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POSITIVE DEFINITENESS OF TRIDIAGONAL MATRICES VIA THE NUMERICAL RANGE*

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Dedicated to Hans Schneider on the occasion of his seventieth birthday.

Abstract. The authors use a recent characterization of the numerical range to obtain several conditions for an Hermitian tridiagonal matrix with positive diagonal entries to be positive definite.

Key words. Positive definite, tridiagonal matrices, numerical range, spread

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1. Introduction. In a paper on the question of unicity of best spline approximations for functions having a positive second derivative, Barrow, Chui, Smith, and Ward proved the following result.

PROPOSITION 1.1. [1, Proposition 1] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a tridiagonal matrix with positive diagonal entries. If*

$$a_{i,i-1}a_{i-1,i} \;\le\; \frac{1}{4}a_{ii}a_{i-1,i-1}\left(1+\frac{\pi^2}{1+4n^2}\right), \quad i = 2,\ldots,n,$$

then $det(A) > 0$.

Johnson, Neumann, and Tsatsomeros improved Proposition 1.1 as follows.

PROPOSITION 1.2. [6, Proposition 2.5, Theorem 2.4] *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be a tridiagonal matrix with positive diagonal entries. If*

$$a_{i,i-1}a_{i-1,i} \;<\; \frac{1}{4}a_{ii}a_{i-1,i-1}\frac{1}{\left(\cos\dfrac{\pi}{n+1}\right)^2}, \quad i = 2,\ldots,n,$$

then $det(A) > 0$. In addition, if A is (also) symmetric, then A is positive definite.

The main purpose of this paper is to derive additional criteria, using the **numerical range**, for an Hermitian tridiagonal matrix A with positive diagonal entries to be positive definite. An example of one of our results is Theorem 2.2.

Recall that the **numerical range of a matrix** $A \in \mathbb{C}^{n,n}$ is the set

$$W(A) \;=\; \{x^*Ax \mid x \in \mathbb{C}^n,\ |x| = 1\}.$$

Many of our results here will be derived with the aid of the following recent characterization of the numerical range due to Chien.

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THEOREM 1.3. [2, Theorem 1] *Let* $A \in \mathbb{C}^{n,n}$ *and suppose* S *be a nonzero subspace of* $\mathbb{C}^n$. *Then*

$$W(A) \;=\; \cup_{x,y}\, W(A_{xy}),$$

where x *and* y *vary over all unit vectors in* S *and* $S^{\perp}$, *respectively, and where*

$$A_{xy} \;=\; \begin{bmatrix} x^*Ax & x^*Ay \\ y^*Ax & y^*Ay \end{bmatrix}.$$

Our main results are developed in Section 2 which contains several conditions for an Hermitian tridiagonal matrix with positive diagonal entries to be positive definite. In Section 3 we give some examples to illustrate different criteria we have obtained in Section 2.

Some notations that we shall employ in this paper are as follows. Firstly, we shall use $e_i^{(k)}$, $i = 1, \ldots, k$, to denote the k–dimensional i–th unit coordinate vector, viz., the i–th entry of $e_i^{(k)}$ is 1 and its remaining entries are 0.
Secondly, for a set $\alpha \subset \{1, 2, \ldots, n\}$ denote by $\alpha' = \{1, 2, \ldots, n\} \setminus \alpha$ its complement in $\{1, 2, \ldots, n\}$. Thirdly, if α and β are subsets of $\{1, 2, \ldots, n\}$ and $A \in \mathbb{C}^{n,n}$, then we shall denote by $A[\alpha, \beta]$ the submatrix of A that is determined by the rows of A indexed by α and by the columns of A indexed by β. If $\alpha = \beta$, then $A[\alpha, \beta]$ will be abbreviated by $A[\alpha]$.

2. Main Results. We come now to the main thrust of this paper which is to develop several criteria for an Hermitian tridiagonal matrix with positive diagonal entries to be positive definite.

