

## ON THE RODMAN-SHALOM CONJECTURE REGARDING THE JORDAN FORM OF COMPLETIONS OF PARTIAL UPPER TRIANGULAR MATRICES\*

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*Dedicated to Hans Schneider on the occasion of his seventieth birthday.*

**Abstract.** Rodman and Shalom [*Linear Algebra and its Applications*, 168:221–249, 1992] conjecture the following statement: Let  $A$  be a lower irreducible partial upper triangular  $n \times n$  matrix over  $F$  such that  $\text{trace}(A) = 0$ . Let  $n_1 \geq n_2 \geq \dots \geq n_p \geq 1$  be a set of  $p$  positive integers such that  $\sum_{i=1}^p n_i = n$ . There exists a nilpotent completion  $A_c$  of  $A$  whose Jordan form consists of  $p$  blocks of

sizes  $n_i \times n_i$ ,  $i = 1, 2, \dots, p$  if and only if  $r(A^k) \leq \sum_{i:n_i \geq k} (n_i - k)$ ,  $k = 1, 2, \dots, n_1$ .

In this paper this conjecture is solved in two cases: when the minimal rank of  $A$  is 2, and for matrices of size  $5 \times 5$ .

**Key words.** Completion problems, Jordan structure, Minimal rank, Majorization.

**AMS subject classifications.** 15A18, 15A21

**1. Introduction.** We consider  $n \times n$  matrices  $A = [a_{ij}]$  over an infinite field  $\mathbb{F}$ . A matrix is said to be a *partial matrix* if some of its entries are given elements from the field  $\mathbb{F}$ , while the rest of them can be arbitrarily chosen and treated as free independent variables. If those last elements are fixed, the resultant matrix is called a *completion*  $A_c$  of the matrix  $A$ . The completion problems consist of finding all the completions of a given partial matrix with prescribed properties. We are interested in *partial upper triangular matrices*, namely, partial matrices  $A = [a_{ij}]$  where  $a_{ij}$ ,  $i \leq j$ , are the given elements. For this kind of matrices some completion problems have been studied. For example, problems related to the rank of the matrix can be found in [9], to eigenvalues in [1, 9], to Jordan form in [2, 7, 8, 9] and to controllability of linear systems in [3, 4].

Let  $A$  be a partial matrix. We denote by  $r(A)$  the *minimal rank* of all possible completions of  $A$ , that is,  $r(A) = \min_{A_c} \text{rank}(A_c)$  where the minimum is taken over the set of all the ranks of possible completions of  $A$ . Moreover, for any positive integer  $k$ , we denote  $r(A^k) = \min_{A_c} \text{rank}((A_c)^k)$ , where the minimum is taken over the set of all the ranks of the  $k$ th power of possible completions of  $A$ .

We recall that a matrix  $A$  is *lower similar* to a matrix  $B$  (we denote  $A \sim B$ ) if there exists a lower triangular matrix  $S$  such that  $A = SBS^{-1}$ . In addition, a matrix  $A$  of size  $n \times n$  is said to be *lower irreducible* if all its  $k \times (n - k)$  submatrices

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$[a_{ij}]_{i=1, j=k+1}^{k, n}$  are nonzero for  $k = 1, 2, \dots, n - 1$ .

Given two nonincreasing sequences of nonnegative integers  $\{k_i\}_{i=1}^p$  and  $\{q_i\}_{i=1}^r$ , we say that  $\{k_i\}_{i=1}^p$  majorizes  $\{q_i\}_{i=1}^r$ , denoted  $\{k_i\}_{i=1}^p \succ \{q_i\}_{i=1}^r$ , if

$$\sum_{i=1}^s k_i \geq \sum_{i=1}^s q_i, \quad s = 1, 2, \dots, p, \quad \text{and} \quad \sum_{i=1}^p k_i = \sum_{i=1}^r q_i.$$

Rodman and Shalom give in [9] the following completion problem,

**CONJECTURE:** *Let  $A$  be a lower irreducible partial upper triangular  $n \times n$  matrix over  $\mathbb{F}$  such that  $\text{trace}(A) = 0$ . Let  $n_1 \geq n_2 \geq \dots \geq n_p \geq 1$  be a set of  $p$  positive integers such that  $\sum_{i=1}^p n_i = n$ . There exists a nilpotent completion  $A_c$  of  $A$  whose Jordan form consists of  $p$  blocks of sizes  $n_i \times n_i$ ,  $i = 1, 2, \dots, p$  if and only if*

$$(1) \quad r(A^k) \leq \sum_{i: n_i \geq k} (n_i - k), \quad k = 1, 2, \dots, n_1.$$

Note that the right hand side of (1) is the rank of  $A_c^k$  provided that  $A_c$  is a nilpotent completion of  $A$  with the described Jordan form. The necessity of (1) is therefore evident.

Rodman and Shalom prove the above conjecture when  $r(A) = 1$  (see [9], Theorem 3.2) or in general for matrices of size  $n \leq 4$  (see [9], Theorem 3.3). In [5] we prove that this conjecture is not true in general for matrices with minimal rank equal to three and for matrices of size  $n \times n$ ,  $n \geq 6$ .

