is the specified constraint the unique minimizer of the algebraic connectivity over all such graphs is also determined. Analogous results are proved for unicyclic graphs with fixed girth. In particular, the unique minimizer and maximizer of the algebraic connectivity is given over all such graphs with girth 3.

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1. Introduction.

A *weighted graph* is an undirected graph G with the additional property that for each edge ε in G, there is an associated positive number, $w(\varepsilon)$, called the *weight of* ε. In the special (but important) case when all the weights are equal to 1, we refer to G as an *unweighted graph*. Given any weighted graph G on vertices $1, 2, \ldots, n$, we define its *Laplacian matrix* $L = (l_{ij})$ as follows,

$$l_{ij} = \begin{cases} -w(\varepsilon), & \text{if } i \neq j \text{ and } \varepsilon \text{ is the edge joining } i \text{ and } j; \\ 0, & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j; \\ -\sum\limits_{k \neq i} l_{ik}, & \text{if } i = j. \end{cases}$$

It is routine to verify that the Laplacian matrix L of a weighted graph G is a symmetric positive semidefinite M-matrix, and since the all ones vector is a null vector, L is always singular. Fiedler [F1] showed that 0 is a simple eigenvalue whenever G is connected. The second-smallest eigenvalue of L, called the *algebraic connectivity* (see [F1]) has received much attention recently; see, e.g., [C, M2, M3, MO2] for surveys and books, [GM1, GM2, KN, KNS, M1], for applications of the algebraic connectivity of trees, and [FK, GMS, MO1], for applications of the algebraic connectivity of graphs. As the name suggests, it seems to provide an algebraic measure of the connectivity of a weighted graph. (See [F1] for a number of results which demonstrate reasons for choosing such a name for this eigenvalue.)

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The eigenvectors corresponding to the algebraic connectivity, called *Fiedler vectors*, also are of interest. Motivated by [F2, Thms. 3.12, 3.14] Fiedler vectors have received much attention recently; see, e.g., [FK, KN, KNS]. Let G be a connected graph with more than one vertex. A vertex v of G is called a *point of articulation* (or *cutpoint*) if $G \setminus v$, the graph obtained from G by removing v and all of its incident edges, is disconnected. A graph with no points of articulation is called *2-connected*, and a *block* in a weighted graph G is a maximal 2-connected subgraph. Equivalently, the blocks of the graph G are the subgraphs induced by the edges in a single equivalence class, given via the following relation: any two edges are equivalent if and only if there is cycle in the graph containing both edges; see also [F2]. We adopt the following terminology from [F2]: for a weighted graph G and Fiedler vector y, we say y gives a *valuation* of the vertices of G, and for each vertex i of G, we associate the number y_i, which is the *valuation of vertex* i. With this terminology in mind we now state the following theorem of Fiedler [F2], which describes some of the structure of a Fiedler vector. We note that the statement of this result in [F2] is for unweighted graphs, but the proof carries over verbatim for the general case.

THEOREM 1.1. [F2] *Let G be a connected weighted graph, and let y be a Fiedler vector of G. Then exactly one of the following two cases occurs.*
Case A: *There is a single block B_0 in G which contains both positively and negatively valuated vertices. Each other block of G has either vertices with positive valuation only, vertices with negative valuation only, or vertices with zero valuation only. Every path P which contains at most two points of articulation in each block, which starts in B_0 and contains just one vertex v in B_0 has the property that the valuations at the points of articulation contained in P form either an increasing, or decreasing, or a zero sequence along this path according to whether $y_v > 0$, $y_v < 0$, or $y_v = 0$, respectively; in the last case all vertices in P have valuation zero.*
Case B: *No block of G contains both positively and negatively valuated vertices. There exists a unique vertex z which has valuation zero and is adjacent to a vertex with non-zero valuation. This vertex z is a point of articulation. Each block contains (with the exception of z) either vertices with positive valuation only, vertices with negative valuation only, or vertices with zero valuation only. Every path P which contains at most two points of articulation in each block, and which starts at z has the property that the valuations at its points of articulation either increase, in which case all valuations of vertices on P are (with the exception of z) positive, or decrease, in which case all valuations of vertices on P are (with the exception of z) negative, or all valuations of vertices on P are zero. Every path containing both positively and negatively valuated vertices passes through z.*

We note here that Theorem 1.1, in either case, seems to identify a "middle" of the graph G (i.e., the special block B_0 in Case A, or the special vertex z in Case B), such that as we move through points of articulation away from that middle, the entries in the Fiedler vector behave monotonically. Throughout this paper we only consider connected weighted graphs.

In the special and well-studied case when G is a weighted tree, every non-pendant vertex is a point of articulation and the blocks are simply the edges of G. Consequently, Theorem 1.1 gives plenty of information on the Fiedler vectors for a tree.

For brevity, when Case B holds, the tree is called a *Type I tree* and the special vertex z is the *characteristic vertex*; when Case A holds, the tree is said to be *Type II* and the vertices which are the end points of the special block B_0 (which is an edge) are called *characteristic vertices* of the tree. Algebraic connectivity and Fiedler vectors for trees have been well studied recently; see, e.g., [GM1, GM2, KN, KNS, M1].

A theme used throughout this paper is to consider the connected components at a vertex and look at the inverses of the principal submatrices of the Laplacian corresponding to those components; see also [FK, KN, KNS]. Each such inverse A is entry-wise positive, and so by Perron's Theorem (see [HJ] for a discussion of the Perron-Frobenius theory) A has a simple positive dominant eigenvalue, called the *Perron value* and is denoted by $\rho(A)$, and a corresponding eigenvector with all entries positive, called the *Perron vector*.

To be more precise, let G be a weighted graph with Laplacian L. For a vertex v of G, we refer to the connected components of $G \setminus v$ as the *connected components at* v, and denote them by $C_1, C_2, \ldots, C_k$ (note that $k \geq 2$ if and only if v is a point of articulation); for each such component let $L(C_i)$ be the principal submatrix of L corresponding to the vertices of C_i. Similarly, if y is a vector and v is a vertex of G, then we denote the entry in y corresponding to vertex v by $y(v)$. For any connected component C_i, we refer to $L(C_i)^{-1}$ as the *bottleneck matrix for* C_i; see [KNS], where a bottleneck matrix is defined in the case of trees. The *Perron value* of C_i is the Perron value of the entry-wise positive matrix $L(C_i)^{-1}$, and we say that C_j is a *Perron component at* v if its Perron value is maximal among $C_1, C_2, \ldots, C_k$.

The following results, taken from [FK], demonstrate how the concepts of algebraic connectivity and Fiedler vectors for weighted graphs can be reformulated in terms of Perron components and bottleneck matrices. We let e denote the all ones vector, and J denote the matrix of all ones ($J = ee^T$).

PROPOSITION 1.2. [FK] *Suppose G is a weighted graph with Laplacian L and algebraic connectivity μ, and that Case A of Theorem 1.1 holds. Let y be a Fiedler vector, and let B_0 be the unique block of G containing both positively and negatively valuated vertices in y. If v is a point of articulation of G, then let C_0 denote the set of vertices in the connected component of $G \setminus v$ which contains the vertices in B_0, and let C_1 denote the vertices in $G \setminus C_0$. If necessary, permute and partition the Laplacian (which we still denote by L) as*

$$L = \left[\begin{array}{c|c} L(C_1) & \begin{matrix} 0 \\ -\theta^T \end{matrix} \\ \hline 0 \;\; -\theta & L(C_0) \end{array}\right]$$

(here vertex v corresponds to the last row of $L(C_1)$), and partition y as $[y_1^T, y_0^T]^T$. If $y(v) \neq 0$, then

$$L(C_1)^{-1} + \frac{\theta^T y_0}{\theta^T e(\theta^T e y(v) - \theta^T y_0)} J$$

is a positive matrix whose Perron value is $1/\mu$ and whose Perron vector is a scalar multiple of y_1.

Proposition 1.3. [FK] *Let G be a weighted graph with algebraic connectivity μ. Case A of Theorem 1.1 holds if and only if there is a unique Perron component at every vertex of G. Case B of Theorem 1.1 holds if and only if there is a unique vertex z such that there are two or more Perron components at z. Further, in this case the algebraic connectivity μ of G is given by $1/\rho(L(C)^{-1})$ for any Perron component C at z, and the algebraic multiplicity of μ is one less than the number of Perron components at z.*

Proposition 1.4. [FK] *Let G be a weighted graph. Then one of the following is always satisfied.*
(i) If Case A of Theorem 1.1 holds with B_0 as the unique block of G containing both positively and negatively valuated vertices, then for every vertex v of G, the unique Perron component at v is the component containing vertices in B_0.
(ii) If Case B of Theorem 1.1 holds with vertex z as the unique vertex which has valuation zero and is adjacent to a vertex with non-zero valuation, then for any vertex $v \neq z$, the unique Perron component at v is the component containing z.

We illustrate Proposition 1.3 with the following example.

Example 1.5. Let G be an unweighted connected graph with cut-point v, and connected components $C_1, C_2, \ldots, C_k$, at v. Then the algebraic connectivity of G is at most one. This can be seen as follows: form $\tilde{G}$ from G by adding edges so that v is adjacent to every other vertex, and so that within each connected component C_i at v, all possible edges are present. A result of Fiedler ([F1]) implies that the algebraic connectivity $\tilde{\mu}$ of $\tilde{G}$, is at least μ, the algebraic connectivity of G. A straightforward calculation shows that each connected component at v (in $\tilde{G}$) has Perron value one. Thus by Proposition 1.3, Case B holds for $\tilde{G}$, and $\tilde{\mu} = 1$ (with algebraic multiplicity $k-1$). Hence $\mu \leq 1$, as desired.

In Section 2 of this paper we investigate the behaviour of the algebraic connectivity of weighted graphs when one or more connected components of the weighted graph is replaced by a single connected component. In the next section we use the results of Section 2, and others to determine the unweighted trees on n vertices with a specified diameter which maximize and minimize the algebraic connectivity over all such trees. These results are then used to determine the minimizer of the algebraic connectivity over all connected graphs with specified radius and number of vertices. Finally, in Section 4 we prove analogous results (as in Section 3) for unicyclic graphs with fixed girth. In particular, when the girth is 3, we exhibit the unique minimizer and maximizer over all such unicyclic graphs.

2. Graph Perturbation Results. We begin with some notation. For square entry-wise nonnegative matrices A and B (not necessarily of the same order), we use the notation $A \ll B$ to mean that there exist permutation matrices P and Q such that PAP^T is entry-wise dominated by a principal submatrix of QBQ^T, with strict inequality in at least one position in the case A and B have the same order. Note that if $A \ll B$, then for all $\varepsilon \geq 0$ such that both $A - \varepsilon J$ and $B - \varepsilon J$ are positive, we have $\rho(A - \varepsilon J) < \rho(B - \varepsilon J)$. For any symmetric matrix M we let $\lambda_1(M)$ denote the largest eigenvalue of M. We begin with two lemmas.