Suppose first that A is an $n \times n$ Hermitian matrix and partition A into

$$A \;=\; \begin{bmatrix} R & B \\ B^* & T \end{bmatrix},$$

where R and T are square. A well known result due to Haynsworth [4] (see also [5, Theorem 7.7.6]) is that A is positive definite if and only if R and the Schur complement of R in A, given by $(A/R) = T - B^*R^{-1}B$, are positive definite. For tridiagonal matrices, we now give a further proof of this fact based on Theorem 1.3 and in the spirit of the *divide and conquer algorithm* due to Sorensen and Tang [9], but see also Gu and Eisenstat [3], in the sense that it is possible to reduce the problem of determining whether A is positive definite into the problem of, first of all, determining whether two smaller principal submatrices of A are positive definite which can be executed in parallel.

THEOREM 2.1. *Let* $A \in \mathbb{C}^{n,n}$ *be an Hermitian tridiagonal matrix with positive diagonal entries and partition* A *into*

$$A \;=\; \begin{bmatrix} A_1 & a_{k,k+1}e_k^{(k)} & 0 \\ a_{k+1,k}e_k^{(k)T} & a_{k+1,k+1} & a_{k+1,k+2}e_1^{(n-k-1)T} \\ 0 & a_{k+2,k+1}e_1^{(n-k-1)} & A_2 \end{bmatrix},$$

where $1 < k < n$, $A_1 \in \mathbb{C}^{k,k}$, and $A_2 \in \mathbb{C}^{n-k-1,n-k-1}$. Then A is positive definite if and only if the symmetric tridiagonal matrix

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} - (1/a_{k+1,k+1}) \times$$

$$\begin{bmatrix} |a_{k,k+1}|^2 e_k^{(k)} e_k^{(k)T} & a_{k,k+1} a_{k+1,k+2} e_k^{(k)} e_1^{(n-k-1)T} \\ a_{k+1,k} a_{k+2,k+1} e_1^{(n-k-1)} e_k^{(k)T} & |a_{k+1,k+2}|^2 e_1^{(n-k-1)} e_1^{(n-k-1)T} \end{bmatrix} \tag{1}$$

is positive definite. Furthermore, if $A = (a_{ij}) \in \mathbb{R}^{n,n}$, then on putting

$$C_1 \;=\; A_1 - \frac{|a_{k,k+1}|^2}{a_{k+1,k+1}} e_k^{(k)} e_k^{(k)T}, \tag{2}$$

a necessary and sufficient condition for the matrix in (1) to be positive definite is that A_1 and A_2 are positive definite and that the inequalities

$$1 - \frac{|a_{k,k+1}|^2}{a_{k+1,k+1}} \left(A_1^{-1}\right)_{k,k} \;>\; 0 \tag{3}$$

and

$$1 - \left(|a_{k+1,k+2}|^2 + \left(\frac{|a_{k,k+1} a_{k+1,k+2}|}{a_{k+1,k+1}} \right)^2 (C_1^{-1})_{k,k} \right) \left(A_2^{-1}\right)_{1,1} \;>\; 0 \tag{4}$$

hold. In addition, if A_1 and A_2 are positive definite, then any one of the following three conditions is also sufficient for A to be positive definite.
(i) *The matrices*

$$A_i - \frac{|a_{k,k+1}|^2 + |a_{k+1,k+2}|^2}{a_{k+1,k+1}} I_i, \quad i = 1, 2, \tag{5}$$

are positive definite,
(ii)

$$1 \;>\; \frac{1}{a_{k+1,k+1}} \left(|a_{k,k+1}|^2 \left(A_1^{-1}\right)_{k,k} + |a_{k+1,k+2}|^2 \left(A_2^{-1}\right)_{1,1} \right), \tag{6}$$

(iii)

$$1 \;>\; \frac{1}{a_{k+1,k+1}} \left(\frac{|a_{k,k+1}|^2}{d_1} + \frac{|a_{k+1,k+2}|^2}{d_{k+2}} \right), \tag{7}$$

where d_1 and d_{k+2} are the smallest eigenvalues of of A_1 and A_2, respectively.