In this paper we prove that this conjecture is true in two remaining cases: when  $r(A) = 2$ , and for matrices of size  $5 \times 5$ .

Let  $A_0$  be the completion obtained by replacing the unspecified elements by zero. We denote by  $J_q(\lambda)$  the  $q \times q$  Jordan block with eigenvalue  $\lambda$ . The block diagonal matrix with diagonal blocks  $A_1, \dots, A_t$  is denoted by  $A_1 \oplus \dots \oplus A_t$ . Finally, given a matrix  $A = [a_{ij}]$  let  $a_{ij}$  represent the corresponding nonzero entries of any similar matrix obtained.

**2. The main result.** We prove the conjecture of Rodman and Shalom for matrices with minimal rank equal to two.

**THEOREM 2.1.** *Let  $A$  be an  $n \times n$  lower irreducible partial upper triangular matrix such that  $\text{trace}(A) = 0$  and  $r(A) = 2$ . Let  $n_1 \geq n_2 \geq \dots \geq n_p \geq 1$  be a set of  $p$  positive integers such that  $\sum_{i=1}^p n_i = n$ . If*

$$(2) \quad r(A^k) \leq \sum_{i: n_i \geq k} (n_i - k), \quad k = 1, 2, \dots, n_1,$$

*then there exists a nilpotent completion  $A_c$  of  $A$  whose Jordan form consists of  $p$  blocks of sizes  $n_i \times n_i$ ,  $i = 1, 2, \dots, p$ .*

*Proof.* By lower similarity we can transform the matrix  $A$  into a partial upper triangular matrix  $A_1$  which has only two nonzero rows. Since  $A$  is lower irreducible,

one of this rows is the first one. We assume that the other one is the  $j$ th row. Then the matrix  $A_1$  has the following structure:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & a_{1j} & a_{1j+1} & \cdots & a_{1n-1} & a_{1n} \\ & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & a_{jj} & a_{jj+1} & \cdots & a_{jn-1} & a_{jn} \\ & & & & & 0 & \cdots & 0 & 0 \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & 0 & 0 \\ & & & & & & & & 0 \end{bmatrix}.$$

Since the matrix  $A_1$  is lower irreducible its  $n$ th column has a nonzero entry. We can distinguish the following cases:

**(a)**  $a_{1n} \neq 0$ .

By lower similarity we cancel the  $(j, n)$ th entry and all the entries in the first row, except the  $(1, 1)$ th entry. Since the matrix  $A_1$  is lower similar to the matrix  $A$ , its minimal rank is equal to two, and then the  $j$ th row has a nonzero entry. Let us assume that:

**(a1)**  $a_{jt}$ ,  $j < t < n$ , is the first nonzero entry beginning from the right.

By lower similarity we can cancel all the entries in this row, except the element  $a_{jj}$ . We obtain the matrix

$$A_2 = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & \cdots & 0 & \cdots & a_{1n} \\ & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & & \ddots & \vdots & & \vdots & & \vdots \\ & & & a_{jj} & \cdots & a_{jt} & \cdots & 0 \\ & & & & \ddots & \vdots & & \vdots \\ & & & & & 0 & \cdots & 0 \\ & & & & & & \ddots & \vdots \\ & & & & & & & 0 \end{bmatrix}.$$

Then:

**(a1.1)** If  $a_{11} = a_{jj} = 0$ , it is easy to prove that  $r(A_2^2) = 0$ . The Jordan form of the completion  $A_{2_0}$  consists of  $n - 2$  blocks of size  $q_1 = q_2 = 2$  and  $q_3 = \cdots = q_{n-2} = 1$ . Therefore,  $r(A) = r(A_2) = \text{rank}(A_{2_0}) = 2$  and  $r(A^2) = r(A_2^2) = \text{rank}(A_{2_0}^2) = 0$ .

If the sequence  $\{n_i\}_{i=1}^p$  satisfies condition (2), then it majorizes the sequence  $\{q_i\}_{i=1}^{n-2}$ . By performing the permutation

$$(1, n, 2, 3, \dots, j - 1, j, t, j + 1, \dots, t - 1, t + 1, \dots, n - 1)$$

of rows and columns in the matrix  $A_2$  and by applying the algorithm of [6] to the new matrix (see Appendix, Part I) we obtain a completion of the matrix  $A_2$  such that its

Jordan form consists of  $p$  blocks of size  $n_i$ ,  $i = 1, 2, \dots, p$ . This completion guarantees the existence of one completion of the matrix  $A$  with the desired characteristics.

**(a1.2)** If  $a_{11} \neq 0$ , we can transform, by lower similarity, the matrix  $A_2$  into the following matrix

$$A_3 = \begin{bmatrix} 0 & 0 & \cdots & a_{1j} & \cdots & a_{1t} & \cdots & a_{1n} \\ & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & & \ddots & \vdots & & \vdots & & \vdots \\ & & & 0 & \cdots & 0 & \cdots & a_{jn} \\ & & & & \ddots & \vdots & & \vdots \\ & & & & & 0 & \cdots & 0 \\ & & & & & & \ddots & \vdots \\ & & & & & & & 0 \end{bmatrix}.$$

This matrix satisfies that  $r(A_3) = 2$  and  $r(A_3^2) = 1$ . The Jordan form of the completion  $A_{3_0}$  consists of  $n - 2$  blocks of size  $q_1 = 3, q_2 = \dots = q_{n-2} = 1$ . Therefore,  $r(A) = r(A_3) = \text{rank}(A_{3_0}) = 2$ ,  $r(A^2) = r(A_3^2) = \text{rank}(A_{3_0}^2) = 1$  and  $r(A^3) = r(A_3^3) = \text{rank}(A_{3_0}^3) = 0$ .