LEMMA 2.1. *Suppose G is a weighted graph with Laplacian L, and that Case A of Theorem 1.1 holds. Let B_0 be the unique block of G containing both positively and negatively valuated vertices. Suppose v is a point of articulation of G. As before let C_0 denote the set of vertices in the connected component of $G \setminus v$ which contains the vertices in B_0, and let C_1 denote the vertices in $G \setminus C_0$. Assume (without loss of generality) that L is partitioned as*

$$L = \left[\begin{array}{c|c} L(C_1) & \begin{array}{c} 0 \\ \\ -\theta^T \end{array} \\ \hline 0 \;\; -\theta & L(C_0) \end{array}\right].$$

If there exists an $x > 0$ such that

$$\frac{1}{\alpha} = \rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J) = \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J),$$

then α is an eigenvalue of L.

Proof. Let y_1 and y_0 be eigenvectors corresponding to $\frac{1}{\alpha}$ for the matrices $L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J$ and $L(C_0)^{-1} - \frac{x}{\theta^T e}J$, respectively. If $e^T y_0 = 0$, then

$$\frac{1}{\alpha}y_0 = (L(C_0)^{-1} - \frac{x}{\theta^T e}J)y_0 = L(C_0)^{-1}y_0,$$

so that $\frac{1}{\alpha}\theta^T y_0 = \theta^T L(C_0)^{-1}y_0 = e^T y_0$. Hence $\theta^T y_0 = 0$. It follows that $[0^T, y_0^T]^T$ is an eigenvector of L corresponding to the eigenvalue α.

So suppose $e^T y_0 \neq 0$, and normalize y_1 and y_0 so that $e^T y_0 = -e^T y_1$. We have

$$L(C_1)^{-1}y_1 - \frac{1}{\theta^T e}Jy_1 + \frac{x}{\theta^T e}Jy_1 = \frac{1}{\alpha}y_1, \tag{1}$$

which yields $\alpha y_1 = L(C_1)y_1 - (x-1)\alpha e_v e^T y_1$, since $L(C_1)e = (\theta^T e)e_v$, where e_v is the standard basis vector with a unique 1 in the entry corresponding to v. Also

$$L(C_0)^{-1}y_0 - \frac{x}{\theta^T e}Jy_0 = \frac{1}{\alpha}y_0, \tag{2}$$

which implies $\alpha y_0 = L(C_0)y_0 + \dfrac{\alpha x e^T y_0}{e^T \theta}\theta$. Now (2) implies $\theta^T L(C_0)^{-1}y_0 - xe^T y_0 = \frac{1}{\alpha}\theta^T y_0$, which in turn implies $(1-x)e^T y_0 = \frac{1}{\alpha}\theta^T y_0$. Thus $L(C_1)y_1 - (x-1)\alpha e_v e^T y_1 = L(C_1)y_1 - (\theta^T y_0)e_v$. Hence $\alpha y_1 = L(C_1)y_1 - (\theta^T y_0)e_v$. Also multiplying on the left of (1) by e_v^T gives $\frac{x}{\theta^T e}e^T y_1 = \frac{1}{\alpha}y(v)$. Thus $\alpha y_0 = L(C_0)y_0 - y(v)\theta$. We now have that $[y_1^T, y_0^T]^T$ is an eigenvector of L corresponding to α. □

LEMMA 2.2. *Suppose G is a weighted graph with Laplacian L and algebraic connectivity μ, and that Case A of Theorem 1.1 holds. Let B_0 be the unique block of G containing both positively and negatively valuated vertices. Suppose v is a point of articulation of G. As before let C_0 denote the set of vertices in the connected component of $G \setminus v$ which contains the vertices in B_0, and let C_1 denote the vertices in $G \setminus C_0$. Assume (without loss of generality) that L is partitioned as*

$$L = \left[\begin{array}{c|c} L(C_1) & \begin{array}{c} 0 \\ \\ -\theta^T \end{array} \\ \hline 0 \;\; -\theta & L(C_0) \end{array}\right].$$

Then there exists an $x > 0$ such that

$$\frac{1}{\mu} = \rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J) = \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J).$$

Proof. Let y be a Fiedler vector and suppose $y(v) \neq 0$. By Proposition 1.2, $L(C_1)^{-1} + \dfrac{\theta^T y_0}{\theta^T e(\theta^T e y(v) - \theta^T y_0)} J$ has Perron value $1/\mu$. Now $L(C_0)y_0 = \mu y_0 + \theta y(v)$, and since $e^T L(C_0) = \theta^T$, it follows that $\frac{1}{\mu} y_0 = L(C_0)^{-1} y_0 + \frac{y(v)}{\mu} e$. Also $e^T L(C_0) y_0 = \theta^T y_0 = \mu e^T y_0 + e^T \theta y(v)$. Note that $e^T y_0 \neq 0$, and therefore $1/\mu = e^T y_0 / (\theta^T y_0 - \theta^T e y(v))$. Then

$$\left(L(C_0)^{-1} + \frac{y(v)}{\theta^T y_0 - \theta^T e y(v)} J\right) y_0 = \frac{1}{\mu} y_0.$$

Next observe that $\dfrac{y(v)}{\theta^T y_0 - \theta^T e y(v)} + \dfrac{\theta^T y_0}{\theta^T e(\theta^T e y(v) - \theta^T y_0)} = -1/\theta^T e$, so setting $x = -\theta^T e(\dfrac{y(v)}{\theta^T y_0 - \theta^T e y(v)})$, we have

$$\frac{1}{\mu} = \rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J) = \lambda_i(L(C_0)^{-1} - \frac{x}{\theta^T e}J), \tag{3}$$

for some i. Now if $\lambda_i(L(C_0)^{-1} - \frac{x}{\theta^T e}J) < \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J)$, then there exists $\tilde{x} > x$ such that

$$\rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{\tilde{x}}{\theta^T e}J) = \lambda_1(L(C_0)^{-1} - \frac{\tilde{x}}{\theta^T e}J) = \frac{1}{\alpha},$$

since the left-hand side of (3) is increasing in x and the right-hand side of (3) is nonincreasing in x, and so α is an eigenvalue of L (by Lemma 2.1). But then $0 < \alpha < \mu$ a contradiction. Hence $i = 1$ as desired.

Finally, suppose that $y(v) = 0$. Then, by Theorem 1.1, $y_1 = 0$ and $e^T y_0 = 0$, so that $\theta^T y_0 = 0$ also. We then have that $L(C_0)^{-1} y_0 = \frac{1}{\mu} y_0$ and for all $x > 0$, $L(C_0)^{-1} y_0 - \frac{x}{\theta^T e} J = \frac{1}{\mu} y_0$, so that $\lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J) \geq \frac{1}{\mu}$, for all $x > 0$. Now

$$\rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J) < \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J)$$

at $x = 0$, since C_0 is the unique Perron component at v; the left-hand side goes to ∞ as $x \to \infty$, while the right-hand side is nonincreasing in x, so for some x,

$$\rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J) = \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J) = \frac{1}{\alpha},$$

and α is an eigenvalue of L. Necessarily $\mu \geq \alpha$, from which we conclude $\mu = \alpha$. ∎

Since we may wish to replace more than one connected component at a single vertex by a new single connected component we need the following result which describes the general form of a bottleneck matrix for more than one connected component.

Proposition 2.3. *Let G be a connected graph, and suppose that v is a point of articulation of G, with connected components $C_1, C_2, \ldots, C_{k+1}$ at v. Let D be a proper subset of the vertices of C_{k+1}. Without loss of generality, we can write $L(C_1 \cup C_2 \cup \cdots \cup C_k \cup D \cup \{v\})$ as*

$$\left[\begin{array}{ccc|c|c} L(C_1) & 0 & 0 & 0 & -L(C_1)e \\ 0 & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & L(C_k) & 0 & -L(C_k)e \\ \hline 0 & \cdots & 0 & L(D) & -\theta \\ \hline -e^T L(C_1) & \cdots & -e^T L(C_k) & -\theta^T & d \end{array}\right],$$

where $d \geq \sum_{i=1}^{k} e^T L(C_i)e$. Then $(L(C_1 \cup C_2 \cup \cdots \cup C_k \cup D \cup \{v\}))^{-1}$ can be written as

$$\left[\begin{array}{c|c|c} M + \delta J & \delta e \theta^T (L(D))^{-1} & \delta e \\ \hline \delta (L(D))^{-1} \theta e^T & (L(D))^{-1} + \delta (L(D))^{-1} \theta \theta^T (L(D))^{-1} & \delta (L(D))^{-1} \theta \\ \hline \delta e^T & \delta \theta^T (L(D))^{-1} & \delta \end{array}\right],$$

where $\delta = 1/(d - \sum_{i=1}^{k} e^T L(C_i)e - \theta^T (L(D))^{-1} \theta)$, and

$$M = \begin{bmatrix} (L(C_1))^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (L(C_k))^{-1} \end{bmatrix}.$$

Proof. This will follow from a direct computation, provided that we can establish that $d - \sum_{i=1}^{k} e^T L(C_i)e - \theta^T (L(D))^{-1} \theta$ is positive. Note that the entire Laplacian matrix can be written as

$$\left[\begin{array}{ccc|c|c|c} L(C_1) & 0 & 0 & 0 & -L(C_1)e & 0 \\ 0 & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & L(C_k) & 0 & -L(C_k)e & 0 \\ \hline 0 & \cdots & 0 & L(D) & -\theta & -X \\ \hline -e^T L(C_1) & \cdots & -e^T L(C_k) & -\theta^T & d & -y^T \\ \hline 0 & \cdots & 0 & -X^T & -y & L(C_{k+1} \setminus D) \end{array}\right].$$

Now $L(D)e = \theta + Xe$, so that $\theta^T (L(D))^{-1} \theta = \theta^T e - \theta^T (L(D))^{-1} Xe$. Since each row sum of L is zero it follows that $d = \sum_{i=1}^{k} e^T L(C_i)e + \theta^T e + y^T e$. Hence $d - \sum_{i=1}^{k} e^T L(C_i)e -$

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$\theta^T(L(D))^{-1}\theta = y^Te + \theta^T(L(D))^{-1}Xe$. Observe that $y^Te > 0$ if and only if there's an edge from v to some vertex in $C_{k+1} \setminus D$, while $\theta^T(L(D))^{-1}Xe > 0$ if and only if there is a walk from v to a vertex in $C_{k+1} \setminus D$ going through a vertex in D. Since G is connected, we see that $y^Te + \theta^T(L(D))^{-1}Xe$ is positive, as desired. □

We are now in a position to prove our main perturbation results. We divide this result into two separate statements in accordance with Theorem 1.1.