Proof. Let S be the subspace of $\mathbb{C}^n$ spanned by $e_{k+1}^{(n)}$. Then, by Theorem 1.3, A is positive definite if and only if A_{xy} is positive definite for all $x \in S$ and $y \in S^{\perp}$. Assume now that $x = [0\ 0 \ldots\ 0\ x_{k+1}\ 0 \ldots\ 0]^t \in S$ and $y = [y_1\ \ldots\ y_k\ 0\ y_{k+2}\ \ldots\ y_n]^t \in S^{\perp}$. Then

$$A_{xy} \;=\; \begin{bmatrix} a_{k+1,k+1} & a_{k+1,k} \bar{x}_{k+1} y_k + a_{k+1,k+2} \bar{x}_{k+1} y_{k+2} \\ a_{k,k+1} \bar{y}_k x_{k+1} + a_{k+2,k+1} \bar{y}_{k+2} x_{k+1} & y^* A y \end{bmatrix}$$

is positive definite if and only if

$$
\begin{array}{rl}
 & a_{k+1,k+1}\ y^*Ay \\
(8)\quad - & (a_{k+1,k}\bar{x}_{k+1}y_k + a_{k+1,k+2}\bar{x}_{k+1}y_{k+2})\,(a_{k,k+1}\bar{y}_k x_{k+1} + a_{k+2,k+1}\bar{y}_{k+2}x_{k+1}) \\
> & 0.
\end{array}
$$

Put $y' := [y_1\ \dots\ y_k\ y_{k+2}\ \dots\ y_n]^t \in \mathbb{C}^{n-1}$. Then (8) becomes

$$
\begin{aligned}
& a_{k+1,k+1}\ y'^* \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} y' - \big(|a_{k,k+1}|^2\bar{y}_k y_k + a_{k+1,k}a_{k+1,k+2}\bar{y}_k y_{k+2} \\
& + a_{k+1,k}a_{k+1,k+2}\bar{y}_{k+2}y_k + |a_{k+1,k+2}|^2\bar{y}_{k+2}y_{k+2}\big) \\
& = a_{k+1,k+1}\ y'^* \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} y' \\
& - y'^* \begin{bmatrix} |a_{k,k+1}|^2 e_k^{(k)} e_k^{(k)T} & a_{k,k+1}a_{k+1,k+2}e_k^{(k)}e_1^{(n-k-1)T} \\ a_{k+1,k}a_{k+2,k+1}e_1^{(n-k-1)}e_k^{(k)T} & |a_{k+1,k+2}|^2 e_1^{(n-k-1)}e_1^{(n-k-1)T} \end{bmatrix} y' > 0
\end{aligned}
$$

This proves the first part of theorem. Namely, that A is positive definite if and only if the $(n-1)\times(n-1)$ matrix in (1) is positive definite.

The equivalence of the positive definiteness of the matrix in (1) to the conditions (3) and (4) is obtained by considering the Schur complement of the matrix C_1 given in (2) and by utilizing the fact that if $u, v \in \mathbb{R}^\ell$, then

$$
\det\left(I + uv^T\right) \ = \ 1 + v^T u. \tag{9}
$$

Suppose now that A_1 and A_2 are positive definite. We next establish (i). For this purpose observe that the subtracted matrix in (1), which has rank 1, has eigenvalues 0 and $\left(|a_{k,k+1}|^2 + |a_{k+1,k+2}|^2\right)/a_{k+1,k+1}$ so that in the positive semidefinite ordering,

$$
\begin{aligned}
\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} - \frac{1}{a_{k+1,k+1}} \begin{bmatrix} |a_{k,k+1}|^2 e_k e_k^T & a_{k,k+1}a_{k+1,k+2}e_k e_1^T \\ a_{k+1,k}a_{k+2,k+1}e_1 e_k^T & |a_{k+1,k+2}|^2 e_1 e_1^T \end{bmatrix} \\
\geq \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} - \frac{|a_{k,k+1}|^2 + |a_{k+1,k+2}|^2}{a_{k+1,k+1}} I_{n-1}.
\end{aligned}
$$

Thus if (5) holds, then A is clearly positive definite because then a submatrix of A, namely $a_{k,k}$, as well as its Schur complement in A given in (1) are positive definite.

To establish the validity of (ii), factor the first matrix in (1) and compute the determinant of the sum of the identity matrix and the rank 1 matrix using the formula in (9). A simple inspection shows that condition (6) implies the positivity of that determinant.