If the sequence  $\{n_i\}_{i=1}^p$  satisfies condition (2), then it majorizes the sequence  $\{q_i\}_{i=1}^{n-2}$ . By applying the method given in Appendix, Part II, we obtain a completion of the matrix  $A_3$ , such that its Segre characteristic is  $\{n_i\}_{i=1}^p$ .

**(a2)**  $a_{jj}$  is the only nonzero entry in the  $j$ th row.  
By lower similarity we can transform the matrix  $A_1$  into

$$A_2 = \begin{bmatrix} 0 & 0 & \cdots & a_{1j} & \cdots & a_{1n} \\ & 0 & \cdots & 0 & \cdots & 0 \\ & & \ddots & \vdots & & \vdots \\ & & & 0 & \cdots & a_{jn} \\ & & & & \ddots & \vdots \\ & & & & & 0 \end{bmatrix}.$$

It is easily proved that the Jordan form of the completion  $A_{2_0}$  consists of  $n - 2$  blocks of size  $q_1 = 3$  and  $q_2 = \dots = q_{n-2} = 1$ , and that if the sequence  $\{n_i\}_{i=1}^p$  satisfies condition (2), then it majorizes the sequence  $\{q_i\}_{i=1}^{n-2}$ . By performing the permutation  $(1, j, n, 2, 3, \dots, n - 1)$  of rows and columns in the matrix  $A_2$ , and by applying the algorithm of [6] (see Appendix, Part I) we obtain a completion of the matrix  $A_2$  such that its Segre characteristic is the sequence  $\{n_i\}_{i=1}^p$ .

**(b)**  $a_{1n} = 0$  and  $a_{jn} \neq 0$ .

By lower similarity we transform the matrix  $A_1$  into

$$A_2 = \begin{bmatrix} a_{11} & 0 & \cdots & a_{1j} & 0 & \cdots & a_{1t} & \cdots & 0 \\ & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ & & \ddots & \vdots & \vdots & & \vdots & & \vdots \\ & & & a_{jj} & 0 & \cdots & 0 & \cdots & a_{jn} \\ & & & & 0 & \cdots & 0 & \cdots & 0 \\ & & & & & \ddots & \vdots & & \vdots \\ & & & & & & 0 & \cdots & 0 \\ & & & & & & & \ddots & \vdots \\ & & & & & & & & 0 \end{bmatrix},$$

where  $a_{1t}$  is the first nonzero entry in the first row, starting from the right. Now, we can distinguish the following subcases.

**(b1)**  $a_{11} = a_{jj} = 0$ .

If  $a_{1j} = 0$  the Jordan form of the completion  $A_{2_0}$  consists of  $n - 2$  blocks of size  $q_1 = q_2 = 2$  and  $q_3 = \cdots = q_{n-2} = 1$ . Again, if the sequence  $\{n_i\}_{i=1}^p$  satisfies (2), then it majorizes the sequence  $\{q_i\}_{i=1}^{n-2}$  and we obtain the desired completion by applying the algorithm of [6] to the matrix obtained performing the permutation

$$(1, t, 2, 3, \dots, j - 1, j, n, j + 1, \dots, t - 1, t + 1, \dots, n - 1)$$

of rows and columns in the matrix  $A_2$  (see Appendix, Part I).

If  $a_{1j} \neq 0$  we apply the method given in (a1.2), to the matrix  $A_2$ .

**(b2)**  $a_{11} \neq 0$  and  $a_{jj} \neq 0$ .

If  $a_{1j} = 0$  we transform, by lower similarity, the matrix  $A_2$  into

$$A_3 = \begin{bmatrix} 0 & 0 & \cdots & a_{1j} & \cdots & a_{1t} & \cdots & 0 \\ & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & & \ddots & \vdots & & \vdots & & \vdots \\ & & & 0 & \cdots & a_{jt} & \cdots & a_{jn} \\ & & & & \ddots & \vdots & & \vdots \\ & & & & & 0 & \cdots & a_{tn} \\ & & & & & & \ddots & \vdots \\ & & & & & & & 0 \end{bmatrix}.$$

We prove easily that the Jordan form of the completion  $A_{3_0}$  consists of  $n - 3$  blocks of size  $q_1 = 4$  and  $q_2 = \cdots = q_{n-3} = 1$ , and again that if the sequence  $\{n_i\}_{i=1}^p$  satisfies (2), then it majorizes the sequence  $\{q_i\}_{i=1}^{n-3}$ . By performing the permutation  $(1, j, t, n, 2, 3, \dots, n - 1)$  of rows and columns in the matrix  $A_3$ , and by applying the algorithm of [6] to this new matrix (see Appendix, Part I) we achieve the desired completion.