THEOREM 2.4. *Suppose G is a weighted graph with Laplacian L and algebraic connectivity μ, and that Case B of Theorem 1.1 holds with vertex z as the unique vertex which has valuation zero and is adjacent to a vertex with non-zero valuation. Let v be any point of articulation, with connected components $C_1, C_2, \ldots, C_k$ at v. Let $C_{i_1}, C_{i_2}, \ldots, C_{i_j}$ be any collection of connected components at v such that the vertex set $C = \cup_{l=1}^{j} C_{i_l}$ does not contain the vertex set of every Perron component at v. Form a new graph $\tilde{G}$ by replacing C by a single connected component $\tilde{C}$ at v. Suppose that the bottleneck matrix of $\tilde{C}$ is denoted $\tilde{M}$. Denote the algebraic connectivity of $\tilde{G}$ by $\tilde{\mu}$. If $(L(C))^{-1} = M \ll \tilde{M}$, then $\tilde{\mu} \leq \mu$.*

Proof. Firstly, suppose $v = z$. If $\rho(\tilde{M}) \leq 1/\mu$, then there are still two or more Perron components at z, and so (by Proposition 1.3) Case B still holds for $\tilde{G}$, with $\tilde{\mu} = \mu$. So suppose $\rho(\tilde{M}) > 1/\mu$. Then in $\tilde{G}$, the unique Perron component at z is $\tilde{C}$. If Case B holds for $\tilde{G}$, then there exists a cut-point $\omega \in \tilde{C}$ such that $\rho(D) = 1/\tilde{\mu}$, for some Perron component at ω with bottleneck matrix D. But if C' is the component at ω containing z, with bottleneck matrix D', then

$$\frac{1}{\tilde{\mu}} = \rho(D) \geq \rho(D') > \rho(D_0) = \frac{1}{\mu},$$

where D_0 is the bottleneck matrix for some Perron component at z in G not containing ω. (The last inequality follows from the fact $D_0 \ll D'$.) In this case $\mu > \tilde{\mu}$. Finally, if Case A holds for $\tilde{G}$, note that at z, $\tilde{C}$ is the unique component at z with both positive and negative valuated vertices in any Fiedler vector. If we let $\bar{C} = \tilde{G} \setminus \tilde{C}$, then by Lemma 2.2 there exists an $x > 0$ such that

$$\frac{1}{\tilde{\mu}} = \rho(L(\bar{C})^{-1} - \frac{1}{\theta^Te}J + \frac{x}{\theta^Te}J) > \rho(L(\bar{C})^{-1} - \frac{1}{\theta^Te}J) \geq \rho(D_0) = \frac{1}{\mu},$$

where D_0 is the bottleneck matrix for some Perron component at z in G not containing the vertices in $\tilde{C}$. Hence again $\mu > \tilde{\mu}$.

Finally, suppose $v \neq z$ and assume without loss of generality that $z \in C_k$. By the definition of C it follows that $z \notin C$. Let C' be the component at z containing v, and let $D = C' \cap C_k$. Observe that

$$(L(C))^{-1} = M = \begin{bmatrix} (L(C_{i_1}))^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (L(C_{i_j}))^{-1} \end{bmatrix},$$

Consider the following matrix $N = (L(C_1 \cup \cdots \cup C_k \cup D \cup \{v\}))^{-1}$, which is the bottleneck matrix for the component at z containing v. Using Proposition 2.3 and the assumption that $M \ll \tilde{M}$ we obtain corresponding entry-wise domination for the matrix N. Thus the result follows from the analysis above. $\square$

THEOREM 2.5. *Suppose G is a weighted graph with Laplacian L and algebraic connectivity μ, and that Case A of Theorem 1.1 holds. Let B_0 be the unique block of G containing both positively and negatively valuated vertices. Let v be a point of articulation of G, with connected components at v being $C_0, A_1, A_2, \ldots, A_k$, where C_0 is the unique Perron component at v (containing the vertices in B_0). Let $C = \cup_{l=1}^{j} A_{i_l}$, where $i_1, i_2, \ldots, i_j \in \{1, 2, \ldots, k\}$. Form a new graph $\tilde{G}$ by replacing C by a single connected component $\tilde{C}$ at v. Suppose the bottleneck matrix of $\tilde{C}$ is denoted $\tilde{M}$. Denote the algebraic connectivity of $\tilde{G}$ by $\tilde{\mu}$. If $(L(C))^{-1} = M \ll \tilde{M}$, then $\tilde{\mu} \le \mu$.*

Proof. By Lemma 2.2, there exists an $x > 0$ such that

$$\frac{1}{\mu} = \rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{x}{\theta^T e}J) = \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J),$$

where $C_1 = G \setminus C_0$. Suppose that $\rho(\tilde{M}) = \rho(L(C_0)^{-1})$. Then Case B holds for $\tilde{G}$, with special vertex v, so

$$\frac{1}{\tilde{\mu}} = \rho(L(C_0)^{-1}) > \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J) = \frac{1}{\mu},$$

so $\tilde{\mu} < \mu$.

Suppose $\rho(\tilde{M}) < \rho(L(C_0)^{-1})$. Then the Perron component at v is still C_0, and the Perron component at every other vertex contains vertices in B_0 (by Proposition 1.4). Hence there exists $\tilde{x} > 0$ such that

$$\frac{1}{\tilde{\mu}} = \rho(L(\tilde{C}_1)^{-1} - \frac{1}{\theta^T e}J + \frac{\tilde{x}}{\theta^T e}J) = \lambda_1(L(C_0)^{-1} - \frac{\tilde{x}}{\theta^T e}J), \tag{4}$$

where $\tilde{C}_1 = \tilde{G} \setminus C_0$. Now $\rho(L(\tilde{C}_1)^{-1} - \frac{1}{\theta^T e}J + \frac{\tilde{x}}{\theta^T e}J) > \rho(L(C_1)^{-1} - \frac{1}{\theta^T e}J + \frac{\tilde{x}}{\theta^T e}J)$ so necessarily we find that $\tilde{x} < x$, since the left-hand side of (4) is increasing in x and the right-hand side of (4) is nonincreasing in x. Consequently,

$$\frac{1}{\tilde{\mu}} = \lambda_1(L(C_0)^{-1} - \frac{\tilde{x}}{\theta^T e}J) \ge \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J) = \frac{1}{\mu},$$

so $\tilde{\mu} \le \mu$.

Finally, suppose that $\rho(\tilde{M}) > \rho(L(C_0)^{-1})$. Then in $\tilde{G}$, if Case B holds for some $z \in \tilde{C}$, then $\frac{1}{\tilde{\mu}} = \rho(D) \ge \rho(L(V)^{-1})$, where D is the bottleneck matrix of some Perron component at z in $\tilde{G}$ and V is the component at z containing v. But $C_0 \subset V$, so $\rho(L(V)^{-1}) > \rho(L(C_0)^{-1}) \ge \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e}J) = 1/\mu$, hence $\tilde{\mu} < \mu$. In $\tilde{G}$ if Case A holds, then at v, $\tilde{C}$ is now the component containing positively and negatively valuated vertices. Thus there exists a $t > 0$ such that

$$\frac{1}{\tilde{\mu}} = \lambda_1(L(\tilde{C})^{-1} - \frac{t}{\theta^T e}J) = \rho(L(\tilde{G} \setminus \tilde{C})^{-1} - \frac{1}{\theta^T e}J + \frac{t}{\theta^T e}J) > \rho(L(C_0)^{-1}),$$

where the last inequality follows from the fact that $\tilde{G} \setminus \tilde{C}$ contains $C_0 \cup \{v\}$, so that by Proposition 2.3 $\left(L(\tilde{G} \setminus \tilde{C})^{-1} - \frac{1}{\theta^T e} J\right) \gg L(C_0)^{-1}$. Note that $\rho(L(C_0)^{-1}) \geq \lambda_1(L(C_0)^{-1} - \frac{x}{\theta^T e} J) = 1/\mu$. Again we have $\tilde{\mu} < \mu$. $\square$

Example 2.6. Suppose G is an unweighted connected graph with algebraic connectivity μ, having cut-point v which is adjacent to k pendant vertices $1, 2, \ldots, k$. Form $\tilde{G}$ from G by replacing vertices $1, 2, \ldots, j$, by a single tree on j vertices. Denoting the algebraic connectivity of $\tilde{G}$ by $\tilde{\mu}$, we have $\tilde{\mu} \leq \mu$. This follows from the fact that if M is the bottleneck matrix for vertices $1, 2, \ldots, j$ in G, and $\tilde{M}$ is the corresponding bottleneck matrix in $\tilde{G}$, then $M = I \ll \tilde{M}$; see also [KNS]. Consequently Theorems 2.4 and 2.5 apply to G in all but one situation: Case B holds for G, $v = z$, $k = j$, and the only Perron components at v are the single vertices $1, 2, \ldots, k$. But in this case we have $\mu = 1 \geq \tilde{\mu}$, the inequality following from the fact that $\tilde{G}$ has a cut-point and Example 1.5.

3. Unweighted Trees with Fixed Diameter. Recall that in any connected graph G on vertices $1, 2, \ldots, n$, the *distance between vertices i and j*, $d(i,j)$ is defined to be the length of the shortest path from i to j. (For convention, we take $d(i,i) = 0$ for all i.) The *diameter* of G is given by $\max_{i,j} d(i,j)$, while the *radius* of G is given by $\min_i \max_j d(i,j)$. It is straightforward to show that if G has diameter d and radius r, then $r \leq d \leq 2r$.

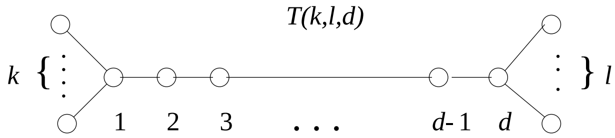

Figure 1.