Finally, condition (7) implies condition (6) because, by interlacing properties of symmetric matrices we have that $\left(A_1^{-1}\right)_{k,k} \geq (1/d_1) > 0$ and $\left(A_2^{-1}\right)_{1,1} \geq (1/d_{k+1}) > 0$. This establishes (iii) and our proof is complete. □

Next from Gershgorin theorem we know that a sufficient condition for an Hermitian matrix $A = (a_{i,j})$ with positive diagonal entries to be positive definite is that

$a_{i,i} > \sum_{j=1,\neq i}^{n} |a_{i,j}|$, $i = 1, 2, \ldots, n$. However, what can be said if, for a tridiagonal matrix, the condition fails to be satisfied for some rows? In this case, we obtain the following criteria.

THEOREM 2.2. *Let $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ be an Hermitian tridiagonal matrix with positive diagonal entries. Then A is positive definite if one of the following conditions is satisfied.*

(i)

$$\left(\max_{i=1,3,\ldots} a_{i,i}\right) - \left(\min_{i=1,3,\ldots} a_{i,i}\right) + \left(\max_{i=1,3,\ldots} \frac{|a_{i+1,i}|}{\sqrt{a_{i,i}}} + \max_{i=3,5,\ldots} \frac{|a_{i-1,i}|}{\sqrt{a_{i,i}}}\right)^2 < \min_{i=2,4,\ldots} a_{i,i}.$$

(ii) $\min_{i=1,3,\ldots} a_{ii} = \max_{i=1,3,\ldots} a_{ii}$, *and*

$$2 \ \max\left\{|a_{i-1,i}|^2 + |a_{i+1,i}|^2 : i = 1, 3, \ldots\right\} \ < \ \left(\min_{i=1,3,\ldots} a_{ii}\right) \ \left(\min_{i=2,4,\ldots} a_{ii}\right).$$

(iii) $\min_{i=1,3,\ldots} a_{ii} \neq \max_{i=1,3,\ldots} a_{ii}$,

$$\begin{aligned} &2\left(\max_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2 - \min_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2\right) \\ &> \ \left(\max_{i=1,3,\ldots} \{a_{i,i}\} - \min_{i=1,3,\ldots} \{a_{i,i}\}\right)^2, \end{aligned}$$

and

$$2 \ \max\left\{|a_{i-1,i}|^2 + |a_{i+1,i}|^2 : i = 1, 3, \ldots\right\} \ < \left(\min_{i=1,3,\ldots} a_{i,i}\right) \ \left(\min_{i=2,4,\ldots} a_{i,i}\right).$$

(iv) $\min_{i=1,3,\ldots} a_{i,i} \neq \max_{i=1,3,\ldots} a_{i,i}$,

$$\begin{aligned} &2\left(\max_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2 - \min_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2\right) \\ &\leq \ \left(\max_{i=1,3,\ldots} \{a_{i,i}\} - \min_{i=1,3,\ldots} \{a_{i,i}\}\right)^2, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{4}\left(\min_{i=1,3,\ldots} a_{i,i} - \max_{i=1,3,\ldots} a_{i,i}\right)^2 \\ &+ \ \max_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+,i}|^2 + \min_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2 \\ &+ \ \left(\frac{\max_{1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2 - \min_{i=1,3,\ldots} |a_{i-1,i}|^2 + |a_{i+1,i}|^2}{\max_{i=1,3,\ldots} a_{i,i} - \min_{i=1,3,\ldots} a_{i,i}}\right)^2 \\ &< \ \left(\min_{i=1,3,\ldots} \{a_{i,i}\}) \ (\min_{i=2,4,\ldots} \{a_{i,i}\}\right). \end{aligned}$$