If  $a_{1j} \neq 0$ , by applying lower similarity to the matrix  $A_3$  we obtain a new matrix as in the case (b1) with  $a_{1j} \neq 0$ .  $\square$

The next theorem proves that the above conjecture is true for matrices of size  $5 \times 5$ .

**THEOREM 2.2.** *Let  $A$  be a lower irreducible partial upper triangular matrix of size  $5 \times 5$  such that  $\text{trace}(A) = 0$ . Let  $n_1 \geq n_2 \geq \dots \geq n_p \geq 1$  be a set of  $p$  positive integers such that  $\sum_{i=1}^p n_i = 5$ . If*

$$r(A^k) \leq \sum_{i: n_i \geq k} (n_i - k), \quad k = 1, 2, \dots, n_1,$$

then there exists a nilpotent completion  $A_c$  of  $A$  whose Jordan form consists of  $p$  blocks of sizes  $n_i \times n_i$ ,  $i = 1, 2, \dots, p$ .

*Proof.* Rodman and Shalom proved this theorem for  $r(A) = 1$  in [9]. As we have seen in Theorem 2.1, this result is also true when  $r(A) = 2$ . If  $r(A) = 4$  the result follows by applying Theorem 2.1 of [9]. Therefore, we can assume that  $r(A) = 3$ .

If  $r(A^4) > 0$  and  $r(A^5) = 0$ , the only possible Jordan form for some nilpotent completion of  $A$  is  $J_5(0)$ . The existence of this completion is guaranteed by Rodman and Shalom in [9].

If  $r(A^3) > 0$  and  $r(A^4) = 0$ , there are two possible Jordan forms for some nilpotent completion of  $A$ ,  $J_4(0) \oplus J_1(0)$  and  $J_5(0)$ . The last one exists by Theorem 2.1 of [9]. The existence of the first one is assured by the initial conditions about minimal rank for  $A^3$  and  $A^4$ .

Now, assume that  $r(A^2) > 0$  and  $r(A^3) = 0$ . There are three possible Jordan forms for some nilpotent completion of  $A$ ,  $J_3(0) \oplus J_2(0)$ ,  $J_4(0) \oplus J_1(0)$  and  $J_5(0)$ . Again, the last one exists by Theorem 2.1 of [9]. The first structure exists as well because otherwise the minimal rank of  $A^3$  cannot be zero. Therefore, we can assure that  $r(A^2) = 1$ . Next we prove the existence of a nilpotent completion of the matrix  $A$  such that its Segre characteristic is  $\{4, 1\}$ .

Let us assume that  $A_{c_0}$  is a completion of  $A$  such that its Segre characteristic is  $\{3, 2\}$ . Then, we distinguish the following cases:

**(a)** If the three first rows of  $A_{c_0}$  are linear independent, this matrix is lower similar to

$$A_{c_0} \sim \left[ \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ c_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ c_{31} & c_{32} & a_{33} & a_{34} & a_{35} \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} A_1 & A_2 \\ 0 & 0 \end{array} \right].$$

Since  $\text{rank}(A_{c_0}^2) = 1$  and  $\text{rank}(A_{c_0}^3) = 0$  we can find the following cases:

**(a1)**  $A_1 \neq 0$ ,  $A_1^2 \neq 0$  and  $A_1^3 = 0$ .

In this case there exists a nonsingular matrix  $T_1$  such that  $T_1 A_1 T_1^{-1} = J_3(0)$ . Therefore,

$$\left[ \begin{array}{cc} T_1 & 0 \\ 0 & I \end{array} \right] \left[ \begin{array}{cc} A_1 & A_2 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{cc} T_1^{-1} & 0 \\ 0 & I \end{array} \right] = \left[ \begin{array}{ccc|cc} 0 & 1 & 0 & a_1 & b_1 \\ 0 & 0 & 1 & a_2 & b_2 \\ 0 & 0 & 0 & a_3 & b_3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

where  $a_3 \neq 0$  or  $b_3 \neq 0$ , because  $\text{rank}(A_{c_0}) = 3$ . But from the last expression we can assure that  $J_{A_{c_0}} = J_4(0) \oplus J_1(0)$ , which is a contradiction.

**(a2)**  $A_1 \neq 0$  and  $A_1^2 = 0$ .

In this case there exists a nonsingular matrix  $T_1$  such that  $T_1 A_1 T_1^{-1} = J_2(0) \oplus J_1(0)$ . Therefore,

$$\begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & I \end{bmatrix} = \left[ \begin{array}{ccc|cc} 0 & 1 & 0 & a_1 & b_1 \\ 0 & 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & a_3 & b_3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

where

$$\text{rank} \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = 2.$$

Next, consider the partial matrix

$$\bar{A} = \left[ \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ c_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ c_{31} & c_{32} & a_{33} & a_{34} & a_{35} \\ \hline 0 & 0 & 0 & 0 & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & 0 \end{array} \right].$$

Then,

$$\begin{bmatrix} T_1 & 0 \\ 0 & I \end{bmatrix} \bar{A} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & I \end{bmatrix} = \left[ \begin{array}{ccc|cc} 0 & 1 & 0 & a_1 & b_1 \\ 0 & 0 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & a_3 & b_3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \bar{x}_{51} & \bar{x}_{52} & \bar{x}_{53} & \bar{x}_{54} & 0 \end{array} \right],$$

with

$$\text{rank} \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = 2.$$

If  $b_2 \neq 0$  we take  $\bar{x}_{54} = 1$  and  $\bar{x}_{51} = \bar{x}_{52} = \bar{x}_{53} = 0$ . The Jordan form of the matrix obtained is  $J_4(0) \oplus J_1(0)$ . This matrix allows us to achieve a completion of the initial matrix  $A$ .