In this section we consider unweighted trees with a specified number of vertices and diameter. Our goal is to determine attainable bounds for the algebraic connectivity of a tree in terms of the number of vertices and the diameter. Of course there has been much analysis done in attempting to provide bounds on the algebraic connectivity in terms of n, the number of vertices, and d, the diameter of G. Specifically, [MO1] proved that the algebraic connectivity of an unweighted graph on n vertices with fixed diameter d is bounded below by $4/nd$. In [M2], it is noted that for trees with n vertices and fixed diameter d the algebraic connectivity is bounded above by $2(1 - \cos(\frac{\pi}{d+1}))$. More recently, in [C] it was shown that the algebraic connectivity of a graph on n vertices with diameter d, and maximum degree Δ is bounded above by $1 - 2\frac{\sqrt{\Delta - 1}}{\Delta}(1 - \frac{2}{d}) + \frac{2}{d}$. In [KN] the authors apply their Perron component approach for trees to obtain the following result. First we need some notation. Let $T(k,l,d)$ be the unweighted tree on n vertices constructed by taking a path on vertices $1, 2, \ldots, d$, and adding k pendant vertices adjacent to vertex 1 and l pendant vertices adjacent to vertex d $(n = k + l + d)$; see also Figure 1.

Lemma 3.1. [KN] *For all unweighted trees on n vertices with fixed diameter $d+1$, the algebraic connectivity is minimized by $T(k,l,d)$, for some $0 \leq k \leq n - d$.*

Our next result refines the above lemma and determines explicitly (up to isomorphism) the graph which minimizes the algebraic connectivity of trees on n vertices and fixed diameter.

THEOREM 3.2. *Among all unweighted trees on n vertices with fixed diameter $d+1$, the minimum algebraic connectivity is attained by $T(\lceil \frac{n-d}{2} \rceil, \lfloor \frac{n-d}{2} \rfloor, d)$.*

Proof. Using Lemma 3.1 it is sufficient to restrict our attention to those trees isomorphic to $T(k,l,d)$. The Laplacian matrix of such a tree is

$$L = \left[\begin{array}{c|ccccc|c} I_k & -e & 0 & 0 & \cdots & 0 & 0 \\ \hline -e^T & k+1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ \vdots & \cdots & 0 & 0 & -1 & l+1 & -e^T \\ \hline 0 & \cdots & 0 & 0 & 0 & -e & I_l \end{array}\right],$$

and it is easy to see that the eigenvalues of L are: 1, with algebraic multiplicity $k+l-2$, along with the eigenvalues of the $(d+2)\times(d+2)$ matrix

$$M_{k,l,d} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -k & k+1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ \vdots & \cdots & 0 & 0 & -1 & l+1 & -l \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Let $p_{k,l,d}$ denote the characteristic polynomial of $M_{k,l,d}$. We want to consider the expression $p_{k,l,d} - p_{k-1,l+1,d}$. Using the fact that the determinant is multilinear, it follows that

$$p_{k,l,d} - p_{k-1,l,d} = -\lambda det \begin{bmatrix} \lambda-2 & 1 & 0 & 0 & 0 & 0 \\ 1 & \lambda-2 & 1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 1 & \lambda-2 & 1 & 0 \\ 0 & 0 & 0 & 1 & \lambda-l-1 & l \\ 0 & 0 & 0 & 0 & 1 & \lambda-1 \end{bmatrix},$$

where the above matrix is of order d. Similarly, we find that

$$p_{k-1,l,d} - p_{k-1,l+1,d} = \lambda det \begin{bmatrix} \lambda-1 & 1 & 0 & 0 & 0 & 0 \\ k-1 & \lambda-k & 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda-2 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & \lambda-2 & 1 \\ 0 & 0 & 0 & 0 & 1 & \lambda-2 \end{bmatrix},$$

which is also equal to (after applying the permutation similarity: i goes to $d-i+1$, for $1 \le i \le d$)

$$\lambda det \begin{bmatrix} \lambda-2 & 1 & 0 & 0 & 0 & 0 \\ 1 & \lambda-2 & 1 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 1 & \lambda-2 & 1 & 0 \\ 0 & 0 & 0 & 1 & \lambda-k & k-1 \\ 0 & 0 & 0 & 0 & 1 & \lambda-1 \end{bmatrix}.$$

Thus

$$p_{k,l,d} - p_{k-1,l+1,d} = \lambda^2(l+1-k) det \begin{bmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & \lambda-2 \end{bmatrix}$$

(the latter matrix has order $d-2$).

Now the linear term in $det(\lambda I - L)$ is the sum of all the $(n-1) \times (n-1)$ principal minors of $-L$. By the matrix tree theorem (see [BM, p. 219]), this sum is equal to $(-1)^{n-1}n$ (recall here that $n = k+l+d$). Furthermore, $det(\lambda I - L) = (\lambda-1)^{k+l-2} p_{k,l,d}$. Since $p_{k,l,d}$ has a factor of λ, we find that the linear term of $p_{k,l,d}$ is $(-1)^{d+1}n$.

Let $\mu_{k,l,d}$ and $\mu_{k-1,l+1,d}$ be the appropriate algebraic connectivities and note that each is at most $2(1-\cos(\frac{\pi}{d+2}))$; see [M2]. It is well known that the smallest eigenvalue of the $(d-2) \times (d-2)$ matrix

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

is $2(1-\cos(\frac{\pi}{d-1}))$. Hence if $0 < \lambda < \min(\mu_{k,l,d}, \mu_{k-1,l+1,d})$, then we have that

$$sgn \left(det \begin{bmatrix} \lambda-2 & 1 & 0 & 0 \\ 1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & 1 & \lambda-2 \end{bmatrix} \right) = (-1)^{d-2}.$$

Suppose that $0 < \lambda < \mu_{k,l,d}$, and note that $p_{k,l,d}$ is increasing (respectively, decreasing) at 0 when d is odd (respectively, even), as is $p_{k-1,l+1,d}$. But for such λ,

$$sgn(p_{k,l,d} - p_{k-1,l+1,d}) = sgn((l+1-k)\lambda^2(-1)^{d-2}).$$

Suppose that $k < l+1$. Then $sgn(p_{k,l,d} - p_{k-1,l+1,d})$ is negative (respectively, positive), and it follows that $\mu_{k,l,d} < \mu_{k-1,l+1,d}$. A similar argument shows that $\mu_{k,l,d} > \mu_{k-1,l+1,d}$, when $k > l+1$. Thus we may conclude that for a tree of diameter $d+1$ with n vertices, the minimum algebraic connectivity is attained for $T(\lceil\frac{n-d}{2}\rceil, \lfloor\frac{n-d}{2}\rfloor, d)$. □

COROLLARY 3.3. *Let T be any tree on n vertices with diameter $d+1$, and with algebraic connectivity μ. Then*

$$\mu \geq \mu_{\lceil\frac{n-d}{2}\rceil, \lfloor\frac{n-d}{2}\rfloor, d},$$

where equality holds if T is isomorphic to $T(\lceil\frac{n-d}{2}\rceil, \lfloor\frac{n-d}{2}\rfloor, d)$.

Consider graphs on n vertices with fixed radius. We begin with a few lemmas. Let e_i denote the i^{th} standard basis vector.

LEMMA 3.4. *Let G be the graph obtained from a path on the vertices $1, 2, \ldots, j+1$ (where $j \geq 1$), by adding k pendant vertices adjacent to vertex 1, and let $F_{k,j}$ be the bottleneck matrix for the (only) component at vertex $j+1$. Then $F_{k,j} \ll F_{k-1,j+1}$.*

Proof. For $F_{k,j}$, label the pendant vertices adjacent to vertex 1 first, then $F_{k,j}$ can be written as

$$\begin{bmatrix} I_k + jJ & ee_1^T P_j \\ P_j e_1 e^T & P_j \end{bmatrix},$$

where

$$P_j = \begin{bmatrix} j & j-1 & j-2 & \cdots & 2 & 1 \\ j-1 & j-1 & j-2 & \cdots & 2 & 1 \\ j-2 & j-2 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 2 & 2 & \cdots & \cdots & 2 & 1 \\ 1 & 1 & \cdots & \cdots & 1 & 1 \end{bmatrix}.$$

For $F_{k-1,j+1}$, also label the pendant vertices adjacent to vertex 1 first, then

$$F_{k-1,j+1} = \begin{bmatrix} I_{k-1} + (j+1)J & (j+1)e & ee_1^T P_j \\ (j+1)e^T & j+1 & e_1^T P_j \\ P_j e_1 e^T & P_j e_1 & P_j \end{bmatrix}.$$

Evidently, $F_{k,j} \ll F_{k-1,j+1}$. □

LEMMA 3.5. *Fix n and consider the tree $T(k,l,d)$ of diameter $d+1$. Let $F_{k,j}$ be the bottleneck matrix for the component at vertex $j+1$ containing vertex j. Let μ_d*

be the algebraic connectivity of $T(\lfloor \frac{n-d}{2} \rfloor, \lceil \frac{n-d}{2} \rceil, d)$. *Then* μ_d *is a strictly decreasing function of* d.

Proof. There are several cases to consider: (i) d odd, n odd; (ii) d odd, n even; (iii) d even, n odd; (iv) d even, n even.

(i) If both d and n are odd, then $T(\lfloor \frac{n-d}{2} \rfloor, \lceil \frac{n-d}{2} \rceil, d)$ is a Type I tree with characteristic vertex $\frac{d+1}{2}$, and by Theorem 2.4 (see also [KN, Thm. 1]), we find that

$$\mu_d = \frac{1}{\rho(F_{\frac{n-d}{2}, \frac{d-1}{2}})}.$$

Also $T(\frac{n-d-2}{2}, \frac{n-d}{2}, d+1)$ is easily seen to be a Type II tree with characteristic vertices $\frac{d+1}{2}$ and $\frac{d+3}{2}$. Hence, by Theorem 2.5 (see also [KN, Thm. 1]), for some $\alpha \in (0,1)$ we have

$$\mu_{d+1} = \frac{1}{\rho(F_{\frac{n-d-2}{2}, \frac{d+1}{2}} - \alpha J)} = \frac{1}{\rho(F_{\frac{n-d}{2}, \frac{d+1}{2}} - (1-\alpha)J)};$$

since $\rho(F_{\frac{n-d}{2}, \frac{d+1}{2}} - J) = \rho(F_{\frac{n-d}{2}, \frac{d-1}{2}})$ we deduce that $\mu_d > \mu_{d+1}$.