Proof. We first note that Bessel's inequality (see, e.g., [7, p.488]) implies that if x and y are n–vectors which are orthogonal to each other, then

$$|x^*Ay|^2 \;\leq\; \|Ax\|_2^2 - |x^*Ax|^2. \tag{10}$$

Next let S be the subspace generated by $e_1, e_3, e_5, \ldots$ and observe that vectors $x \in S$ and $y \in S^\perp$ have the form $x = [x_1\ 0\ x_3\ 0\ x_5 \cdots]^T$ and $y = [0\ y_2\ 0\ y_4 \cdots]^T$, respectively. Then by Theorem 1.3, A is positive definite if and only if A_{xy} is positive definite for $x \in S$ and $y \in S^\perp$, which is equivalent to

$$x^*Ax\ y^*Ay \;>\; |x^*Ay|^2. \tag{11}$$

Assume now that condition (i) is satisfied. We first recall the following inequality. Let $p_1, \cdots, p_n$ be positive numbers. Then for any real numbers $q_1, q_2, \ldots, q_n$,

$$(q_1 + q_2 + \cdots + q_n)/(p_1 + p_2 + \cdots + p_n) \;\leq\; \max\{q_i/p_i : i = 1, 2, \ldots, n\} \tag{12}$$

Then by (12) we have that

$$\frac{\sum_{i=1,3,\ldots} |a_{i,i}x_i|^2}{\sum_{i=1,3,\ldots} a_{i,i}|x_i|^2} \;\leq\; \max_{i=1,3,\ldots} a_{i,i}, \tag{13}$$

and

$$\begin{aligned}
&\frac{\sum_{i=1,3,\ldots} |a_{i+1,i}x_i + a_{i+1,i+2}x_{i+2}|^2}{\sum_{i=1,3,\ldots} a_{i,i}|x_i|^2} \\
&\leq \left(\sqrt{\frac{\sum_{i=1,3,\ldots} |a_{i+1,i}x_i|^2}{\sum_{i=1,3,\ldots} a_{i,i}|x_i|^2}} + \sqrt{\frac{\sum_{i=1,3,\ldots} |a_{i+1,i+2}x_{i+2}|^2}{\sum_{i=1,3,\ldots} a_{i,i}|x_i|^2}}\right)^2 \\
&\leq \left(\sqrt{\max_{i=1,3,\ldots} \frac{|a_{i+1,i}|^2}{a_{i,i}}} + \sqrt{\max_{i=3,5,\ldots} \frac{|a_{i-1,i}|^2}{a_{i,i}}}\right)^2 \\
&= \left(\max_{i=1,3,\ldots} \frac{|a_{i+1,i}|}{\sqrt{a_{i,i}}} + \max_{i=3,5,\ldots} \frac{|a_{i-1,i}|}{\sqrt{a_{i,i}}}\right)^2. \qquad (14)
\end{aligned}$$

From (13), (14), and the hypothesis we now obtain,

$$\begin{aligned}
&\frac{\|Ax\|_2^2 - |x^*Ax|^2}{x^*Ax} \\
&= \frac{\sum_{i=1,3,\ldots} |a_{i,i}x_i|^2}{\sum_{i=1,3,\ldots} a_{i,i}|x_i|^2} + \frac{\sum_{i=1,3,\ldots} |a_{i+1,i}x_i + a_{i+1,i+2}x_{i+2}|^2}{\sum_{i=1,3,\ldots} a_{ii}|x_i|^2} - \sum_{i=1,3,\ldots} a_{ii}|x_i|^2 \\
&\leq \max_{i=1,3,\ldots} \{a_{i,i}\} + \left(\max_{i=1,3,\ldots} \frac{|a_{i+1,i}|}{\sqrt{a_{i,i}}} + \max_{i=3,5,\ldots} \frac{|a_{i-1,i}|}{\sqrt{a_{i,i}}}\right)^2 - \min_{i=1,3,\ldots} \{a_{i,i}\} \\
&< \min_{i=2,4,\ldots} \{a_{i,i}\} \;\leq\; \sum_{i=2,4,\ldots} a_{i,i}|y_i|^2 \;=\; y^*Ay.
\end{aligned}$$

The result now follows from (11) and (10).