If  $b_2 = 0$  we take  $\bar{x}_{52} = 1$  and  $\bar{x}_{51} = \bar{x}_{53} = \bar{x}_{54} = 0$ . The Jordan form of the matrix obtained is  $J_4(0) \oplus J_1(0)$ . Again, this matrix allows us to get a completion of the initial matrix  $A$ .

**(b)** If the independent rows of  $A_{c_0}$  are the first one, the second one and the fourth one, this matrix is lower similar to

$$A_{c_0} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ c_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ c_{41} & c_{42} & c_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By performing the permutation  $(1, 2, 4, 3, 5)$  of rows and columns in  $A_{c_0}$  and by applying the above method, we obtain the desired completion.

(c) Finally, if the independent rows of  $A_{c_0}$  are the first one, the third one and the fourth one, this matrix is lower similar to

$$A_{c_0} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & 0 & 0 \\ c_{31} & c_{32} & a_{33} & a_{34} & a_{35} \\ c_{41} & c_{42} & c_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, by performing the permutation  $(1, 3, 4, 2, 5)$  of rows and columns in  $A_{c_0}$  and by applying the method given in (a) we achieve the desired completion.  $\square$

**3. Appendix. Part I:** Let  $Y$  be a matrix of size  $n \times m$  such that its  $m$ -diagonal sums are  $s_1, s_2, \dots, s_m$ . It is well known, see [10], that the matrix

$$\begin{bmatrix} J_m & O \\ Y & J_n \end{bmatrix} \text{ is lower similar to } \begin{bmatrix} J_m & O \\ L & J_n \end{bmatrix},$$

where

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ s_1 & s_2 & \cdots & s_m \end{bmatrix}.$$

The matrix  $L$  is said to be the *lower concentrated form* of the matrix  $Y$ .

In the same way, if  $c_1, c_2, \dots, c_n$  are the  $n$ -diagonal sums the matrix

$$\begin{bmatrix} J_m & O \\ Y & J_n \end{bmatrix} \text{ is lower similar to } \begin{bmatrix} J_m & O \\ C & J_n \end{bmatrix},$$

where

$$C = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ c_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_n & 0 & \cdots & 0 \end{bmatrix}.$$

The matrix  $C$  is called *left concentrated form* of the matrix  $Y$ .

In general given a matrix

$$A = \begin{bmatrix} J_{n_1} & O & \cdots & O & O \\ Y_{21} & J_{n_2} & \cdots & O & O \\ \vdots & \vdots & & \vdots & \vdots \\ Y_{p-11} & Y_{p-12} & \cdots & J_{n_{p-1}} & O \\ Y_{p1} & Y_{p2} & \cdots & Y_{pp-1} & J_{n_p} \end{bmatrix}$$



by lower similarity we can transform  $A$  into the matrix

$$\bar{A} = \begin{bmatrix} J_{n_1} & O & \cdots & O & O \\ C_{21} & J_{n_2} & \cdots & O & O \\ \vdots & \vdots & & \vdots & \vdots \\ C_{p-11} & C_{p-12} & \cdots & J_{n_{p-1}} & O \\ C_{p1} & C_{p2} & \cdots & C_{pp-1} & J_{n_p} \end{bmatrix},$$

where

$$C_{ij} = \begin{bmatrix} c_{11}^{(ij)} & 0 & \cdots & 0 \\ c_{21}^{(ij)} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{n_i 1}^{(ij)} & 0 & \cdots & 0 \end{bmatrix}_{n_i \times n_j}$$

for  $i > j$ ,  $i = 2, 3, \dots, p$  and  $j = 1, 2, \dots, p-1$ , and  $C_{ii-1}$  is the left concentrated form of the matrix  $Y_{ii-1}$ , for  $i = 2, 3, \dots, p$ . We call the matrix  $\bar{A}$  *block reduction to the first column* of the matrix  $A$ .

The algorithm given in [6] is designed to work on a partial block upper triangular matrices, that is, matrices with the following structure

$$\tilde{A} = \begin{bmatrix} J_{n_1} & O & \cdots & O & O \\ X_{21} & J_{n_2} & \cdots & O & O \\ \vdots & \vdots & & \vdots & \vdots \\ X_{p-11} & X_{p-12} & \cdots & J_{n_{p-1}} & O \\ X_{p1} & X_{p2} & \cdots & X_{pp-1} & J_{n_p} \end{bmatrix},$$

where the matrix  $X_{ij}$  has all its elements unknown, for  $i > j$ ,  $i = 2, 3, \dots, p$  and  $j = 1, 2, \dots, p-1$ .