(ii) For d odd and n even, $T(\frac{n-d-1}{2}, \frac{n-d+1}{2}, d)$ is a Type II tree with characteristic vertices $\frac{d+1}{2}$ and $\frac{d+3}{2}$, so for some $\alpha \in (0,1)$,

$$\mu_d = \frac{1}{\rho(F_{\frac{n-d-1}{2}, \frac{d+1}{2}} - \alpha J)} = \frac{1}{\rho(F_{\frac{n-d+1}{2}, \frac{d-1}{2}} - (1-\alpha)J)}.$$

On the other hand, $T(\frac{n-d-1}{2}, \frac{n-d-1}{2}, d+1)$ has the property that the Perron component at $\frac{d+1}{2}$ is isomorphic to the Perron component at $\frac{d+3}{2}$. It follows that $T(\frac{n-d-1}{2}, \frac{n-d-1}{2}, d+1)$ is a Type II tree with characteristic vertices $\frac{d+1}{2}$ and $\frac{d+3}{2}$, and that

$$\mu_{d+1} = \frac{1}{\rho(F_{\frac{n-d-1}{2}, \frac{d+1}{2}} - \frac{1}{2}J)}.$$

If $\alpha > 1/2$, then

$$\mu_d = \frac{1}{\rho(F_{\frac{n-d-1}{2}, \frac{d+1}{2}} - \alpha J)} > \mu_{d+1},$$

while if $\alpha \leq 1/2$, then

$$\begin{aligned} \mu_d = \frac{1}{\rho(F_{\frac{n-d+1}{2}, \frac{d-1}{2}} - (1-\alpha)J)} &\geq \frac{1}{\rho(F_{\frac{n-d+1}{2}, \frac{d-1}{2}} - \frac{1}{2}J)} \\ &> \frac{1}{\rho(F_{\frac{n-d-1}{2}, \frac{d+1}{2}} - \frac{1}{2}J)} = \mu_{d+1}, \end{aligned}$$

where the last inequality follows from Lemma 3.4.

(iii) When d is even and n is odd, $T(\frac{n-d-1}{2}, \frac{n-d+1}{2}, d)$ is a Type II tree with characteristic vertices $\frac{d}{2}$ and $\frac{d+2}{2}$, so for some $\alpha \in (0,1)$

$$\mu_d = \frac{1}{\rho(F_{\frac{n-d-1}{2},\frac{d}{2}} - \alpha J)} = \frac{1}{\rho(F_{\frac{n-d+1}{2},\frac{d}{2}} - (1-\alpha)J)}.$$

However, $T(\frac{n-d-1}{2}, \frac{n-d-1}{2}, d+1)$ is a Type I tree with characteristic vertex $\frac{d+2}{2}$, so that

$$\mu_{d+1} = \frac{1}{\rho(F_{\frac{n-d-1}{2},\frac{d}{2}})} < \frac{1}{\rho(F_{\frac{n-d-1}{2},\frac{d}{2}} - \alpha J)} = \mu_d.$$

(iv) When both d and n are even, $T(\frac{n-d}{2}, \frac{n-d}{2}, d)$ is a Type II tree with characteristic vertices $\frac{d}{2}$ and $\frac{d+2}{2}$ and isomorphic Perron components at the characteristic vertices, so we find that $\mu_d = \frac{1}{\rho(F_{\frac{n-d}{2},\frac{d}{2}} - \frac{1}{2}J)}$. But $T(\frac{n-d-2}{2}, \frac{n-d}{2}, d+1)$ is a Type II tree with characteristic vertices $\frac{d+2}{2}$ and $\frac{d+4}{2}$. So for some $\alpha \in (0,1)$,

$$\mu_{d+1} = \frac{1}{\rho(F_{\frac{n-d-2}{2},\frac{d+2}{2}} - \alpha J)} = \frac{1}{\rho(F_{\frac{n-d}{2},\frac{d}{2}} - (1-\alpha)J)}.$$

If $\alpha > 1/2$, then we have

$$\mu_{d+1} = \frac{1}{\rho(F_{\frac{n-d}{2},\frac{d}{2}} - (1-\alpha)J)} < \frac{1}{\rho(F_{\frac{n-d}{2},\frac{d}{2}} - \frac{1}{2}J)} = \mu_d,$$

while if $\alpha \le 1/2$ then we have

$$\mu_{d+1} = \frac{1}{\rho(F_{\frac{n-d-2}{2},\frac{d+2}{2}} - \alpha J)} \le \frac{1}{\rho(F_{\frac{n-d-2}{2},\frac{d+2}{2}} - \frac{1}{2}J)} < \frac{1}{\rho(F_{\frac{n-d}{2},\frac{d}{2}} - \frac{1}{2}J)} = \mu_d,$$

by the entry-wise domination of the bottleneck matrices. Thus in all four cases, we find that $\mu_{d+1} < \mu_d$. $\square$

THEOREM 3.6. *Among all graphs on n vertices with fixed radius r, the minimum algebraic connectivity is attained by $T(\lfloor\frac{n-(2r-1)}{2}\rfloor, \lceil\frac{n-(2r-1)}{2}\rceil, 2r-1)$.*

Proof. It is easily seen that there is a spanning subtree of G, T with diameter d satisfying $r \le d \le 2r$. Hence $\mu(G) \ge \mu(T) \ge \mu_{d-1}$, where $\mu(G)$ denotes the algebraic connectivity of G. Note the last inequality follows from Theorem 3.2. But by Lemma 3.5, $\mu_{d-1} \ge \mu_{2r-1}$ (with equality holding if and only if $d = 2r$). Observing that $T(\lfloor\frac{n-(2r-1)}{2}\rfloor, \lceil\frac{n-(2r-1)}{2}\rceil, 2r-1)$ is indeed a graph with radius r, the proof is complete. $\square$

Consider maximizing the algebraic connectivity of trees on n vertices with fixed diameter. Note that if T is any tree on n vertices with fixed diameter $d+1$, then its algebraic connectivity is at most $2(1-\cos(\frac{\pi}{d+2}))$; see, e.g., [CDS, p. 187]. Furthermore, if d is odd, then (by considering Perron components) the tree constructed by taking a path on vertices $1, 2, \ldots, d+2$, and adding $n-d-2$ pendant vertices to vertex $\frac{d+3}{2}$, is a

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Type I tree with characteristic vertex $\frac{d+3}{2}$ and algebraic connectivity $2(1-\cos(\frac{\pi}{d+2}))$. This completely solves the problem when d is odd.

Throughout the rest of this section, we take d to be even. Let $P_{l,d}$ be the tree constructed by taking a path on vertices $1, 2, \ldots, d+2$, and adding l pendant vertices to vertex $\frac{d+2}{2}$ and $n-l-d-2$ pendant vertices to vertex $\frac{d+4}{2}$; see Figure 2.

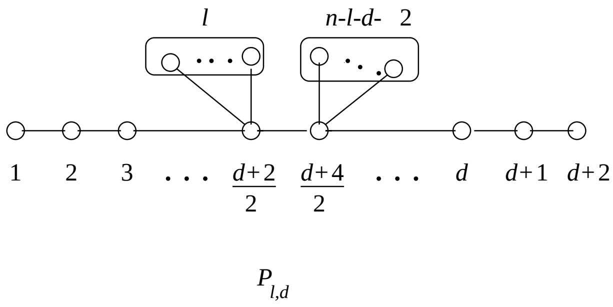

FIGURE 2.

LEMMA 3.7. *For all unweighted trees on n vertices with fixed diameter $d+1$, the algebraic connectivity is maximized by $P_{l,d}$, for some $0 \le l \le n-d-2$.*

Proof. Let T be any tree on n vertices with fixed diameter $d+1$, and with algebraic connectivity μ. Label the vertices on a path of length $d+1$, 1 up to $d+2$. Suppose that there are $\frac{d+4}{2}+l$ vertices in the component at vertex $\frac{d+4}{2}$ containing vertex $\frac{d+2}{2}$, and $n-l-(\frac{d+4}{2})$ vertices in the component at vertex $\frac{d+2}{2}$ containing vertex $\frac{d+4}{2}$. It is easily seen (by considering Perron components; see also [KNS]) that the characteristic vertices of $P_{l,d}$ are $\frac{d+2}{2}$ and $\frac{d+4}{2}$. Form $\tilde{T}$ from $P_{l,d}$ by replacing the component at $\frac{d+4}{2}$ containing $\frac{d+2}{2}$ by the corresponding component of T. There is entry-wise domination of the corresponding bottleneck matrices, so by Theorem 2.5 (see also [KN, Thm. 1]), the algebraic connectivity $\tilde{\mu}$ of $\tilde{T}$, is at most $\mu_{l,d}$, the algebraic connectivity of $P_{l,d}$, and using [KN, Thm. 1] the characteristic vertices of $\tilde{T}$ lie in the new component, or are still $\frac{d+2}{2}$ and $\frac{d+4}{2}$. Now form T from $\tilde{T}$ by replacing a component at a characteristic vertex containing $\frac{d+4}{2}$ by the corresponding component of T. Again we have entry-wise domination of the bottleneck matrices, so that $\mu \le \tilde{\mu}$. (We note here that Prop. 1 of [KNS] may be used to explicitly compute the entries of the bottleneck matrices that appear in this proof.) Hence $\mu \le \tilde{\mu} \le \mu_{l,d}$. We conclude that over trees on n vertices with diameter $d+1$, the algebraic connectivity is maximized by $P_{l,d}$ for some l. ☐

As before, we now explicitly determine (up to isomorphism) the tree which attains the maximum algebraic connectivity over trees on n vertices with diameter $d+1$.

THEOREM 3.8. *Among all unweighted trees on n vertices with fixed diameter $d+1$, the maximum algebraic connectivity is attained by $P_{n-d-2,d}$ (i.e., the path on vertices $1, 2, \ldots, d+2$, with $n-d-2$ pendant vertices adjacent to vertex $\frac{d+2}{2}$).*

Proof. For simplicity of notation let $L_{l,k}$ be the Laplacian matrix for $P_{l,d}$, where $k = n-l-d-2$. We find that $L_{l,k}$ has the eigenvalue 1 with multiplicity $k+l-2$, and the other eigenvalues of $L_{l,k}$ correspond to the eigenvalues of

$$M_{l,k} = \left[\begin{array}{ccc|cccc|ccc} & P & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & l+2 & -l & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -k & k+2 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & DPD^{-1} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \end{array}\right],$$

where P is the $\frac{d}{2} \times \frac{d}{2}$ matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

and $D = [d_{ij}]$ is the $\frac{d}{2} \times \frac{d}{2}$ permutation matrix in which

$$d_{ij} = \begin{cases} 1, & \text{if } j = \frac{d}{2} - i + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Let $q_{l,k}$ be the characteristic polynomial of $M_{l,k}$, so that $det(\lambda I - L_{l,k}) = (\lambda - 1)^{k+l-2} q_{l,k}$. Note that the constant term of $det(\lambda I - L_{l,k})$ is equal to $(-1)^{d+2+k+l-1}$ $(d + 2 + k + l)$; but this is the same as $(-1)^{k+l-2} \times \{constant\ \ term\ \ of\ \ q_{l,k}\}$, so that the constant term of $q_{l,k}$ is $(-1)^{d-1}(d + 2 + k + l) = -(d + 2 + k + l)$ since d is even. As before we consider the expression $q_{l,k} - q_{l-1,k+1}$. Using similar calculations as those in the proof of Theorem 3.2 it follows that

$$q_{l,k} - q_{l-1,k+1} = -(l - k - 1)(\lambda^2)(det(\lambda I - P))^2.$$

Let $\mu_{l,k}$ be the algebraic connectivity of $L_{l,k}$. Firstly, suppose that $l < k + 1$. Then $q_{l,k} - q_{l-1,k+1} > 0$, for all $\lambda > 0$, and since both $q_{l,k}$ and $q_{l-1,k+1}$ are decreasing at 0, it follows that $\mu_{l,k} < \mu_{l-1,k+1}$. A similar argument applies when $l > k+1$ (since $\mu_{l,k} = \mu_{k,l}$), and we find that when $l + k$ is fixed and $l, k \geq 1$, then $\mu_{l,k}$ is maximized by $\mu_{1,l+k-1}$. Finally, let $q_{0,k+1}$ be the characteristic polynomial of

$$\left[\begin{array}{ccc|cccc|ccc}
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & P & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -k-1 & k+3 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -1 & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & DPD^{-1} & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & &
\end{array}\right].$$