Let us now set $\alpha_1 := \min_{i=1,3,\dots} \{a_{i,i}\}$, $\beta_1 := \max_{i=1,3,\dots} \{a_{i,i}\}$, $\alpha_2 := \min_{i=1,3,\dots} |a_{i-1,i}|^2+|a_{i+1,i}|^2$, and $\beta_2 := \max_{i=1,3,\dots} |a_{i-1,i}|^2+|a_{i+1,i}|^2$. Then

$$\begin{aligned}
&\|Ax\|^2 - |x^*Ax|^2 \\
&= \sum_{i=1,3,\dots} |a_{i,i}x_i|^2 + \sum_{i=1,3,\dots} |a_{i+1,i}x_i + a_{i+1,i+2}x_{i+2}|^2 - \left(\sum_{i=1,3,\dots} a_{i,i}|x_i|^2\right)^2 \\
&\le \sum_{i=1,3,\dots} |a_{i,i}x_i|^2 - \left(\sum_{i=1,3,\dots} a_{i,i}|x_i|^2\right)^2 \\
&+ \left(\sqrt{\sum_{i=1,3,\dots} |a_{i+1,i}|^2|x_i|^2} + \sqrt{\sum_{i=1,3,\dots} |a_{i+1,i+2}|^2|x_{i+2}|^2}\right)^2 \\
&\le \sum_{i=1,3,\dots} |a_{i,i}x_i|^2 - \left(\sum_{i=1,3,\dots} a_{i,i}|x_i|^2\right)^2 + 2\left[\sum_{i=1,3,\dots} (|a_{i-1,i}|^2+|a_{i+1,i}|^2)|x_i|^2\right] \\
&= t\alpha_1^2 + (1-t)\beta_1^2 - [t\alpha_1 + (1-t)\beta_1]^2 + 2[t\alpha_2 + (1-t)\beta_2] \\
&= t(1-t)(\alpha_1-\beta_1)^2 + 2(t\alpha_2 + (1-t)\beta_2) \\
&= -(\alpha_1-\beta_1)^2t^2 + \left[(\alpha_1-\beta_1)^2 + 2(\alpha_2-\beta_2)\right]t + 2\beta_2, \qquad (15)
\end{aligned}$$

for some $t \in [0,1]$. If $\alpha_1 = \beta_1$ then the value of (15) is $2(\alpha_2-\beta_2)t + 2\beta_2$. If $\alpha_1 \ne \beta_1$ then (15) assumes its maximum $\frac{1}{4}(\alpha_1-\beta_1)^2 + (\alpha_2+\beta_2) + (\frac{\alpha_2-\beta_2}{\alpha_1-\beta_1})^2$ at $t = \frac{1}{2} + \frac{\alpha_2-\beta_2}{2(\alpha_1-\beta_1)^2}$ when $2(\beta_2-\alpha_2) \le (\beta_1-\alpha_1)^2$ and assumes it maximum $2\beta_2$ when $2(\beta_2-\alpha_2) > (\beta_1-\alpha_1)^2$. Therefore

$$\|Ax\|_2^2 - |x^*Ax|^2 \le \begin{cases} \frac{1}{4}(\alpha_1-\beta_1)^2 + (\alpha_2+\beta_2) + \left(\frac{\alpha_2-\beta_2}{\alpha_1-\beta_1}\right)^2, \\ \qquad\qquad if\ \alpha_1 \ne \beta_1\ and\ 2(\beta_2-\alpha_2) \le (\beta_1-\alpha_1)^2, \\ \\ 2\beta_2,\ if\ \alpha_1 = \beta_1\ or\ \alpha_1 \ne \beta_1\ and\ 2(\beta_2-\alpha_2) > (\beta_1-\alpha_1)^2. \end{cases} \qquad (16)$$

If one of the conditions (ii), (iii), or (iv) is satisfied, then by (16),

$$\|Ax\|_2^2 - |x^*Ax|^2 < (\min_{i=1,3,\dots} a_{ii})\,(\min_{i=2,4,\dots} a_{ii}) \le x^*Ax\ y^*Ay.$$

Thus, by (11) and (10), A is positive definite. $\square$

We remark that a similar result can be obtained by switching the odd and even indices in statement of theorem and in the proof.

Recall now that for an Hermitian matrix $A \in \mathbb{C}^{n,n}$ with eigenvalues $\lambda_1 \ge \dots \ge \lambda_n$, the *spread* of A is $\lambda_1 - \lambda_n$ and is denoted by $Sp(A)$. For references on eigenvalue spreads, see, e.g., Nylen and Tam [8], and Thompson [10].