Now consider the following partial block matrix

$$M_1 = \begin{bmatrix} D_1 & E_{12} & \cdots & E_{1p-1} & E_{1p} \\ B_{21} & D_2 & \cdots & E_{2p-1} & E_{2p} \\ \vdots & \vdots & & \vdots & \vdots \\ B_{p-11} & B_{p-12} & \cdots & D_{p-1} & E_{p-1p} \\ B_{p1} & B_{p2} & \cdots & B_{pp-1} & D_p \end{bmatrix},$$

where

- For  $i = 1, 2, \dots, p$

$$D_i = \begin{bmatrix} 0 & a_{12}^{(i)} & a_{13}^{(i)} & \cdots & a_{1n_i-1}^{(i)} & a_{1n_i}^{(i)} \\ x & 0 & a_{23}^{(i)} & \cdots & a_{2n_i-1}^{(i)} & a_{2n_i}^{(i)} \\ x & x & 0 & \cdots & a_{3n_i-1}^{(i)} & a_{3n_i}^{(i)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x & x & x & \cdots & 0 & a_{n_i-1 n_i}^{(i)} \\ x & x & x & \cdots & x & 0 \end{bmatrix}_{n_i \times n_i}$$

and  $\text{rank}(D_{i_0}) = n_i - 1$ .

- For  $i > j$ ,  $i = 2, 3, \dots, p$  and  $j = 1, 2, \dots, p - 1$ , the matrix  $B_{ij}$  has at least its first column formed by unspecified elements, being the rest of its elements unspecified or zero.

- For  $i < j$ ,  $i = 1, 2, \dots, p - 1$  and  $j = 2, 3, \dots, p$ , the elements of block  $E_{ij}$  can be unknown or equal to zero.

It is easy to see that the Segre characteristic of the completion  $M_{1_0}$  is  $\{n_1, n_2, \dots, n_p\}$ . Let  $\{m_j\}_{j=1}^s$  be a sequence such that  $\{n_i\}_{i=1}^p \prec \{m_j\}_{j=1}^s$ . We obtain a completion of the matrix  $M_1$  such that its Segre characteristic is  $\{m_j\}_{j=1}^s$  by applying the following process:

(a) Replace by zeros all the unspecified elements of  $M_1$ , except the elements in the first column of blocks  $B_{ij}$ , for  $i > j$ ,  $i = 2, 3, \dots, p$  and  $j = 1, 2, \dots, p - 1$ . Therefore we obtain:

$$M_2 = \begin{bmatrix} D_{1_0} & O & O & \cdots & O & O \\ \tilde{B}_{21} & D_{2_0} & O & \cdots & O & O \\ \tilde{B}_{31} & \tilde{B}_{32} & D_{3_0} & \cdots & O & O \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \tilde{B}_{p1} & \tilde{B}_{p2} & \tilde{B}_{p3} & \cdots & \tilde{B}_{pp-1} & D_{p_0} \end{bmatrix}.$$

(b) By admissible similarity, the matrix  $M_2$  is transformed into the matrix

$$M_3 = T^{-1}M_2T = \begin{bmatrix} J_{n_1} & O & O & \cdots & O \\ S_{21} & J_{n_2} & O & \cdots & O \\ S_{31} & S_{32} & J_{n_3} & \cdots & O \\ \vdots & \vdots & \vdots & & \vdots \\ S_{p1} & S_{p2} & S_{p3} & \cdots & J_{n_p} \end{bmatrix},$$

where the matrix  $S_{ij}$ , for  $i > j$ ,  $i = 2, 3, \dots, p$  and  $j = 1, 2, \dots, p - 1$ , has its first column formed by unknown elements, and the other entries equal to zero.

(c) Apply the Algorithm given in [6] to the matrix  $\tilde{A}$  in order to obtain a completion  $\tilde{A}_c$  such that its Segre characteristic is  $\{m_j\}_{j=1}^s$ .

(d) Obtain the block reduction to the first column of  $\tilde{A}_c$  and call  $\tilde{A}_{c_1}$  the matrix obtained.

The matrix  $\tilde{A}_{c_1}$  is also a completion of  $M_3$ . Since the matrix  $T$  is an admissible similarity,  $T\tilde{A}_{c_1}T^{-1}$  is a completion of  $M_2$  and therefore of  $M_1$  with the desired characteristic.

EXAMPLE 3.1. Consider the partial block matrix

$$M_1 = \left[ \begin{array}{cccc|cc|ccc} 0 & 1 & 1 & 1 & x & 0 & x & x & 0 \\ x & 0 & 2 & 1 & 0 & x & 0 & x & x \\ x & x & 0 & 1 & x & x & x & 0 & x \\ x & x & x & 0 & 0 & x & 0 & x & 0 \\ \hline x & 0 & 0 & 0 & 0 & 2 & x & 0 & x \\ x & x & x & 0 & x & 0 & 0 & x & 0 \\ \hline x & x & 0 & 0 & x & 0 & 0 & 1 & 2 \\ x & x & x & 0 & x & x & x & 0 & 2 \\ x & 0 & x & 0 & x & 0 & x & x & 0 \end{array} \right].$$

The Segre characteristic of the matrix  $M_{1_0}$  is  $\{4, 2, 3\}$ . We want to find a completion  $M_{1_c}$  of  $M_1$  such that its Segre characteristic is the sequence  $\{5, 4\}$ .