Then, again using a similar calculation it follows that

$$q_{1,k} - q_{0,k+1} = k(\lambda^2)(det(\lambda I - P))^2.$$

As above, we now find that $\mu_{0,k+1} > \mu_{1,k}$. Thus for trees on n vertices with diameter $d+1$, $P_{n-d-2,d}$ maximizes the algebraic connectivity. ☐

Corollary 3.9. *Let T be any tree on n vertices with fixed diameter $d+1$, and algebraic connectivity μ. Then*

$$\mu \leq \mu_{0,n-d-2}$$

where equality holds if T is isomorphic to $P_{n-d-2,d}$.

4. Graphs with Fixed Girth. In this section we consider unweighted connected graphs containing a cycle of length 3 or more. Recall that the *girth* of such a graph G is the length (number of vertices, or edges) of the shortest cycle in G. For a graph G, suppose that we have an edge $\{i,j\}$ which is not on any cycle (hence this edge is also a block). We say that vertices i and j have *mutual Perron components* if the unique Perron component at vertex i contains vertex j and the unique Perron component at vertex j contains vertex i. As was the case in Section 3, we are concerned with describing the graphs which extremize the algebraic connectivity over all graphs on n vertices, with specified girth s. First we consider the graph which minimizes the algebraic connectivity over the set of such graphs. In [F1] it was shown that the algebraic connectivity is monotone on spanning subgraphs, hence it is clear that the minimum algebraic connectivity of graphs on a fixed number of vertices, and with specified girth s is attained for a unicyclic (i.e., exactly one cycle in the graph) graph with girth s. Note that we may think of a unicyclic graph as a cycle, with trees hanging off some (or all) of the vertices on that cycle. Recall that in [KNS] it is shown that the bottleneck matrix for a path entry-wise dominates (after a suitable simultaneous permutation of the rows and columns) the bottleneck matrix of any other tree on the same number of vertices. We begin with the following proposition.

PROPOSITION 4.1. *Let G be a connected graph, and suppose that C is a connected component at some vertex of G; let M denote the principal submatrix of the Laplacian matrix of G corresponding to the vertices in C. Modify C to form $\tilde{C}$ as follows: Choose vertices $1, 2, \ldots, k$ in C, and add one or more connected components at vertex i, $1 \le i \le k$. For each $1 \le i \le k$, let B_i denote the collection of new vertices added at vertex i, and suppose that in B_i, vertices $v_{j_1(i)}, v_{j_2(i)}, \ldots, v_{j_{m_i}(i)}$ are adjacent to vertex i. Let the new graph be $\tilde{G}$, with Laplacian matrix L. Then we have the following expression for $(L(\tilde{C}))^{-1}$,*

$$\left[\begin{array}{c|c|c|c|c}
(L(B_1))^{-1} + (M^{-1})_{11}J & (M^{-1})_{12}J & \cdots & (M^{-1})_{1k}J & ee_1^T M^{-1} \\ \hline
(M^{-1})_{21}J & \ddots & \ddots & (M^{-1})_{2k}J & ee_2^T M^{-1} \\ \hline
\vdots & \ddots & \ddots & \vdots & \vdots \\ \hline
(M^{-1})_{k1}J & \cdots & (M^{-1})_{k,k-1}J & (L(B_k))^{-1} + (M^{-1})_{kk}J & ee_k^T M^{-1} \\ \hline
M^{-1}e_1e^T & M^{-1}e_2e^T & \cdots & M^{-1}e_ke^T & M^{-1}
\end{array}\right],$$

where $(M^{-1})_{ij}$ denotes the $(i,j)^{th}$ entry of M^{-1}.

Proof. Note that $L(\tilde{C})$ can be written as

$$\left[\begin{array}{c|c|c|c|c}
L(B_1) & 0 & \cdots & 0 & -(\sum_{p=1}^{m_1} e_{j_p(1)})e_1^T \\ \hline
0 & L(B_2) & \ddots & \vdots & -(\sum_{p=1}^{m_2} e_{j_p(2)})e_2^T \\ \hline
\vdots & \ddots & \ddots & 0 & \vdots \\ \hline
0 & \cdots & 0 & L(B_k) & -(\sum_{p=1}^{m_k} e_{j_p(k)})e_k^T \\ \hline
-e_1(\sum_{p=1}^{m_1} e_{j_p(1)}^T) & -e_2(\sum_{p=1}^{m_2} e_{j_p(2)}^T) & \cdots & -e_k(\sum_{p=1}^{m_k} e_{j_p(k)}^T) & M + \sum_{p=1}^{k} m_p e_p e_p^T
\end{array}\right].$$

The result can now be easily verified by direct computation, and recalling that $L(B_i)e = \sum_{p=1}^{m_i} e_{j_p(i)}$. ◻

COROLLARY 4.2. *Let G be a connected unicyclic graph with Laplacian matrix L, and let C be a connected component at some vertex u, which contains vertices on the cycle. For each vertex i on the cycle, suppose that there are m_i components at vertex i not including the vertices on the cycle. Let B_i be the collection of vertices of the m_i components at vertex i. Modify C to form $\tilde{C}$ by replacing those m_i components by a single path on $|B_i|$ vertices at vertex i. Let the new graph be denoted by $\tilde{G}$, and suppose the Laplacian matrix for $\tilde{G}$ is denoted $\tilde{L}$. Then $(L(C))^{-1} \ll (\tilde{L}(\tilde{C}))^{-1}$.*

Proof. Suppose (without loss of generality) that the vertices on the cycle are labeled $1, 2, \ldots, k$, with vertex 1 being the vertex closest (in distance) to u. Let M^{-1} ($\tilde{M}^{-1}$) denote the bottleneck matrix for the collection of vertices $C \setminus \cup_{i=2}^{k} B_i$ ($\tilde{C} \setminus \cup_{i=2}^{k} B_i$). For brevity, let $S = C \setminus \cup_{i=2}^{k} B_i$ and $\tilde{S} = \tilde{C} \setminus \cup_{i=2}^{k} B_i$. From [KNS, Thm. 5] we have $(L(B_i))^{-1} \ll (\tilde{L}(B_i))^{-1}$, for $i = 1, 2, \ldots, k$, so in particular, if $u = 1$, by Proposition 4.1, we obtain our result. If $u \ne 1$, then we need only show that $M^{-1} \ll \tilde{M}^{-1}$. Suppose v is the vertex in C (and $\tilde{C}$) which is adjacent to u. By [FK, Lemma 2],

$$M^{-1} = (L(S))^{-1} = \left[\begin{array}{c|c} (L(S \setminus v))^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] + J.$$

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It is now a straightforward induction to show that $(L(S \setminus v))^{-1} \ll (\tilde{L}(\tilde{S} \setminus v))^{-1}$. This completes the proof. □

PROPOSITION 4.3. *Among all connected graphs on n vertices with fixed girth s, the algebraic connectivity is minimized by a unicyclic graph with girth s, with the following property: There are at most two connected components at every vertex on the cycle, and the component not including the vertices on the cycle (if one exists), is a path.*

Proof. Let G be any connected unicyclic graph on n vertices with girth s. Suppose the vertices on the cycle are labeled $1, 2, \ldots, s$. For each vertex j $(1 \leq j \leq s)$ the connected components at j not containing the vertices on the cycle are trees (possibly empty). In other words, the union of these components is, by definition, a *forest*. Hence let F_j be the union of the connected components at j, not containing the other vertices on the cycle. Fix j $(1 \leq j \leq s)$ and suppose F_j is not a path. If Case A holds for G and $B_0 \not\subseteq F_j$, or Case B holds and $z \notin F_j \cup \{j\}$, then replace F_j by a path on $|F_j|$ vertices and apply Theorems 2.5 and 2.4 to produce a unicyclic graph whose algebraic connectivity is at most that of G. If Case A holds and B_0 is an edge $\{u, v\}$ (with $d(u, j) < d(v, j)$) in F_j, then since F_j is not a path, either the component at u containing v is not a path, or in the component at v containing u, the tree at j not containing the vertices on the cycle is not a path. In either case we can replace the corresponding components by a path and apply Theorem 2.5 (and Corollary 4.2 in the latter case). If Case B holds and $z \in F_j \cup \{j\}$, then either some component at z not containing the vertices on the cycle is not a path (in which case we can apply Theorem 2.4), or $z \neq j$ and in the component at z containing the cycle, the tree at j containing z is not a path (in which case we can use Theorem 2.5 and Corollary 4.2), or all components at z have one of the above two forms, and the degree of z is at least three (otherwise F_j is a path). In this case at least one Perron component at z is a path, so by replacing the remaining components at z by a single (unicyclic) graph in which each of the remaining paths are adjoined to the end of the path connected to the cycle, we form an unicyclic graph whose algebraic connectivity is at most that of G (by Theorem 2.4). This completes the proof. □

We conjecture that the unique minimizer for girth s, will be isomorphic to an s-cycle with a path of length $n - s$ joined at exactly one vertex on the cycle. In what follows, we verify this conjecture for the case $s = 3$.

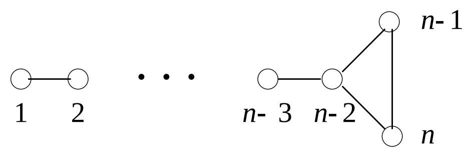

FIGURE 3.