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Theorem 2.3. *Let $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ be an Hermitian tridiagonal matrix with positive diagonal entries. If*

$$\left(\min_{i=1,3,\ldots} \{a_{i,i}\}\right) \left(\min_{i=2,4,\ldots} \{a_{i,i}\}\right) > \frac{n(n-1)}{8} \sum_{i<j} \left(|a_{i,i} - a_{j,j}|^2 + 4|a_{i,j}|^2\right),$$

then A is positive definite.

Proof. Let S be the subspace generated by $e_1, e_3, \ldots$. For $x \in S$ and $y \in S^{\perp}$, $y^*x = 0$. By Remark 1 in Tsing [11] ,

$$W_0(A) \equiv \{u^*Av \mid u, v \in \mathbb{C}^n,\ |u| = |v| = 1,\ u^*v = 0\}$$

is a circular disk centered at the origin of radius $(\lambda_1 - \lambda_n)/2$. Thus $|y^*Ax|^2 \leq \frac{1}{4}(\lambda_1 - \lambda_n)^2 = \frac{1}{4}Sp(A)^2$. By [8, Corollary 4],

$$Sp(A)^2 \leq \frac{n}{(n-2)^2} \sum_{i=1}^{n} Sp(A[\{i\}'])^2, \tag{17}$$

But then from (17),

$$Sp(A)^2 \leq \frac{n}{(n-2)^2} \frac{n-1}{(n-3)^2} \frac{n-2}{(n-4)^2} \cdots \frac{5}{3^2} \frac{4}{2^2} \frac{3}{1^2} \sum_{i,j} ((n-2)!) Sp(A[\{i,j\}])^2,$$

from which we now get that

$$\frac{1}{4} Sp(A)^2 \leq \frac{n(n-1)}{8} \sum_{i<j} \left(|a_{i,i} - a_{j,j}|^2 + 4|a_{i,j}|^2\right).$$

It now follows immediately from (11) and the fact that

$$x^*Ax\ y^*Ay \geq \left(\min_{i=1,3,\ldots} \{a_{i,i}\}\right) \left(\min_{i=2,4,\ldots} \{a_{i,i}\}\right),$$

that A is positive definite. □

3. Examples. In this section, we give some examples to illustrate different criteria obtained in the paper.

Example 3.1. Consider the matrix

$$A = \begin{bmatrix} a & 2 & 0 & 0 \\ 2 & b & 3 & 0 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 5 & b \end{bmatrix},$$

where a and b are positive numbers. Table 1 lists the applicability for positive definiteness of A with $a = 4$ so that $\min_{i=1,3,\ldots} a_{ii} = \max_{i=1,3,\ldots} a_{ii}$, while Table 2 shows criteria for $a = 3.9$ and $\min_{i=1,3,\ldots} a_{ii} \neq \max_{i=1,3,\ldots} a_{ii}$.

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TABLE 1
Test for positive definiteness of A (Example 3.1)

Applicability	a=4, b=18	a=4, b=17	a=4, b=16.1
Theorem 2.2 (i)	yes	yes	yes
Theorem 2.2 (ii)	yes	no	no
Proposition 1.2	yes	yes	no
Gershgorin Theorem	no	no	no

TABLE 2
Test for positive definiteness of A (Example 3.1)

Applicability	a=3.9, b=18	a=3.9, b=17	a=3.9, b=16.2
Theorem 2.2 (i)	yes	yes	yes
Theorem 2.2 (iii)	yes	no	no
Proposition 1.2	yes	yes	no
Gershgorin Theorem	no	no	no

EXAMPLE 3.2. Consider the matrix

$$A = \begin{bmatrix} 10 & 0.5 & 0 \\ 0.5 & 8 & 8.5 \\ 0 & 8.5 & 10 \end{bmatrix}.$$

It is easy to verify the principal minors of A are positive, A is positive definite. Table 3 compares three other criteria, and shows that the condition Theorem 2.3 works.

TABLE 3
Test for positive definiteness of A (Example 3.2)

Applicability	
Theorem 2.3	yes
Proposition 1.2	no
Gershgorin Theorem	no

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