Firstly we consider the following matrix

$$M_2 = \left[ \begin{array}{cccc|cc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x & 0 & 0 & 0 & x & 0 & 0 & 1 & 2 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 2 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \end{array} \right].$$

By using the admissible similarity

$$T = \left[ \begin{array}{cccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{array} \right],$$

we transform  $M_2$  into the matrix

$$M_3 = T^{-1}M_2T = \left[ \begin{array}{cccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x & 0 & 0 & 0 & x & 0 & 0 & 1 & 0 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 1 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \end{array} \right].$$

By applying the Algorithm given in [6] to

$$\tilde{A} = \begin{bmatrix} J_4 & O & O \\ X_{21} & J_2 & O \\ X_{31} & X_{32} & J_3 \end{bmatrix}$$

where  $X_{21}, X_{31}$  and  $X_{32}$  have all their elements unspecified, we obtain  $\tilde{A}_c$ , which Segre characteristic is  $\{5, 4\}$ . By doing its block reduction to the first column we obtain

$$\tilde{A}_{c_1} = \left[ \begin{array}{cccc|ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The matrix

$$M_{1_c} = T\tilde{A}_{c_1}T^{-1} = \left[ \begin{array}{cccc|ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

is a completion of  $M_1$  with the desired characteristic.

**Part II:** Consider the following lower irreducible partial upper triangular matrix,

$$A = \begin{bmatrix} 0 & 0 & \cdots & a_{1j} & \cdots & a_{1t} & \cdots & a_{1n} \\ & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & & \ddots & \vdots & & \vdots & & \vdots \\ & & & 0 & \cdots & 0 & \cdots & a_{jn} \\ & & & & \ddots & \vdots & & \vdots \\ & & & & & 0 & \cdots & 0 \\ & & & & & & \ddots & \vdots \\ & & & & & & & 0 \end{bmatrix}.$$

This matrix satisfies that  $r(A) = 2$ ,  $r(A^2) = 1$  and  $r(A^3) = 0$ . The Jordan form of the completion  $A_0$  consists of  $n - 2$  blocks of size  $q_1 = 3$  and  $q_i = 1$ ,  $i = 2, 3, \dots, n - 2$ . Therefore,  $r(A) = \text{rank}(A_0) = 2$ ,  $r(A^2) = \text{rank}(A_0^2) = 1$  and  $r(A^3) = \text{rank}(A_0^3) = 0$ .

If the sequence  $\{n_i\}_{i=1}^p$  satisfies condition (2) then it majorizes the sequence  $\{q_i\}_{i=1}^{n-2}$ . We are going to obtain a completion  $A_c$  of the matrix  $A$  whose Segre characteristic is  $\{n_i\}_{i=1}^p$ .

By performing the permutation  $(1, j, n, 2, \dots, j-1, j+1, \dots, n-1)$  of rows and columns in  $A$ , we obtain

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{11} = \begin{bmatrix} 0 & a_{1j} & a_{1n} \\ * & 0 & a_{jn} \\ * & * & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & a_{1t} & \cdots & 0 \\ * & \cdots & * & 0 & \cdots & 0 & \cdots & 0 \\ * & \cdots & * & * & \cdots & * & \cdots & * \end{bmatrix},$$

$A_{21}$  is a matrix of size  $(n-3) \times 3$  whose first column is formed by unspecified elements and  $A_{22}$  is a partial upper triangular matrix whose known elements are equal to zero. Next, consider the matrix

$$\bar{A}_1 = \begin{bmatrix} A_{11} & \bar{A}_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where  $\bar{A}_{12}$  is like  $A_{12}$  with  $a_{1t} = 0$ . The structure of  $\bar{A}_1$  allows us to apply the algorithm given in [6] and to obtain a completion

$$\bar{A}_{1c} = \begin{bmatrix} A_{11o} & 0 \\ A_{21c} & A_{22c} \end{bmatrix},$$

whose Segre characteristic is  $\{n_i\}_{i=1}^p$ . We can distinguish two cases:

**(a)** The row  $t+1$  is zero.

Then the matrix

$$A_{1c} = \begin{bmatrix} A_{11o} & A_{12o} \\ A_{21c} & A_{22c} \end{bmatrix},$$

is a completion of  $A_1$  similar to  $\bar{A}_{1c}$ .

**(b)** The row  $t+1$  of  $\bar{A}_{1c}$  has an 1.

In this case, consider the matrix

$$A_2 = \begin{bmatrix} \tilde{A}_{11} & A_{12} \\ A_{21c} & A_{22c} \end{bmatrix},$$

where  $\tilde{A}_{11}$  is like  $A_{11}$  but its entry  $(3, 2)$  is equal to 0. We are going to find a completion  $A_{2c}$  similar to  $\bar{A}_{1c}$ .