LEMMA 4.4. *Let G be the graph as in Figure 3. Then vertices $\frac{n}{2}$ and $\frac{n+2}{2}$ have mutual Perron components, if n is even, otherwise vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ have mutual Perron components.*

Proof. First we note that the Perron value of the bottleneck matrix for the component at vertex k $(k = 1, 2, \ldots, n-2)$ containing the triangle is the same as that for the component at k containing vertex n of the graph G with the edge $\{(n-1), n\}$

deleted (this is because in the latter, vertices $n-1$ and n are isomorphic, hence the entries in the Perron vector are equal, from which it follows that the Perron vector also serves as a Perron vector for the former). For even n, the components at vertex $\frac{n}{2}$ are a path on $\frac{n-2}{2}$ vertices, as seen in Figure 4.

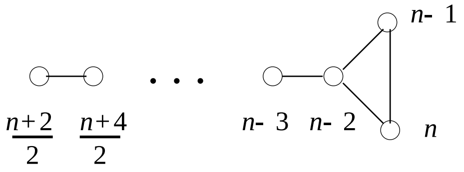

FIGURE 4.

The Perron value of the second component remains the same even if the edge $\{(n-1), n\}$ is deleted. But this component (with the deleted edge) is a tree which contains a path on $\frac{n-2}{2}$ vertices. It follows that the Perron component at vertex $\frac{n}{2}$ contains vertex $\frac{n+2}{2}$. At vertex $\frac{n+2}{2}$, the components are a path on $\frac{n}{2}$ vertices, and a graph on $\frac{n-2}{2}$ vertices; applying similar arguments as in the case above for vertex $\frac{n}{2}$ it follows that the path on $\frac{n}{2}$ vertices is a Perron component, so the Perron component at vertex $\frac{n+2}{2}$ contains vertex $\frac{n}{2}$. The argument for the case n odd is similar. □

COROLLARY 4.5. *Let P_k be the bottleneck matrix of order k for a component which is a path on k vertices. Let G be as in Figure 3, with algebraic connectivity μ. If n is even, then $\mu = \frac{1}{\rho(P_{\frac{n}{2}} - tJ)}$ for some $t \in (0,1)$, while if n is odd, then $\mu = \frac{1}{\rho(P_{\frac{n-1}{2}} - tJ)}$ for some $t \in (0,1)$.*

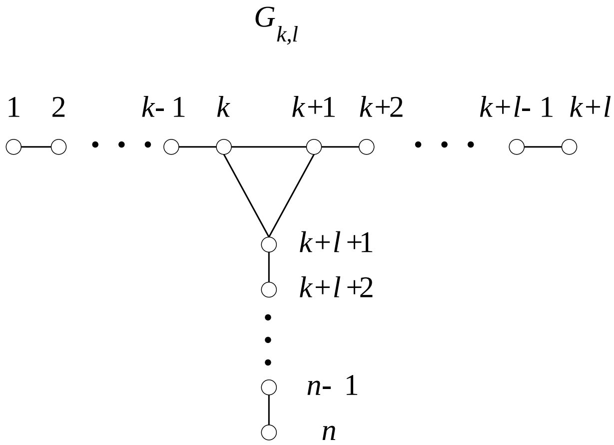

FIGURE 5.

LEMMA 4.6. *Consider the graph $G_{k,l}$ as in Figure 5. Suppose that some Fiedler vector gives vertices $k+l+1, k+l+2, \ldots, n$ a 0 valuation, vertices $1, 2, \ldots, k$ a positive valuation and vertices $k+1, k+2, \ldots, k+l$ a negative valuation. Then $k = l$ and the algebraic connectivity of $G_{k,k}$ is given by $\mu = \frac{1}{\rho(P_k - \frac{2}{3}J)}$.*

Proof. We may partition the Laplacian for $G_{k,l}$ as

$$L = \begin{bmatrix} P_k^{-1} + e_k e_k^T & -e_k e_l^T & -e_k e_{n-l-k}^T \\ -e_l e_k^T & P_l^{-1} + e_l e_l^T & -e_l e_{n-l-k}^T \\ -e_{n-l-k} e_k^T & -e_{n-l-k} e_l^T & P_{n-l-k}^{-1} + e_{n-l-k} e_{n-l-k}^T \end{bmatrix},$$

and the Fiedler vector as

$$L = \begin{bmatrix} u \\ -v \\ 0 \end{bmatrix},$$

where u and v are entry-wise positive. Then $P_k^{-1}u + (u_k + v_l)e_k = \mu u$, so that $P_k u - (\frac{u_k+v_l}{\mu})e = \frac{1}{\mu}u$. Similarly, $P_l v - (\frac{u_k+v_l}{\mu})e = \frac{1}{\mu}v$. Further, $e^T P_k^{-1} u + (u_k + v_l) = \mu e^T u$, so that $2u_k + v_l = \mu e^T u$, and similarly $u_k + 2v_l = \mu e^T v$. Thus we have

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_k \\ v_l \end{bmatrix} = \mu \begin{bmatrix} e^T u \\ e^T v \end{bmatrix},$$

yielding

$$\begin{bmatrix} u_k \\ v_l \end{bmatrix} = \frac{\mu}{3} \begin{bmatrix} e^T u \\ e^T v \end{bmatrix}$$

(since $e^T u = e^T v$). Consequently, $\frac{u_k+v_l}{\mu} = \frac{2}{3}e^T u$, so that

$$(P_k - \frac{2}{3}J)u = \frac{1}{\mu}u, \text{ and } (P_l - \frac{2}{3}J)v = \frac{1}{\mu}v.$$

Since u and v are Perron vectors, and the corresponding Perron values are both $\frac{1}{\mu}$, it follows that $k = l$ and $\mu = \frac{1}{\rho(P_k - \frac{2}{3}J)}$. ◻

Corollary 4.7. *Suppose that we have the graph $G_{k,k}$ with algebraic connectivity μ, and let μ' be the algebraic connectivity of G, the graph in Figure 3. Then $\mu' < \mu$.*

Proof. If n is even (hence $n - 2k \geq 2$), then

$$\mu = \frac{1}{\rho(P_k - \frac{2}{3}J)} \geq \frac{1}{\rho(P_{\frac{n-2}{2}} - \frac{2}{3}J)} > \frac{1}{\rho(P_{\frac{n}{2}} - tJ)},$$

for all $t \in (0,1)$. Thus $\mu' < \mu$. If n is odd, then

$$\mu = \frac{1}{\rho(P_k - \frac{2}{3}J)} \geq \frac{1}{\rho(P_{\frac{n-1}{2}} - \frac{2}{3}J)},$$

while $\mu' = \frac{1}{\rho(P_{\frac{n-1}{2}} - tJ)}$, for some $t \in (0,1)$. Evidently, we obtain the result if $0 < t < \frac{2}{3}$. Suppose that $t \geq \frac{2}{3}$, and note that then $\mu' = \frac{1}{\rho(B-(1-t)J)}$, where B is the bottleneck matrix for the component at vertex $\frac{n+1}{2}$ containing vertex n. But then a principal submatrix of B dominates $P_{\frac{n-1}{2}}$, and the result follows. ◻

THEOREM 4.8. *Let $G_{k,l}$ be as in Figure 5. Suppose that Case A holds for $G_{k,l}$, and that some Fiedler vector has all nonzero entries, with the triangle being the unique block with both positive and negative valuations. If μ is the algebraic connectivity of $G_{k,l}$, then $\mu' < \mu$, where μ' is the algebraic connectivity of G (as in Figure 3).*

Proof. Without loss of generality, we can write the Laplacian matrix of $G_{k,l}$ as

$$\begin{bmatrix} P_k^{-1} + e_k e_k^T & -e_k e_1^T & -e_k e_1^T \\ -e_1 e_k^T & P_l^{-1} + e_1 e_1^T & -e_1 e_1^T \\ -e_1 e_k^T & -e_1 e_1^T & P_{n-l-k}^{-1} + e_1 e_1^T \end{bmatrix},$$

and the Fiedler vector as

$$L = \begin{bmatrix} u \\ -v \end{bmatrix},$$

where u is of order $k+l$, and v is of order $n-k-l$ and both are positive vectors. From the eigenvalue/eigenvector equation, we have

$$\left(\begin{bmatrix} P_k^{-1} & 0 \\ 0 & P_l^{-1} \end{bmatrix} + (e_k - e_{k+1})(e_k - e_{k+1})^T \right) u + v_1(e_k + e_{k+1}) = \mu u, \tag{5}$$

and

$$P_{n-k-l}^{-1} v + (v_1 + u_k + u_{k+1}) e_1 = \mu v. \tag{6}$$

From (6), we find that $P_{n-k-l} v - \dfrac{(v_1 + u_k + u_{k+1})}{\mu} e = \dfrac{1}{\mu} v$, and that $e^T P_{n-k-l}^{-1} v + v_1 + u_k + u_{k+1} = \mu e^T v$, hence $2v_1 + u_k + u_{k+1} = \mu e^T v$. Thus we find that

$$\left(P_{n-k-l} - \left(\frac{v_1 + u_k + u_{k+1}}{2v_1 + u_k + u_{k+1}} \right) J \right) v = \frac{1}{\mu} v.$$

Therefore $\mu = 1/\rho(P_{n-k-l} - tJ)$ for some $t \in (\frac{1}{2}, 1)$. In particular, if

$$n - k - l \le \begin{cases} \frac{n-2}{2}, & \text{if } n \text{ even;} \\ \frac{n-3}{2}, & \text{if } n \text{ odd,} \end{cases}$$

then $\mu' < \mu$. Now consider (5) and observe that

$$\left(\begin{bmatrix} P_k^{-1} & 0 \\ 0 & P_l^{-1} \end{bmatrix} + (e_k - e_{k+1})(e_k - e_{k+1})^T \right)^{-1}$$
$$= \begin{bmatrix} P_k & 0 \\ 0 & P_l \end{bmatrix} - \frac{1}{3} \begin{bmatrix} J & -J \\ -J & J \end{bmatrix}.$$

From (5) we find that

$$\left(\begin{bmatrix} P_k & 0 \\ 0 & P_l \end{bmatrix} - \frac{1}{3} \begin{bmatrix} J & -J \\ -J & J \end{bmatrix} \right) u - \frac{v_1}{\mu} e = \frac{1}{\mu} u,$$

so that

$$\left(\begin{bmatrix} P_k & 0 \\ 0 & P_l \end{bmatrix} - \frac{1}{3}\begin{bmatrix} J & -J \\ -J & J \end{bmatrix} - \frac{v_1}{2v_1+u_k+u_{k+1}}J\right)u = \frac{1}{\mu}u.$$