If the 1 of  $\bar{A}_{1c}$  is in position  $(t+1, 1)$ , the matrix  $A_{2c}$  is

$$A_{2c} = \begin{bmatrix} \tilde{A}_{11c} & A_{12o} \\ A_{21c} & A_{22c} \end{bmatrix},$$

where

$$\tilde{A}_{11c} = \begin{bmatrix} 0 & a_{1j} & a_{1n} \\ \alpha & 0 & a_{jn} \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{with } \alpha = -a_{1t}/a_{1j}.$$

If the 1 of  $\bar{A}_{1c}$  is in position  $(t + 1, h)$ , with  $3 < h \leq j + 1$ , the matrix  $A_{2c}$  is

$$A_{2c} = \begin{bmatrix} \tilde{A}_{11c} & A_{12c} \\ A_{21c} & A_{22c} \end{bmatrix},$$

where

$$A_{12c} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & a_{1t} & \cdots & 0 \\ 0 & \cdots & \alpha & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

with  $\alpha = -a_{1t}/a_{1j}$  in position  $(2, h)$ .

Finally, if the 1 is in position  $(t + 1, h)$ , with  $j + 1 < h \leq t$ , by elementary transformations we obtain a completion  $A_{2c}$  similar to  $\bar{A}_{1c}$ .

EXAMPLE 3.2. Consider the partial upper triangular matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 1 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 \end{bmatrix}.$$

We are going to find a completion of  $A$  such that its Segre characteristic is  $\{7, 1\}$ .

First, by performing the permutation  $(1, 3, 8, 2, 4, 5, 6, 7)$  we obtain the matrix

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & | & 0 & 0 & 0 & 1 & 0 \\ * & 0 & 1 & | & * & 0 & 0 & 0 & 0 \\ * & * & 0 & | & * & * & * & * & * \\ \hline * & 0 & 0 & | & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & | & * & 0 & 0 & 0 & 0 \\ * & * & 0 & | & * & * & 0 & 0 & 0 \\ * & * & 0 & | & * & * & * & 0 & 0 \\ * & * & 0 & | & * & * & * & * & 0 \end{bmatrix}.$$

By applying the algorithm of [6] to the matrix  $\bar{A}_1$  we obtain the following matrix

whose Segre characteristic is  $\{7, 1\}$

$$\bar{A}_{1_c} = \left[ \begin{array}{ccc|cccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Since there is an 1 in the row 7, consider the matrix

$$A_2 = \left[ \begin{array}{ccc|cccc} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ * & 0 & 1 & * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & * & * & * & * & * \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

By applying the elementary transformations  $C7 - C2 \rightarrow C7$ ,  $C6 - C3 + C2 \rightarrow C6$  and  $C5 + C3 - C2 \rightarrow C5$  (where  $C_i$  is the column  $i$  of  $A_2$ ), and by its corresponding transformations by rows, we obtain the matrix

$$A_{2_c} = \left[ \begin{array}{ccc|cccc} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

This matrix is a completion of  $A_2$ , similar to  $\bar{A}_{1_c}$ . By performing the permutation  $(1, 4, 2, 5, 6, 7, 8, 3)$  of rows and columns in the matrix  $A_{2_c}$  we obtain a completion  $A_c$  of the matrix  $A$  whose Segre characteristic is  $\{7, 1\}$ .

REFERENCES

- [1] Joseph A. Ball, Israel Gohberg, Leiba Rodman and Tamir Shalom. On the eigenvalues of matrices with given upper triangular part. *Integral Equations and Operator Theory*, 13:488–497, 1990.
- [2] Israel Gohberg and Sorin Rubinstein. A classification of upper equivalent matrices: the generic case. *Integral Equations and Operator Theory*, 14:533–544, 1991.
- [3] Leonid Gurvits, Leiba Rodman and Tamir Shalom. Controllability and completion of partial upper triangular matrices over rings. *Linear Algebra and its Applications*, 172:135–149, 1992.

- [4] Leonid Gurvits, Leiba Rodman and Tamir Shalom. Controllability of completion of partial upper triangular matrices. *Mathematics of Control, Signals and Systems*, 6:30–40, 1993.
- [5] Cristina Jordán, Juan R. Torregrosa and Ana M. Urbano. On the Jordan Form of Completions of Partial Upper Triangular Matrices. *Linear Algebra and its Applications*, 254:241–250, 1997.
- [6] Cristina Jordán, Juan R. Torregrosa and Ana M. Urbano. An Algorithm for Nilpotent Completions of Partial Jordan Matrices. *Linear Algebra and its Applications*, 275/276:315–325, 1998.
- [7] Mark Krupnik. Geometric multiplicities of completions of partial triangular matrices. *Linear Algebra and its Applications*, 220:215–227, 1995.
- [8] Mark Krupnik and Leiba Rodman. Completions of partial Jordan and Hessenberg matrices. *Linear Algebra and its Applications*, 212/213:267–287, 1994.
- [9] Leiba Rodman and Tamir Shalom. Jordan form of completions of partial upper triangular matrices. *Linear Algebra and its Applications*, 168:221–249, 1992.
- [10] Ion Zaballa. Matrices with prescribed rows and invariant factors. *Linear Algebra and its Applications*, 87:113–146, 1987.