In particular,

$$\mu = 1/\rho\left(\begin{bmatrix} P_k & 0 \\ 0 & P_l \end{bmatrix} - \frac{1}{3}\begin{bmatrix} J & -J \\ -J & J \end{bmatrix} - tJ\right)$$

for some $t \in (0, \frac{1}{2})$. Note that

$$\rho\left(\begin{bmatrix} P_k & 0 \\ 0 & P_l \end{bmatrix} - \frac{1}{3}\begin{bmatrix} J & -J \\ -J & J \end{bmatrix}\right) \le \rho\left(\begin{bmatrix} P_k & 0 \\ 0 & P_l \end{bmatrix}\right).$$

If follows then that if

$$max(k,l) \le \begin{cases} \frac{n-2}{2}, & \text{if } n \text{ even}, \\ \frac{n-3}{2}, & \text{if } n \text{ odd}, \end{cases}$$

then $\mu' < \mu$. On the other hand, if

$$max(k,l) \ge \begin{cases} \frac{n}{2}, & \text{if } n \text{ even}, \\ \frac{n-1}{2}, & \text{if } n \text{ odd}, \end{cases}$$

then we have

$$n-k-l \le n-1-max(k,l) \le \begin{cases} \frac{n-2}{2}, & \text{if } n \text{ even}, \\ \frac{n-1}{2}, & \text{if } n \text{ odd}. \end{cases}$$

From our work above, we see that the only case in which we can have $\mu < \mu'$ is n odd, $min(k,l) = 1$ and $max(k,l) = \frac{n-1}{2}$. In this case it is not difficult to see that every Fiedler vector yields a valuation of 0 for the vertex on the triangle which is not a point of articulation, contrary to our hypothesis. Consequently, we have $\mu' < \mu$, as desired. □

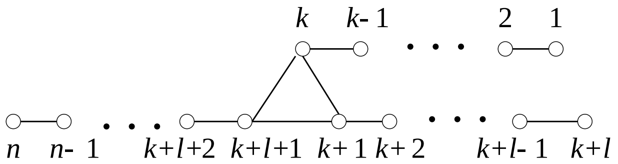

FIGURE 6.

LEMMA 4.9. *Consider the graph H given in Figure 6 with Laplacian matrix L, and let $C = \{1, 2, \ldots, k+l\}$. Consider also the graph G as in Figure 3, but relabeled via $i \to n-i+1$, for $1 \le i \le n$, with Laplacian matrix M. Then $(L(C))^{-1} \ll (M(C))^{-1}$.*

Proof. We have

$$L(C) = \begin{bmatrix} P_k^{-1} & 0 \\ 0 & P_l^{-1} \end{bmatrix} + (e_k - e_{k+1})(e_k - e_{k+1})^T,$$

so that

$$(L(C))^{-1} = \begin{bmatrix} P_k - \frac{1}{3}J & \frac{1}{3}J \\ \frac{1}{3}J & P_l - \frac{1}{3}J \end{bmatrix}.$$

Also

$$(M(C))^{-1} = F - \frac{1}{3}F(e_1 - e_2)(e_1 - e_2)^T F,$$

where

$$F = \left[\begin{array}{cc|c} k+l-1 & k+l-2 & \multirow{2}{*}{$ee_1^T P_{k+l+2}$} \\ k+l-2 & k+l-1 & \\ \hline \multicolumn{2}{c|}{P_{k+l+2}e_1e^T} & P_{k+l+2} \end{array}\right]$$

so that $(M(C))^{-1} = F - \frac{1}{3}(e_1 - e_2)(e_1 - e_2)^T$. It's now straightforward to verify that $(L(C))^{-1} \ll (M(C))^{-1}$. □

THEOREM 4.10. *The graph on n vertices with girth 3 of minimum algebraic connectivity is G as in Figure 3.*

Proof. From Proposition 4.3 we have that the graph of girth 3 with minimum algebraic connectivity has the form of $G_{k,l}$ as in Figure 5. If some Fiedler vector valuates the 3-cycle with both positive and negative valuations, then the algebraic connectivity for $G_{k,l}$ exceeds that for G, by Theorem 4.8 and Lemma 4.6. It follows that the minimizer either falls under Case B, or that there is an edge off the 3-cycle in which the endpoints of that edge have mutual Perron components. In either case, there is a vertex x on the 3-cycle with the property that we may replace the component containing the other vertices on the 3-cycle at x by the corresponding component at x of G, which will lower the algebraic connectivity, by Lemma 4.9 and Theorem 2.4. Hence G has minimum algebraic connectivity. □

Determination of the graph on n vertices with fixed girth s that maximizes the algebraic connectivity appears to be more difficult. However, in the case of unicyclic graphs we can say the following. The proof of the next result is similar to the proof of Corollary 4.2, and employing the fact that the bottleneck matrix for k pendant vertices is entry-wise dominated (after a suitable simultaneous permutation of the rows and columns) by the bottleneck matrix of any other tree on k vertices; see [KNS].

LEMMA 4.11. *Let G be a connected unicyclic graph with Laplacian matrix L, and let C be a connected component at some vertex u, which contains vertices on the cycle. For each vertex i on the cycle, suppose that there are m_i components at vertex i not including the vertices on the cycle. Let B_i be the collection of vertices of the m_i components at vertex i. Modify C to form $\tilde{C}$ by replacing those m_i components by $|B_i|$ pendant vertices at vertex i. Let the new graph be denoted by $\tilde{G}$, and suppose the Laplacian matrix for $\tilde{G}$ is denoted $\tilde{L}$. Then $(\tilde{L}(\tilde{C}))^{-1} \ll (L(C))^{-1}$.*

PROPOSITION 4.12. *Among all connected unicyclic graphs on n vertices with fixed girth s, the algebraic connectivity is maximized by a graph with girth s, with the* *following property: Each vertex on the cycle is adjacent to a nonnegative number of pendant vertices.*

Proof. Suppose G is a connected unicyclic graph with girth s, and algebraic connectivity μ. Assume the vertices on the cycle are labeled $1, 2, \ldots, s$. As in the proof of Proposition 4.3, let F_j denote the union of the connected components at vertex j $(1 \le j \le s)$ except for the unique component at j which contains the vertices on the cycle. Fix j $(1 \le j \le s)$. Suppose F_j is not the union of $|F_j|$ pendant vertices. Form $\tilde{G}$ by replacing F_j with $|F_j|$ pendant vertices each adjacent to vertex j, and denote the algebraic connectivity of $\tilde{G}$ by $\tilde{\mu}$. In $\tilde{G}$, if any of the $|F_j|$ pendant vertices adjacent to j is a Perron component at j, then it follows that $\tilde{\mu} \ge 1$. This inequality follows directly from Proposition 1.3 if Case B holds for $\tilde{G}$, otherwise it follows from Lemma 2.2 in the event Case A holds for $\tilde{G}$. Since G has a cutpoint $(|F_j| \ge 1)$, using Example 1.5, it follows that $\mu \le 1$. Hence $\mu \le \tilde{\mu}$. Otherwise, none of the $|F_j|$ pendant vertices each adjacent to vertex j in $\tilde{G}$ is a Perron component at j, in which case we can use the entry-wise domination of the bottleneck matrices for F_j and the $|F_j|$ pendant vertices, and apply Theorems 2.4 and 2.5 to obtain $\mu \le \tilde{\mu}$. ∎

In the case when the girth is 3, we can prove the following result.

LEMMA 4.13. *Let G be the unicyclic graph on n vertices with girth 3, by taking a 3-cycle and appending $n-3$ pendant vertices to a single vertex on the cycle. Then the algebraic connectivity of G is equal to 1.*

Proof. Let x denote the vertex of degree $n-1$ in G. Then it is clear that there are $n-1$ Perron components at x, each with Perron value 1. Hence Case B holds for G and by Theorem 2.4, the algebraic connectivity of G is equal to 1. ∎

THEOREM 4.14. *The unique unicyclic graph on n vertices with girth 3 of maximum algebraic connectivity is the graph G as in Lemma 4.13.*

Proof. Note that we may assume $n \ge 4$. It follows from Example 1.5 that any such unicyclic graph with girth 3 has algebraic connectivity at most 1, and by Lemma 4.13, the algebraic connectivity of G is equal to 1. Now we show that G is the unique (up to isomorphism) such unicyclic graph with algebraic connectivity 1. Let H be any unicyclic graph with girth 3, and note that by Proposition 4.12, we need only consider the case in which each vertex not on the 3-cycle is pendant. If there are two vertices on the 3-cycle which are adjacent to pendant vertices, then by considering Perron components, we find that Case A holds, with the 3-cycle as the unique block containing both positively and negatively valuated vertices. Applying Lemma 2.2, reveals that the algebraic connectivity of H is strictly less than 1. Hence G is the unique unicyclic graph with girth 3, and algebraic connectivity 1. ∎

REFERENCES

[BM] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan, London, 1977.

[C] F.R.K. Chung, *Spectral Graph Theory*, Conference Board of the Mathematical Sciences, No. 92, American Mathematical Society, Providence, RI, 1997.

[CDS] D. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980

[FK] S. Fallat and S. Kirkland, Perron components and algebraic connectivity for weighted graphs, To appear in *Linear and Multilinear Algebra*.

[F1] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.* **23**:298-305 (1973).

ELA

[F2] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, *Czechoslovak Math. J.* **25**:607-618 (1975).

[GM1] R. Grone and R. Merris, Algebraic connectivity of trees, *Czechoslovak Math. J.* **37**:660-670 (1987).

[GM2] R. Grone and R. Merris, Ordering trees by algebraic connectivity, *Graphs and Combinatorics* **6**:229-237 (1990).

[GMS] R. Grone, R. Merris and V.S. Sunder, The Laplacian spectrum of a graph, *SIAM J. on Matrix Analysis and Applications* **11**:218-238 (1990).

[HJ] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[KN] S. Kirkland and M. Neumann, Algebraic connectivity of weighted trees under perturbation, *Linear and Multilinear Algebra* **42**:187-203 (1997).

[KNS] S. Kirkland, M. Neumann and B. Shader, Characteristic vertices of weighted trees via Perron values, *Linear and Multilinear Algebra* **40**:311-325 (1996).

[M1] R. Merris, Characteristic vertices of trees, *Linear and Multilinear Algebra* **22**:115-131 (1987).

[M2] R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra Appl.* **197, 198**:143-176 (1994).

[M3] R. Merris, A survey of graph Laplacians, *Linear and Multilinear Algebra* **39**:19-31 (1995).

[MO1] B. Mohar, Eigenvalues, diameter, and mean distance in graphs, *Graphs and Combinatorics* **7**:53-64 (1991).

[MO2] B. Mohar, Laplace eigenvalues of graphs–a survey, *Discrete Mathematics* **109**:171-183 (1992).